Real Analysis

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Week 1

1 Prerequisites

1.1 Sets and mappings

1.2 Relations

Let S be a set. A **relation** R on S is a subset of $S \times S$. It is customery to write $x\,R\,y$ instead of $(x, y) \in R$ while talking about relations.

A relation R is said to be

1. **reflexive** if $\boxed{x\,R\,x \text{ for all } x \in S}$.
2. **symmetric** if $\boxed{x\,R\,y \text{ implies } y\,R\,x}$.
3. **transitive** if $\boxed{x\,R\,y \text{ and } y\,R\,z \text{ imply } x\,R\,z}$.
4. an **equivalence relation** if $\boxed{\text{it is reflexive, symmetric and transitive}}$.
5. an **order relation** if it is transitive and for any two elements x and y, exactly one of the following occurs: $x = y$, $x\,R\,y$, $y\,R\,x$.

An **ordered set** is a pair (S, R), where S is a set and R is an order relation defined on it.

Notation. Let (S, R) be an ordered set, and let $a, b \in S$. Then

$$(a, b) := \{x \in S : a\,R\,x, x\,R\,b\}, \quad [a, b] = (a, b) \cup \{a, b\}.$$

$$[a, b) = (a, b) \cup \{a\}, \quad (a, b] := (a, b) \cup \{b\}.$$

$$(-\infty, b) := \{x \in S : x\,R\,b\}, [a, b) = (a, b) \cup \{a\}, \quad (a, \infty) := \{x \in S : a\,R\,x\},$$

$$(-\infty, b] := (-\infty, b) \cup \{b\}, \quad [a, \infty) := (a, \infty) \cup \{a\}.$$

Let S be an ordered set, R being an order relation defined on it. Let A be a subset of S. An element $a \in S$ is said to be an **upper bound** for A if for any $x \in A$, either $x\,R\,a$ or $x = a$. An element $b \in S$ is said to be a **least upper bound** (or **supremum**) for A if

1. b is an upper bound of A, and

2. if c is an upper bound for A, then either $b = c$ or $b\,R\,c$.

Exercise 1.1 Show that a subset A of an ordered set S can not have more than one supremums.

Notation. If x is the least upper bound of a set E, we write $x = \sup E$.

Notation. Let $(S, <)$ be an ordered set and $x, y \in S$. By $x > y$, we mean $y < x$, by $x \leq y$, we mean 'either $x < y$ or $x = y$', and by $x \geq y$, we mean 'either $y < x$ or $y = x$'.

Exercise 1.2 Let $(S, <)$ be an ordered set and let $E, F \subseteq S$.

1. Suppose $a = \sup E$, and $b < a$. Show that there is an element $x \in E$ such that $b < x$.
2. If $x \leq y$ for all $x \in E$ and $y \in F$, then show that $\sup E < \sup F$ or $\sup E = \sup F$.

Least upper bound property:
Let S be an ordered set. It is said to have the least upper bound property if whenever a nonempty subset S has an upper bound, it has a least upper bound in S.

1.3 Binary operations

A **binary operation** on a set S is a map from $S \times S$ to S.

2 Real Numbers

Definition 2.1 The space of real numbers $\mathbb{R}$ is a set, equipped with two operations, addition $(+)$ and multiplication $(.)$, and a relation $<$, satisfying properties 1–5 below:

1. Properties of addition:

A1 $(x + y) + z = x + (y + z)$,

A2 $x + 0 = x = 0 + x$,

A3 for each $x \in \mathbb{R}$, there is a $-x \in \mathbb{R}$ such that $x + (-x) = 0 = (-x) + x$.

A4 $x + y = y + x$,

In short, $\mathbb{R}$ is an abelian group under addition.

2. Properties of multiplication:

M1 $(x.y).z = x.(y.z)$,

M2 $x.1 = x = 1.x$,

M3 for each $x \neq 0 \in \mathbb{R}$, there is a $1/x \in \mathbb{R}$ such that $x.(1/x) = 1 = (1/x).x$.

M4 $x.y = y.x$,

In short, $\mathbb{R}\backslash\{0\}$ is an abelian group under multiplication.

3. Distributive law holds: $(x + y)z = xz + yz$.

Any space, equipped with two operations satisfying 1, 2 and 3 above is called a **field**.

Exercise 2.2 Shw that

1. $0.x = 0$ for all $x \in \mathbb{R}$.
2. if $x + y = x + z$, then $y = z$.
3. if $x \cdot y = 0$ and $x \neq 0$, then $y = 0$.
4. if $xy = xz$ and $x \neq 0$, then $y = z$.
5. $-(-x) = x$ for all $x \in \mathbb{R}$.
6. $\frac{1}{1/x} = x$.

Definition 2.3 An **ordered field** is a field, with an order relation $<$ such that

1. $a < b \Rightarrow a + c < b + c$,
2. $0 < a,\ 0 < b \Rightarrow 0 < ab$.

4. $(\mathbb{R}, <)$ **is an ordered field**

Exercise 2.4 Show that

1. $0 < 1$.
2. $0 < x < y$ implies $0 < 1/y < 1/x$.
3. What happens if $x < y < 0$?
4. An ordered field is always infinite.
5. A field that is ordered is not necessarily an ordered field.

5. $\mathbb{R}$ **has the LUB property**.

Remark 2.5 At this point, we can give the definitions of the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{Q}$. Call a subset E of $\mathbb{R}$ **inductive** if whenever it contains a real x, it also contains $x+1$. $\mathbb{R}$ itself is one such set. Now define $\mathbb{N}$ to be the intersection of all inductive sets that contain 0. Let us first convince ourselves that this will indeed define what we normally mean by the set of natural numbers. Let $n := \underbrace{1 + 1 + \cdots + 1}_{\text{n times}}$. First, if $a < 0$, then $a \notin \mathbb{N}$. Because the set $\mathbb{R}_+$ is inductive, and it does not contain a. Next, if $0 < a < 1$, then $a \notin \mathbb{N}$; for the set $\{0\} \cup [1, \infty)$ is inductive and does not contain a. Similarly, if $n < a < n+1$, then $a \notin \mathbb{N}$, because $\{0, 1, \ldots, n\} \cup [n+1, \infty)$ is inductive and does not contain a.

Having defined $\mathbb{N}$, it is easy to define the other three sets.

Lemma 2.6 *Between any two real numbers, there is another.*

Proof: □

Lemma 2.7 *Let $x, y \in \mathbb{R}$. If $x \leq y + \epsilon$ for all $\epsilon > 0$, then $x \leq y$.*

Proof: □

Lemma 2.8 *Let $E, F \subseteq \mathbb{R}$, and $G = \{x + y : x \in E, y \in F\}$. Then $\sup G = \sup E + \sup F$.*

Proof: $\sup E + \sup F$ will be an upper bound for G. So $\sup G \leq \sup E + \sup F$. Therefore it is enough to show that $\sup G \geq \sup E + \sup F$. Take $\epsilon > 0$. We will show that $\sup G + \epsilon \geq \sup E + \sup F$. Choose $x \in E$, $y \in F$ such that $x > \sup E - \epsilon/2$ and $y > \sup F - \epsilon/2$. Then $x + y > \sup E + \sup F - \epsilon$. Hence $\sup G > \sup E + \sup F - \epsilon$. □

Exercise 2.9 Show that $x > y > 0$ implies $x^2 > y^2$. Using this, show that if $x > 0$, $y > 0$ and $x^2 > y^2$, then $x > y$.

