Indian Statistical Institute, Delhi Centre

Branching Processes

Fall 2009

Assignment # 3

Date Given: October 1, 2009 (Thursday) **Total Points: 20**
Date Due: October 8, 2009 (Thursday)

1. Suppose $(X_n, \mathcal{F}_n)_{n \geq 0}$ is a martingale such that $\mathbf{E}\left[X_n^2\right] < \infty$ for all $n \geq 0$. Then prove that

   (a) For any $m \geq 1$ and any square integrable $\mathcal{F}_m$-measurable random variable Y we must have $\mathbf{E}\left[(X_n - X_m)\,Y\right] = 0$ for all $n \geq m$.

   Note: So in particular we have $\mathbf{E}\left[(X_n - X_m)\,X_m\right] = 0$ for all $n \geq m \geq 1$. Thus a *martingale increment* is *orthogonal* to its current value.

   (b) $\mathbf{E}\left[(X_n - X_m)^2 \,\middle|\, \mathcal{F}_m\right] = \mathbf{E}\left[X_n^2 \,\middle|\, \mathcal{F}_m\right] - X_m^2$. [5 + 3 = 8]

2. Suppose $(Z_n)_{n \geq 1}$ are i.i.d. Normal$\,(0, 1)$ random variables and θ is another random random variable with finite first moment which is independent of $(Z_n)_{n \geq 1}$. Define $X_n = \theta + Z_n$ for $n \geq 1$. Show that

$$\mathbf{E}\left[\theta \,\middle|\, X_1, X_2, \ldots, X_n\right] \xrightarrow[\text{a.s.}]{\mathcal{L}_1} \theta\,.$$

   [7]

3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{X_\beta\}_{\beta \in \mathcal{I}}$ be a collection of random variables such that there exists a non-negative random variable Y which is integrable such that $|X_\beta| \leq Y$ for all $\beta \in \mathcal{I}$. Then show that the collection $\{X_\beta\}_{\beta \in \mathcal{I}}$ is *uniformly integrable.* [5]