Example 2.10

1. Let $E = \{r \in \mathbb{Q} : r < 1/2\}$. Then $\sup E = 1/2$.
2. Let $E = \{r \in \mathbb{Q} : r^2 < 2\}$. Then $(\sup E)^2 = 2$.
   Sketch of proof: Write $y = \sup E$. We will show that we can not have $y^2 < 2$ or $y^2 > 2$. First, suppose if posssible that $y^2 > 2$. Choose an h such that $(y - h)^2 > 2$. Then $y - h$ is an upper bound, which contradicts the definition of y. Next assume if possible that $y^2 < 2$. This time choose an $h > 0$ such that $y + h$ is rational and $(y + h)^2 < 2$. This would contradict the fact that y is an upper bound. So one must have $y^2 = 2$.

Theorem 2.11 (Principle of Mathematical Induction) *Let E be a set of positive integers such that*
(i) $1 \in E$,
(ii) if an integer $k \in E$, then $k + 1 \in E$.
Then E contains all positive integers.

Proof: $E \cup \{0\}$ is an inductive set containing the element 0. Hence $\mathbb{N} \subseteq E \cup \{0\}$. Hence $\mathbb{Z}_+ = \mathbb{N} \backslash \{0\} \subseteq E$. □

Theorem 2.12 (Well Ordering Principle) *Every nonempty subset of $\mathbb{N}$ has a smallest member.*

Proof: Suppose if possible S is a nonempty subset of $\mathbb{N}$ that has no smallest element. Let $E = \{x \in \mathbb{N} : x < n \text{ for all } n \in S\}$, i. e. E is the set of strict lower bounds of S in $\mathbb{N}$. Then $0 \in E$, because, otherwise we must have $0 \in S$, which would mean 0 is the smallest element of S. Next, if $x \in E$, then $x + 1 \in E$. For, if $x + 1 \notin E$ then there is a $t \in S$ such that $t \leq x + 1$. Since S has no smallest member, there is an $s \in S$ such that $s < t$. But then $s < x + 1$, and therefore $s \leq x$ so that $x \notin E$. Thus E is inductive, and hence contains all the natural numbers. In particular $S \subseteq E = \mathbb{N}$. Now take a $t \in S$. Since $t \in E$ also, this implies $t < t$, which can not happen. □

Elementary consequences:

Theorem 2.13 *1. $\mathbb{N}$ is not bounded above.*

2. If x and y are positive reals, then $\exists n \in \mathbb{N}$ such that $y < nx$.

3. If x is a positive real, then $\exists n \in \mathbb{N}$ such that $0 < 1/n < x$.

4. Let E be a bounded subset of $\mathbb{N}$. Then it has a maximum element.

5. Between any two distinct real numbers, there is a rational number.

6. Given a positive real x and a positive integer n, there is one and only one positive real y such that $y^n = x$.

7. Between any two distinct real numbers, there is an irrational number.

Proof:

1. Suppose if possible $\mathbb{N}$ has an upper bound. By the LUB property of $\mathbb{R}$, it has an upper bound x in $\mathbb{R}$. Since then $x - 1$ is not an upper bound for $\mathbb{N}$, there is an $n \in \mathbb{N}$ such that $n > x - 1$. But then $n + 1 > x$, i.e. x can not be an upper bound for $\mathbb{N}$.

2. By (1), there is an $n \in \mathbb{N}$ such that $y/x < n$.

3. Take $y = 1$ in (2).

4. Let $t \in \mathbb{R}$ be an upper bound for E so that $n < t$ for all $n \in E$. By (1), there is an integer $k \in \mathbb{N}$ such that $t < k$. Define $F = \{k - n : n \in E\}$. Since for all $n \in E$, $n < k$, this is a subset of $\mathbb{N}$. Hence by the well ordering principle, F has a minimum element, say $k - m$, where $m \in E$. This means $k - m \leq k - n$ for all $n \in E$, i. e. $m \geq n$ for all $n \in E$. But $m \in E$. So m is the maximum element of E.

5. Consider the case $0 < x < y$. By (3), there is an $n \in \mathbb{N}$ such that $0 < 1/n < y - x$. Now look at the set $\{m/n : m \in \mathbb{N}\}$. Let $m_0 = max\{m : m/n \leq x\}$. Then we have

$x < \frac{m_0+1}{n} < y$ (the first inequality follows from the definition of m_0, and the second follows from the fact that $\frac{m_0+1}{n} - x = 1/n - (x - \frac{m_0}{n}) \leq 1/n < y - x$).

The case $x < y < 0$ is similar. In the other two cases, viz. $x < 0 < y$ and $y < 0 < x$, 0 would be a rational between the two.

6. *Proof for the case n=2*: Let $E = \{t \in \mathbb{R} : t \geq 0, t^2 \leq x\}$. Observe that $(\frac{x}{1+x})^2 = \frac{1}{(1+\frac{1}{x})^2} < \frac{1}{1/x} = x$, and $(1+x)^2 > x$. Therefore the set E is nonempty and bounded above. Let $y = \sup E$.
   Case I. $y^2 < x$: For any h, $(y+h)^2 = y^2 + 2hy + h^2 = y^2 + h(2y+h)$. Choose h in such a way that $0 < h < y$, and $h < \frac{x-y^2}{3y}$. Then $(y+h)^2 < y^2 + \frac{x-y^2}{3y} 3y = x$. So y can not be the supremum of E.
   Case II. $y^2 > x$: $(y-h)^2 = y^2 - 2hy + h^2 = y^2 - h(2y-h)$. Choose $h \in (0, y)$ such that $h < \frac{y^2-x}{3y}$. Then $(y-h)^2 = y^2 - h(2y-h) > y^2 - h(2y+h) > y^2 - (y^2 - x) = x$. So $y - h$ is an upper bound, which is not the case.

   Therefore we must have $y^2 = x$.

   Proof for general n: As before, let $E = \{t \in \mathbb{R} : t \geq 0, t^n \leq x\}$ and $y = \sup E$. If $y^n < x$, choose h such that $0 < h < y$ and $h < \frac{x-y^n}{2^n y^{n-1}}$. Then

$$\begin{aligned}
(y+h)^n &= y^n + h\left(\binom{n}{1}y^{n-1} + \binom{n}{2}y^{n-2}h + \cdots + h^{n-1}\right) \\
&< y^n + \frac{x-y^n}{2^n y^{n-1}}\left(\binom{n}{1} + \cdots + \binom{n}{n}\right)y^{n-1} \\
&< x.
\end{aligned}$$

   But this is not possible.

   Similarly if $y^n > x$, choose h such that $0 < h < y$, $h < \frac{y^n-x}{2^n y^{n-1}}$. Then

$$\begin{aligned}
(y-h)^n &= y^n - h\left(\binom{n}{1}y^{n-1} - \binom{n}{2}y^{n-2}h + \cdots \pm h^{n-1}\right) \\
&> y^n - h\left(\binom{n}{1}y^{n-1} + \binom{n}{2}y^{n-2}h + \cdots + h^{n-1}\right) \\
&> y^n - \frac{y^n-x}{2^n y^{n-1}}\left(\binom{n}{1} + \cdots + \binom{n}{n}\right)y^{n-1} \\
&> x.
\end{aligned}$$

   This would imply $y - h$ is an upper bound for E.

   So we must have $y^n = x$.

7. Use part 5. □

Week 2

2.1 Rational powers of positive numbers

For $x > 0$ and p a positive rational, define x^p to be $(x^m)^{1/n}$ where $p = m/n$ with $m, n \in \mathbb{Z}_+$,.

Exercise 2.14 Show that the above definition is independent of the choice of m and n.

For $x > 0$ and p a negative rational, define x^p to be $\frac{1}{x^{-p}}$. Define x^0 to be 1 for all $x > 0$.

Exercise 2.15 Let $x, y > 0$ and $p, q \in \mathbb{Q}$. Show that

$$x^p x^q = x^{p+q}, \quad (x^p)^q = x^{pq}, \quad (x^p)(y^p) = (xy)^p.$$

Show that if $x > 1$ and $0 < p < q$, then $x^p < x^q$.

Exercise 2.16 Let E, $F \subseteq \mathbb{R}_+$, with $\sup E > 0$, $\sup F > 0$. Assume $EF \subseteq G$. Show that $(\sup E)(\sup F) \leq \sup G$.

2.2 Arbitrary powers of positive numbers

Let $x > 1$, $p > 0$. Define

$$E_{p,x} = \{x^r : r \in \mathbb{Q}, 0 \leq r < p\}.$$

Take some rational $s > p$. Then $y < x^s$ for all $y \in E_{p,x}$. Thus $E_{p,x}$ is bounded above. Define $f_p(x) = \sup E_{p,x}$.

Proposition 2.17 *Let $x > 1$, $p, q > 0$, and $f_p(x)$ be as above. Then*

1. $f_p(x) = x^p$ *if* $p \in \mathbb{Q}$,
2. $f_p(x)f_q(x) = f_{p+q}(x)$,
3. $f_q(f_p(x)) = f_{pq}(x)$,
4. $f_p(x) < f_q(x)$ *if* $p < q$,
5. $f_p(x)f_p(y) = f_p(xy)$.

Proof: We will prove the first two parts here. Others would be left as exercises.

Part (1) will follow from the following lemma.

Lemma 2.18 *Let $x > 1$, and p be a positive rational. Given any $\epsilon > 0$, there exists a rational $r < p$ such that $x^p - \epsilon < x^r$.*

Proof: Choose $N \in \mathbb{N}$ such that $x^p(\frac{x-1}{N}) < \epsilon$. Now write p in the form m/n where $n > N$. Take $r = \frac{m-1}{n}$. Then

$$\begin{aligned} x^p - x^r &= x^{\frac{m}{n}} - x^{\frac{m-1}{n}} \\ &= x^{\frac{m-1}{n}}(x^{\frac{1}{n}} - 1) \\ &< x^p \frac{x-1}{n} \\ &< \epsilon. \end{aligned}$$

□

Next, we will show that $f_p(x)f_q(x) = f_{p+q}(x)$ for all $p, q > 0$.

It is easy to see that $E_pE_q \subseteq E_{p+q}$ for all $p, q > 0$. Hence it follows that $f_p(x)f_q(x) \leq f_{p+q}(x)$ We now need to prove the reverse inequality.

Take an $\epsilon > 0$. Choose a rational t such that $t < p + q$ and $x^t > f_{p+q}(x) - \epsilon$ (in the case $p+q$ is a rational, we have to use the above lemma for this). Let r and s be rationals such that $r \leq p$, $s \leq q$ and $r + s = t$. Then $x^r x^s = x^t > f_{p+q}(x) - \epsilon$. Hence $f_p(x)f_q(x) > f_{p+q}(x) - \epsilon$. Since this is true for all $\epsilon > 0$, we have $f_p(x)f_q(x) \geq f_{p+q}(x)$. □

This number $f_p(x)$ will be called the **$\mathbf{p}^{\mathbf{th}}$ power of** x and will normally be denoted by x^p. For $0 < x < 1$, define x^p to be $\frac{1}{(1/x)^p}$. For $x > 0$ and $p < 0$, define x^p to be $\frac{1}{x^{-p}}$.

Exercise 2.19 Show that if $x > 0$ and p, q are any real numbers, then

$$x^p x^q = x^{p+q}, \quad (x^p)^q = x^{pq}.$$

3 Countable Sets

Suppose we are given two sets E and F and asked to show that E contains en element that is not contained in F. One way of course would be to check each and every element of E if it belongs to F or not. But that may not always be possible. If we could somehow show that the set E is 'bigger' than F, that would automatically imply the existence of one such element. Thus it is natural to try and compare the 'sizes' of different sets. For finite sets, this is easy. One just has to count the number of elements in each set and compare them. But this way of comparing can not be readily applied for infinite sets. For finite sets, another equivalent alternative way is to see if there is any bijection between the two sets. They are of the same size if there is some bijection. This notion extends readily to infinite sets as well, and we say two sets are of the same size if there exists a bijection between the two. Now the standard accepted terminology for 'size' is **cardinality**.

Two sets E and F are said to be of the same cardinality if there is a bijective map from one to the other.

For finite sets, in addition to comparing the cardinality of two sets, we can also talk about the value of their cardinality, which is given by a natural number. For example, we say that a finite set is of cardinality n if there is a bijection between that set and the set $\{1, 2, \ldots, n\}$. For infinite sets, the way we have defined the concept of cardinals being equal, it is clear that the 'value' has to be some known set, rather than any integer, or real number. The most natural candidate that comes to mind immediately is the set $\mathbb{N}$ of natural numbers. There is a special name for sets whose cardinality 'equals' $\mathbb{N}$, or, whose cardinality is the same as that of $\mathbb{N}$.

Definition 3.1 A set E is said to be **countable** if there is a bijection between E and $\mathbb{N}$. It is said to be **at most countable** if it is finite or countable.

Example 3.2 1. $\mathbb{Z}$. We can take the following bijection:

$$k \mapsto \begin{cases} 2k & \text{if } k \geq 0, \\ -2k-1 & \text{if } k < 0. \end{cases}$$

2. The set of all even integers.

3. The set of all odd integers.

Proposition 3.3 *Any subset of a countable set is at most countable.*

Proof: It is enough to prove that any infinite subset of a countable set is countable. Let F be a countable set and E be an infinite subset of F. Let $f : \mathbb{N} \to F$ be a bijection. Then

$$F = \{f(0), f(1), \ldots\}.$$

Let us define a function $g : \mathbb{N} \to E$ as follows:

$g(0) = f(k)$ if k is the least integer n for which $f(n) \in E$,
(i. e. $k = \min\{n : f(n) \in E\} = \min f^{-1}(E)$)

Having chosen $g(0), g(1), \ldots, g(r)$, let

$$g(r+1) = f(k) \quad \text{where } k = \min f^{-1}(E \backslash \{g(0), g(1), \ldots, g(r)\}).$$

Corollary 3.4 *If there is an injective map f from a set E into a countable set F, then E is at most countable.*

Exercise 3.5 Prove the converse of the above corollary.

Proposition 3.6 *Countable union of at most countable sets is at most countable.*

Proof: Let E_i, $i \in \mathbb{N}$ be a countable family of sets that are at most countable. Since each of E_i is at most countable, there are injective maps $f_i : E_i \to \mathbb{N}$. Define a map $f : \cup_{i\in\mathbb{N}} E_i \to \mathbb{N} \times \mathbb{N}$ such that

$$f(a) = (j, f_j(a)) \quad \text{where } j = \min\{k : a \in E_k\}.$$

Check that f is 1–1. Since $\mathbb{N} \times \mathbb{N}$ is countable, it follows that $\cup_{i\in\mathbb{N}} E_i$ is at most countable. □

Proposition 3.7 *Finite Cartesian products of countable sets is countable.*

Proof: It is enough to show that $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable. Define $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{N}$ by

$$f(m, n) = \sum_{i=1}^{m+n-2} i + m.$$

We will show that f is 1–1. Suppose $f(m, n) = f(r, s)$. This means

$$\sum_{i=1}^{m+n-2} i + m = \sum_{i=1}^{r+s-2} i + r. \tag{3.1}$$

If $m + n > r + s$, then we get $\sum_{i=r+s}^{m+n-2} i = r - m$. But clearly the left hand side is greater than or equal to the initial term $r + s - 1$ and $r + s - 1 \geq r > r - m$. Thus we can not have $m + n > r + s$. Similarly $r + s > m + n$ is also not possible. Hence $r + s = m + n$, and therefore using (3.1), we get $m = r$ and $n = s$. Thus f is 1–1. Now by corollary 3.4, $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable. □

Corollary 3.8 $\mathbb{Q}$ *is countable.*

Proof: Write each element r of $\mathbb{Q}$ in the form m/n, where $m \in \mathbb{Z}$, $n \in \mathbb{Z}_+$ and m, n have no common divisor. Now define $f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}_+$ by

$$f(r) = (m, n).$$

Since $\mathbb{Z}$ and $\mathbb{Z}_+$ are countable, and f is 1–1, $\mathbb{Q}$ is countable. □

Remark 3.9 Infinite cartesian products of finite sets may fail to be countable.

For example, take the space S of all sequences each of whose terms are either 0 or 1. Check that this is just the cartesian product of countably many copies of $\{0, 1\}$. Let us show that this is uncountable. If it is countable, there would exist a bijection between $\mathbb{N}$ and S. Now take any map $f : \mathbb{N} \to S$. Let us write all the elements in the range of f in a column, stacked one below the other:

$$\begin{aligned} f(0) &= a_{00}, a_{01}, a_{02}, \ldots, \\ f(1) &= a_{10}, a_{11}, a_{12}, \ldots, \\ f(2) &= a_{20}, a_{21}, a_{22}, \ldots, \\ \cdots & \quad \cdots \end{aligned}$$

Now choose a sequence $\{b_n\}$ as follows: choose b_i so that $b_i \neq a_{ii}$ for all i. Then $\{b_n\}$ can not be equal to any of the sequences $f(0), f(1), \ldots$. Thus $\{b_n\} \in S$, but is not in the range of f. So f can not be onto, and hence can not be bijective.

Notice that in order that you can choose b_i different from a_{ii}, the set where the terms of the sequence take values must have more than one element.

The argument employed above is due to mathematician George Cantor, and is known as the **Cantor's diagonal argument**.

3.1 Decimal Expansion of a Real Number

A real **sequence** is just a map from $\mathbb{Z}_+$ to $\mathbb{R}$.

Let $x \in [0, 1)$. We will associate with x a sequence each term of which is a nonnegative integer and lies between 0 and 9.

Choose $k_1 \in \mathbb{N}$ such that $\frac{k_1}{10} \leq x < \frac{k_1+1}{10}$.

Choose $k_2 \in \mathbb{N}$ such that $\frac{k_2}{10^2} \leq x - \frac{k_1}{10} < \frac{k_2+1}{10^2}$.

Having chosen $k_1, \ldots, k_{n-1}$, choose k_n such that $\frac{k_n}{10^n} \leq x - \sum_{i=1}^{n-1} \frac{k_i}{10^i} < \frac{k_n+1}{10^n}$.

Then $\{k_1, k_2, \ldots\}$ is the sequence that we associate with the number x and one says that $.k_1k_2k_3\ldots$ the decimal expansion of x. It is easy to see that $0 \leq k_i \leq 9$ for each i.

Remark 3.10 1. x is the supremum of the set $E = \{x_1, x_2, \ldots\}$ where $x_n = \sum_{i=1}^{n} \frac{k_i}{10^i}$.

By our choice of k_1, k_2 etc. we have $x \geq x_n$ for all $n \geq 1$. Hence $\sup E \leq x$. Now, given any $\epsilon > 0$, there is an $n \in \mathbb{Z}_+$ such that $10^n > 1/\epsilon$. But by our choice, $x < \sum_{i=1}^{n-1} \frac{k_i}{10^i} + \frac{k_n+1}{10^n}$. Hence $x - \epsilon < x - \frac{1}{10^n} < \sum_{i=1}^{n} \frac{k_i}{10^i} = x_n$. Thus $x = \sup E$.

2. As a corollary of the above remark, it follows that if the decimal expansions of two real numbers coincide, then they are equal.

3. We now have a way of recovering the number x if we know its decimal expansion. But, given a sequence whose terms are all integers and lie between 0 and 9, how do we know in the first place whether it is indeed the decimal expansion of a number or not? First of all, if a given sequence $k_1, k_2, \ldots$ is to be the decimal expansion of a real, then that number has to be $x := \sup\{x_1, x_2, \ldots\}$, where $x_n = \sum_{i=1}^{n} \frac{k_i}{10^i}$. And secondly, we must

have $x_n \leq x < x_n + \frac{1}{10^n}$ for all n. The first inequality here is clear from the definition of x. So all we need to do is, verify whether $x < x_n + \frac{1}{10^n}$ for all values of n.

For this, we need the following simple fact: If $k_j < 9$ for some $j > n$, then $x < x_n + \frac{1}{10^n}$.

For any $m > j$, we have

$$\begin{aligned} x_m &= \sum_{i=1}^{m} \frac{k_i}{10^i} = x_n + \frac{k_{n+1}}{10^{n+1}} + \ldots + \frac{k_j}{10^j} + \ldots + \frac{k_m}{10^m} \\ &\leq x_n + \frac{k_{n+1}}{10^{n+1}} + \ldots + \frac{k_j}{10^j} + \frac{9}{10^{j+1}} + \ldots + \frac{9}{10^m} \\ &\leq x_n + \frac{k_{n+1}}{10^{n+1}} + \ldots + \frac{k_j}{10^j} + \frac{1}{10^j}(1 - \frac{1}{10^{m-j}}) \\ &\leq x_n + \frac{k_{n+1}}{10^{n+1}} + \ldots + \frac{k_j + 1}{10^j}. \end{aligned}$$

Hence $x \leq x_n + \frac{k_{n+1}}{10^{n+1}} + \ldots + \frac{k_j+1}{10^j}$. But

$$\begin{aligned} & x_n + \frac{k_{n+1}}{10^{n+1}} + \ldots + \frac{k_j + 1}{10^j} \\ &\leq x_n + \frac{9}{10^{n+1}} + \ldots + \frac{9}{10^j} \\ &= x_n + \frac{1}{10^n}(1 - \frac{1}{10^{j-n}}) \\ &< x_n + \frac{1}{10^n}. \end{aligned}$$

Hence $x < x_n + \frac{1}{10^n}$.

Now, if infinitely many of the k_i's are different from 9, then for any n, there will always exist a $j > n$ such that $k_j < 9$. Therefore we will have $x_n \leq x < x_n + \frac{1}{10^n}$ for all n. But this means $k_1, k_2, \ldots$ is the decimal expansion of x. Thus the answer to the question is: *if infinitely many of the k_i's are different from 9, then the sequence $k_1, k_2, \ldots$ is the decimal expansion of a real number $x \in [0, 1)$.* In other words, we have the following proposition:

Proposition 3.11 *Let $\{k_1, k_2, \ldots\}$ be a sequence such that each $k_i \in \mathbb{N}$, $0 \leq k_i \leq 9$ for all i, and $k_i \neq 9$ for infinitely many i's. Let $x_n = \sum_{i=1}^{n} \frac{k_i}{10^i}$, $n \geq 1$ and let E be the set $\{x_1, x_2, \ldots\}$. Then E is bounded above (by $\sup\{\sum_{i=1}^{n} \frac{9}{10^i} : n \in \mathbb{Z}_+\} = 1$) and the decimal expansion for $\sup E$ is $.k_1 k_2 \ldots$.*

4. Converse of the above statement is also true, i. e. if the k_i's are all 9 from a certain stage onwards, then the sequence $k_1, k_2, \ldots$ is not the decimal expansion of any real.

Exercise 3.12 Let $\{k_1, k_2, \ldots\}$ be a sequence such that each $k_i \in \mathbb{N}$, $0 \leq k_i \leq 9$ for all i. Let $m \in \mathbb{Z}_+$ be such that $k_m \neq 9$ and $k_i = 9$ for all $i > m$. Let E be as earlier. Then show that the decimal expansion of $\sup E$ is $.k_1 \ldots k_{m-1}(k_m + 1)00 \ldots$.

5. There is nothing special about the number 10 above. One could equally well have chosen k_1, k_2 etc in such a manner that $\frac{k_1}{2} \leq x < \frac{k_1+1}{2}$, $\frac{k_2}{2^2} \leq x - \frac{k_1}{2} < \frac{k_2+1}{2^2}$, $\ldots$, $\frac{k_n}{2^n} \leq x - \sum_{i=1}^{n-1} \frac{k_i}{2^i} < \frac{k_n+1}{2^n}$, and each k_i is either 0 or 1. The resulting expansion then is called the **diadic expansion** of x

Theorem 3.13 $\mathbb{R}$ *is uncountable.*

Proof: It is enough to prove that the interval $(0,1)$ is uncountable. Let $f : \mathbb{Z}_+ \to (0,1)$ be a map, and let $.k_{i1}k_{i2}\ldots$ be the decimal expansion of the number $f(i)$. Thus the range of f consists of the elements

$$\begin{aligned} f(1) &= .k_{11}k_{12}\ldots \\ f(2) &= .k_{21}k_{22}\ldots \\ \ldots & \quad\; \ldots \end{aligned}$$

Now choose a sequence $\{r_n\}$ as follows:

$$r_n = \begin{cases} 0 & \text{if } 1 \leq k_{nn} \leq 9, \\ 1 & \text{if } k_{nn} = 0. \end{cases}$$

Then there is a real x whose decimal expansion is $.r_1r_2\ldots$. Clearly this number x is different from all the numbers $f(i)$ (otherwise their decimal expansions would have been the same). Thus f can not be onto. Therefore there can not exist any bijection between $\mathbb{Z}_+$ and $(0,1)$. □

Exercise 3.14 Which of the following sets are countable and which are uncountable:

$\mathbb{R}$, the set of all irrational numbers, the set of all open intervals in $\mathbb{R}$ with rational endpoints, the set of all rectangles in $\mathbb{R}^2$ whose vertices have rational endpoints.

Week 3

4 Topological Notions

Let $E \subseteq \mathbb{R}$. A point $x \in E$ is said to be an **interior point** of E if there is an $\epsilon > 0$ such that $N_\epsilon(x) := (x-\epsilon, x+\epsilon) \subseteq E$. The set of all interior points of E is called the **interior of** E and is denoted by $\operatorname{int} E$.

A subset E of $\mathbb{R}$ is called **open** if every point of E is an interior point of E. A subset E of $\mathbb{R}$ is said to be **closed** if E^c is open

A **neighbourhood** of a point a will always mean an open set of the form (b, c) containing the point a.

Proposition 4.1 *Arbitrary union of open sets is again an open set.*

Finite intersection of open sets is open.

Proof: Let U_α be a family of open sets, and let $U = \cup_\alpha U_\alpha$. Take any $x \in U$. Then $x \in U_\alpha$ for some α. Since U_α is open, there is an $\epsilon > 0$ such that $N_\epsilon(x) \subseteq U_\alpha$. But then $N_\epsilon(x) \subseteq U$. Therefore x is an interior point of U. Thus U is open.

Let $U_1, U_2, \ldots, U_n$ be open, and let $U = \cap_{i=1}^n U_i$. Take any $x \in U$. Then $x \in U_i$ for all $i \in \{1, 2, \ldots, n\}$. Since each U_i is open, there exist positive numbers $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ such that $N_{\epsilon_i}(x) \subseteq U_i$. Now take $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$. Then $N_\epsilon(x) \subseteq N_{\epsilon_i}(x) \subseteq U_i$ for each i. Therefore $N_\epsilon(x) \subseteq U$. Thus x is an interior point of U. Since this is true for any point x in U, U is open. □

Infinite intersection of open sets may fail to be open. For example, $[0,1] = \cap_{n=1}^\infty (-\frac{1}{n}, 1+\frac{1}{n})$.

Proposition 4.2 *Let $E \subseteq \mathbb{R}$ and let V be the union of all open sets contained in E. Then $V = \operatorname{int} E$.*

Proof: Let us first show that $\operatorname{int} E$ is open. Take an $x \in \operatorname{int} E$. Then there is some $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq E$. We need to show that any point in $(x-\epsilon, x+\epsilon)$ is an interior point of E. Take $y \in (x-\epsilon, x+\epsilon)$. Let $\delta = \min\{x+\epsilon-y, y-x+\epsilon\}$. Then $\delta > 0$ and $(y-\delta, y+\delta) \subseteq (x-\epsilon, x+\epsilon) \subseteq E$. Hence $y \in \operatorname{int} E$. Thus $(x-\epsilon, x+\epsilon) \subseteq \operatorname{int} E$, i. e. $\operatorname{int} E$ is open.

Since $\operatorname{int} E$ is open and is contained in E, one has $\operatorname{int} E \subseteq V$. On the other hand, since V itself is open, for any point x of V, there is an $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq V$. Also, $V \subseteq E$. Hence one has $(x-\epsilon, x+\epsilon) \subseteq E$. Thus $x \in \operatorname{int} E$. Thus $V \subseteq \operatorname{int} E$. □

Proposition 4.3 *Arbitrary intersection of closed sets is closed.*

Finite union of closed sets is closed.

Proof: Exercise. □

Exercise 4.4 Show that infinite union of closed sets may fail to be closed.

Definition 4.5 For a subset E of $\mathbb{R}$, the **closure** $\bar{E}$ of E is defined to be the intersection of all closed sets F such that $E \subseteq F$.

The following proposition says that the interior of a set is the maximal open set contained in it and the closure of a set is the minimal closed set containing it.

Proposition 4.6 *Let $E \subseteq \mathbb{R}$.*

1. *Let V be an open subset of $\mathbb{R}$ such that (1) $V \subseteq E$ and (2) if U is open and $U \subseteq E$ then $U \subseteq V$. Then $V = \operatorname{int} E$.*

2. *Let F be a closed subset of $\mathbb{R}$ satisfying (1) $E \subseteq F$, and (2) if G closed and $E \subseteq G$ then $F \subseteq G$, then $F = \bar{E}$.*

Proof: Exercise. □

4.1 Limit point

Let $E \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is called a limit point of E if for any $\epsilon > 0$, the neighbourhood $N_\epsilon(x)$ contains infinitely many points of E. A limit point is also called an **accumulation point** or a **cluster point**.

We denote the set of all limit points of a set E by E' and call it the first derived set of E.

Lemma 4.7 *A number $x \in \mathbb{R}$ is a limit point of E if and only if for every $\epsilon > 0$, $N_\epsilon(x)$ contains a point of E other than x..*

Proof: Take any $\epsilon > 0$. If $N_\epsilon(x)$ contains infinitely many points of E, then it has to contain at least one point of E other than x. So if x is a limit point then the given condition holds.

For the converse, assume that x is not a limit point of E. Then there is an $\epsilon > 0$ for which there are only finitely many points, say $x_1, x_2, \dots x_n$ of E in $N_\epsilon(x)$. Let $\delta = \min\{|x - x_i| : 1 \leq i \leq n, |x - x_i| > 0\}$. Then the neighbourhood $N_\delta(x)$ does not contain any point x_i for which $x \neq x_i$, i. e. does not contain any point of E other than x. □

Exercise 4.8 A finite subset of $\mathbb{R}$ can not have any limit point.

Example 4.9 *1. 0 is a limit point of the set* $\{\frac{1}{n} : n \in \mathbb{Z}_+\}$.

2. $E = (0,1)$. *Any point of* E *is a limit point of* E.

3. $E = \{\frac{1}{n} : n \in \mathbb{Z}_+, n \geq 2\} \cup \{1 - \frac{1}{n} : n \in \mathbb{Z}_+, n \geq 2\}$. *0 and 1 are two limit points of* E.

4. $E = [0,1) \cup \{1 + \frac{1}{n} : n \in \mathbb{Z}_+\}$. *Any point in* $[0,1]$ *is a limit point of* E.

5. $E = \mathbb{Q}$. *Any real number is a limit point of* E.

Thus there can be several limit points of a set.

Proposition 4.10 *Let* E *and* F *be subsets of* $\mathbb{R}$ *such that* $E \subseteq F$. *Then* $E' \subseteq F'$.

Proof: Exercise. □

Proposition 4.11 *Let* $E_1, E_2 \subseteq \mathbb{R}$. *Then*

1. $(E_1 \cup E_2)' = E_1' \cup E_2'$,

2. $(E_1 \cap E_2)' \subseteq E_1' \cap E_2'$.

Proof: Take $x \in E_1' \cup E_2'$. Then either $x \in E_1'$ or $x \in E_2'$. In the first case, for every $\epsilon > 0$, $N_\epsilon(x)$ contains infinitely many points of E_1, and in the second case, for every $\epsilon > 0$, $N_\epsilon(x)$ contains infinitely many points of E_2. In either case, $N_\epsilon(x)$ contains infinitely many points from $E_1 \cup E_2$. So $x \in (E_1 \cup E_2)'$. Thus $E_1' \cup E_2' \subseteq (E_1 \cup E_2)'$.

Observe that to prove the reverse inclusion, it is enough to show that if $x \in (E_1 \cup E_2)'$, and $x \notin E_1'$, then $x \in E_2'$. So assume $x \in (E_1 \cup E_2)'$, and $x \notin E_1'$. Since $x \notin E_1'$, there is an $\epsilon_0 > 0$ for which $N_{\epsilon_0}(x) \cap E_1$ is at most finite. But then for any $\epsilon \leq \epsilon_0$, $N_\epsilon(x) \cap E_1$ is finite. But since $N_\epsilon(x) \cap (E_1 \cup E_2)$ is infinite for all $\epsilon > 0$, we must have $N_\epsilon(x) \cap E_2$ infinite for all $\epsilon \leq \epsilon_0$. This implies $N_\epsilon(x) \cap E_2$ is infinite for all $\epsilon > 0$. Hence $x \in (E_2)'$.

This proves part (1). Proof of part (2) is left as an exercise. □

The following examples show that one does not have equality in part (2) in general.

Take $E_1 = (0,1)$, $E_2 = (1,2)$, or $E_1 = \mathbb{Q}$, $E_2 = \mathbb{R} \backslash \mathbb{Q}$.

Proposition 4.12 *Let* E_α *be a family of subsets of* $\mathbb{R}$. *Then*

$$\cup_\alpha E_\alpha' \subseteq (\cup_\alpha E_\alpha)', \quad (\cap_\alpha E_\alpha)' \subseteq \cap_\alpha E_\alpha'.$$

Proof: Exercise. □

The following example illustrates that one does not have equality in the above.

Let $\mathbb{Q} = \{r_1, r_2, \ldots\}$. Let $E_n = \{r_n\}$.

Proposition 4.13 *$E \cup E'$ is closed.*

Proof: Take $x \in (E \cup E')^c$. Consider $N_\epsilon(x)$, where $\epsilon > 0$. Our claim now is that if $N_\epsilon(x)$ contains a point of $E \cup E'$, then it contains a point of E. Suppose $y \in E'$ be a point in $N_\epsilon(x)$. Since $N_\epsilon(x)$ is open, there is a $\delta > 0$ such that $N_\delta(y) \subseteq N_\epsilon(x)$. Since $y \in E'$, $N_\delta(y)$ contains a point of E, say z, that is different from x. But then $z \in N_\epsilon(x)$.

From the above, it now follows that if every neighbourhood of x intersects $E \cup E'$, then every neighbourhood will contain an element of E (other than x) and hence $x \in E'$, which is not the case. So there exists a neighbourhood $N_\epsilon(x)$ such that $N_\epsilon(x) \cap (E \cup E') = \Phi$. But this means $N_\epsilon(x) \subseteq (E \cup E')^c$. □

Proposition 4.14 *$\bar{E} = E \cup E'$.*

Proof: Let F be a closed set containing E. We will show that $E \cup E' \subseteq F$. Take $x \in F^c$. Since F^c is open, there is a neighbourhood $N_\epsilon(x)$ of x such that $N_\epsilon(x) \subseteq F^c$. Since $E \subseteq F$, $N_\epsilon(x)$ does not contain any point of E, i. e. $x \notin E'$. Thus $F^c \subseteq E'^c$, so that $E' \subseteq F$. Thus $E \cup E' \subseteq F$. □

Definition 4.15 Let $E \subseteq \mathbb{R}$. The **boundary** of E is defined to be the set $\bar{E} \setminus \operatorname{int} E$ and is denoted by ∂E.

Proposition 4.16 *Let $E \subseteq \mathbb{R}$. A point x is in the boundary of E if and only if for any $\epsilon > 0$, the neighbourhood $N_\epsilon(x)$ intersects both E and E^c.*

Proof: First assume that $x \in \partial E = \bar{E} \setminus \operatorname{int} E$. Since $x \in \bar{E} = E \cup E'$, every neighbourhood $N_\epsilon(x)$ intersects E. And since $x \notin \operatorname{int} E$, there does not exist any $\epsilon > 0$ for which $N_\epsilon(x) \subseteq E$; in other words, for all $\epsilon > 0$, $N_\epsilon(x)$ intersects E^c.

Conversely, let x be a real satisfying the condition of the proposition. Since $N_\epsilon(x)$ intersects E for every $\epsilon > 0$, either $x \in E$ or $x \in E'$, i.e. $x \in \bar{E}$. Next, since $N_\epsilon(x)$ always intersects E^c, x is not an interior point of E. □

Exercise 4.17 Let E be a subset of $\mathbb{R}$. Show that $\sup E$ (resp. $\inf E$) is either a point of E or a limit point of E.

Show that $E'' \subseteq E'$. Hence prove that E' is closed.

Theorem 4.18 (Bolzano-Weierstrass Theorem) *Any bounded infinite subset of $\mathbb{R}$ has a limit point.*

Proof: Let E be a bounded subset of $\mathbb{R}$. Let $a_0, b_0 \in \mathbb{R}$ be such that $E \subseteq [a_0, b_0]$. Consider the two intervals $[a_0, \frac{a_0+b_0}{2}]$ and $[\frac{a_0+b_0}{2}, b_0]$. At least one will contain infinitely many points of

E. If exactly one of them contain infinitely many points of E, call it $[a_1, b_1]$. If both contain infinitely many points of E, then write $[a_1, b_1]$ for the left interval. Now repeat the procedure with the interval $[a_1, b_1]$ to get $[a_2, b_2]$. Having obtained the intervals $[a_0, b_0], \ldots, [a_n, b_n]$, get $[a_{n+1}, b_{n+1}]$ by dividing $[a_n, b_n]$ into two equal parts.

Observe the following now: $a_0 \leq a_1 \leq \ldots \leq a_n \leq \ldots \leq b_n \leq \ldots \leq b_1 \leq b_0$. Hence the set $L = \{a_0, a_1, \ldots\}$ has a supremum, and $U = \{b_0, b_1, \ldots\}$ has an infimum. Also, observe that $b_n - a_n = \frac{b_0 - a_0}{2^n}$ for all $n \in \mathbb{N}$. We will show that $\sup L = \inf U$, and this number is a limit point for the set E.

Let us write $a = \sup L$, and $b = \inf U$. Suppose if possible, $a < b$. Since $a_n \leq a$ and $b_n \geq b$ for all n, we have $b_n - a_n \geq b - a$ for all n. By our observation made earlier, this implies $\frac{b_0 - a_0}{2^n} \geq b - a$ for all n. But this is not possible, as the set $\{2^n : n \in \mathbb{N}\}$ is not bounded above. Next, assume $a > b$. Then there exists c such that $a > c > b$. Since $a = \sup L$, $a_n > c$ for some n, and $b_m < c$ for some m. Let $k = \max\{m, n\}$. Then we have $a_k > c$ and $b_k < c$; in other words, $b_k < a_k$. This is not possible. So we can not have $a > b$ either. Thus we have $a = b$.

Take an interval $N_\epsilon(a)$ around a. Choose n such that $a_n > a - \epsilon$, and m such that $b_m < a + \epsilon$. Taking $k = \max\{m, n\}$, we get $[a_k, b_k] \subseteq N_\epsilon(a)$. Hence $N_\epsilon(a)$ contains infinitely many points of E. $\square$

Remark 4.19 *In the course of the above proof, we have almost proved another very important theorem, the* ***Cantor intersection theorem****, which says that if we have a decreasing sequence of closed intervals, with lengths shrinking to zero, then their intersection is a singleton. We will study it later, once we have introduced the notion of convergence of a sequence.*

Week 4

4.2 Extended Real Numbers

Often it is helpful to deal with the following set rather than $\mathbb{R}$: $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$. Define addition and multiplication on this new set as follows: if $a, b \in \mathbb{R}$, define $a + b$ and ab to be their sum and product respectively in $\mathbb{R}$. For $a \in \mathbb{R}$, define

$$a + \infty = \infty = \infty + a, \quad a + (-\infty) = -\infty = (-\infty) + a,$$

$$\infty + \infty = \infty, (-\infty) + (-\infty) = -\infty,$$

$$a \cdot (\pm\infty) = \begin{cases} \pm\infty & \text{if } a > 0 \\ \mp\infty & \text{if } a < 0 \end{cases}$$

$$\infty \cdot \infty = \infty, \ \ (-\infty) \cdot (-\infty) = \infty, \ \ \infty \cdot (-\infty) = -\infty, \ \ (-\infty) \cdot \infty = -\infty$$

$$a/0 = \begin{cases} \infty & \text{if } a > 0 \\ -\infty & \text{if } a < 0 \end{cases}, \qquad a/\pm\infty = 0$$

We leave the following quantities undefined: $\infty + (-\infty)$, $0 \cdot \pm\infty$, $\pm\infty/\pm\infty$.

By a neighbourhood of $+\infty$ in $\bar{\mathbb{R}}$, we mean a set of the form $(a, \infty]$, and by a neighbourhood of $-\infty$, we mean a set of the form $[-\infty, b)$.

For unbounded subsets of $\mathbb{R}$, we agree on the following convention: if $E \subseteq \mathbb{R}$ is not bounded above, then we say that $+\infty$ is a limit point of E, and if E is not bounded below, then we say that $-\infty$ is a limit point of E.

Remark 4.20 *Let $E \subseteq \mathbb{R}$. Then $x \in \bar{\mathbb{R}}$ is a limit point of E if and only if every neighbourhood of x intersects E at a point other than x. Equivalently, $x \in \bar{\mathbb{R}}$ is a limit point of E if and only if every neighbourhood of x contains infinitely many points of E.*

If we agree to use the extended real number system, then we can talk about supremum and infimum of unbounded sets also.

4.3 Lim sup and lim inf

Definition 4.21 Let E be a subset of $\mathbb{R}$. Define the lim sup (limit superior) and lim inf (limit inferior) of the set E as follows:

$$\limsup E = \begin{cases} \sup E' & \text{if } E' \text{ is nonempty,} \\ -\infty & \text{if } E' \text{ is empty.} \end{cases}$$

$$\liminf E = \begin{cases} \inf E' & \text{if } E' \text{ is nonempty,} \\ \infty & \text{if } E' \text{ is empty.} \end{cases}$$

Observe that from Exercise 4.17 it follows that $\sup E'$ and $\inf E'$ both are limit points of E.

Proposition 4.22 *Let $E \subseteq \mathbb{R}$ be bounded and infinite and let $a = \limsup E$. Then for any $\epsilon > 0$, $N_\epsilon(a)$ contains infinitely many points of E, but there are at most finitely many points x in E with $x \geq a + \epsilon$.*

Similarly, if $b = \liminf E$, then there are infinitely many points of E in $N_\epsilon(b)$, but only finitely many points x satisfying $x \leq b - \epsilon$.

Proof: Observe that from Exercise 4.17 it follows that $\limsup E := \sup E'$ and $\liminf E := \inf E'$ both are limit points of E. Since $a = \limsup E$ is a limit point of E, for any $\epsilon > 0$, the neighbourhood $(a - \epsilon, a + \epsilon)$ contains infinitely many points of E. In particular, there are infinitely many points $x \in E$ with $x > a - \epsilon$.

Suppose if possible, there is an $\epsilon > 0$ such that there are infinitely many points $x \in E$ with $x > a + \epsilon$. Let M be an upper bound of E. Then the set $F := [a + \epsilon, M] \cap E$ is infinite and bounded. Hence by Bolzano-Weirstrass theorem, it has a limit point, say, b. Since $F \subseteq E$, b is a limit point of E as well. But $b \geq a + \epsilon$, which contradicts the fact that $a = \limsup E$. □

The next proposition shows that we could have used the above conditions as the defining conditions for lim sup and lim inf.

Proposition 4.23 *Let E be a subset of $\mathbb{R}$, and let a (resp. b) be a real satisfying the following condition: for any $\epsilon > 0$, $N_\epsilon(a)$ (resp. $N_\epsilon(b)$) contains infinitely many points of E, but there are at most finitely many points x in E with $x \geq a + \epsilon$ (resp. $x \leq b - \epsilon$).*

Then $a = \limsup E$ (resp. $b = \liminf E$).

Exercise 4.24 Prove/disprove the following:

$$\begin{aligned} \limsup(E + F) &\geq \limsup E + \limsup F, \\ \limsup(EF) &\geq (\limsup E)(\limsup F) \text{ if } E \subseteq \mathbb{R}_+, F \subseteq \mathbb{R}_+. \end{aligned}$$

5 Sequences and Series

5.1 Sequences

A real **sequence** is just a map from $\mathbb{N}$ (or $\mathbb{Z}_+$) to $\mathbb{R}$. A sequence $f : n \mapsto a_n$ is usually denoted by $\{a_n\}_{n\in\mathbb{N}}$, or by simply $\{a_n\}$. Let $\{a_n\}$ be a sequence and let $\{k_n\}_n$ be a sequence such that $k_1 < k_2 < \ldots < k_n < \ldots$. Then we say $\{b_n\}_n$is a **subsequence** of $\{a_n\}$ if $b_n = a_{k_n}$ for all n.

A sequence $\{a_n\}$ is said to converge to a limit a if for any neighbourhood $N_\epsilon(a)$, from a certain stage onwards, all the terms of the sequence lie in $N_\epsilon(a)$. In other words, given any $\epsilon > 0$, if there exists an $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n > N$, then we say that the sequence $\{a_n\}$ converges to a $(-\infty < a < \infty)$. In such a case, we write $\lim_{n\to\infty} a_n = a$. For example, check that the sequence $\{\frac{1}{n}\}$ converges to zero. The number N may and in general will depend on the value of ϵ.

We say a sequence is convergent if it converges to a finite limit. Note that a convergent sequence can have at most one limit.

A sequence $\{a_n\}$ is said to diverge to ∞ if for any $M > 0$, there exists an $N \in \mathbb{N}$ such that $a_n > M$ for all $n > N$. In such a case, one writes $\lim_{n\to\infty} a_n = \infty$. It is said to diverge to $-\infty$ if for any $M > 0$, there is an $N \in \mathbb{N}$ such that $a_n < -M$ for all $n > N$. We write $\lim_{n\to\infty} a_n = -\infty$.

Exercise 5.1 Give examples of sequences that diverge to ∞ or $-\infty$.

A sequence $\{a_n\}$ is said to be bounded above if there is an $M > 0$ such that $a_n \leq M$ for all n. It is said to be bounded below if there is an $M > 0$ such that $a_n > -M$ for all n. A sequence that is bounded below as well as above is said to be bounded.

Lemma 5.2 *Any convergent sequence is bounded.*

The converse need not be true. (Give examples)

Proposition 5.3 *Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences, with* $\lim a_n = a$, $\lim b_n = b$. *Then*

1. $\lim(a_n + b_n) = a + b$,
2. $\lim(a_n b_n) = ab$,
3. *if* $b \neq 0$, *then* $\lim(\frac{a_n}{b_n}) = \frac{a}{b}$,
4. *for any* $\lambda \in \mathbb{R}$, $\lim(\lambda a_n) = \lambda a$,
5. *if* $a_n \leq b_n$ *for all* n, *then* $a \leq b$.

Examples

1. If $p > 0$, then $\lim \frac{1}{n^p} = 0$.

2. If $p > 0$, then $\lim \sqrt[n]{p} = 1$.

3. $\lim \sqrt[n]{n} = 1$. (Use $n = (1 + \sqrt[n]{n} - 1)^n > \frac{n(n-1)}{2}(\sqrt[n]{n} - 1)^2$)

4. If $p > 0$ and $\alpha \in \mathbb{R}$, then
$$\lim \frac{n^\alpha}{(1+p)^n} = 0.$$
($(1+p)^n > \binom{n}{k}p^k = n(n-1)\dots(n-k+1)\frac{p^k}{k!} > (\frac{n}{2})^k\frac{p^k}{k!}$ if $n > 2k$. Choose $k > \alpha$. Then for all $n > 2k$, $\frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k}\frac{1}{n^{k-\alpha}}$.)

5. $$\lim x^n = \begin{cases} 0 & \text{if } -1 < x < 1, \\ 1 & \text{if } x = 1, \\ \infty & \text{if } x > 1, \end{cases}$$
and the limit does not exist if $x \leq -1$.

Week 5

Proposition 5.4 *If a_n converges to a, then any subsequence converges to a.*
If a_n diverges to ∞, then any subsequence diverges to ∞.
If a_n diverges to $-\infty$, then any subsequence diverges to $-\infty$.

Proposition 5.5 *If a is a limit point of the set E, then there is a sequence $\{a_n\}$ of distinct points from E such that $\lim a_n = a$.*

Proof: Since a is a limit point of E, the neighbourhood $N_1(a)$ contains a point, say x of E such that $x \neq a$. Define $a_1 = x$. Next, let $\epsilon = \min\{\frac{1}{2}, |a_1 - a|\}$, and let a_2 be a point of E lying in $N_\epsilon(a)$ such that $a_2 \neq a$. Having chosen $a_1, a_2, \ldots, a_n$, let a_{n+1} be a point of E different from a such that it lies in $N_\epsilon(a)$ where $\epsilon = \min\{\frac{1}{n+1}, |a_1 - a|, \ldots, |a_n - a|\}$.

First, observe that by our choice, $|a_{n+1} - a| < |a_i - a|$ for $1 \leq i \leq n$, so that a_{n+1} is different from $a_1, a_2, \ldots, a_n$. Thus all the elements of the sequence $\{a_n\}$ are distinct. We also have $|a_n - a| < \frac{1}{n}$ for all n. Hence given any $\epsilon > 0$, if we take N to be an integer greater than $1/\epsilon$, then for all $n > N$, we will have $|a_n - a| < \frac{1}{n} < \frac{1}{N} < \epsilon$. Thus $\lim a_n = a$. □

Observe that if a is a limit point of a set E. then for any $b \in E$, a is a limit point of $E \setminus \{b\}$ also.

Exercise 5.6 Prove that the converse of the above proposition is also true.

Proposition 5.7 *Let $E \subseteq \mathbb{R}$. Then $a \in \bar{E}$ if and only if there is a sequence $(a_n)_n$ such that $a_n \in E$ for all n and $\lim a_n = a$.*

5.2 Monotone sequences

A sequence $\{a_n\}$ is said to be monotone increasing if for all $n \in \mathbb{N}$, we have $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$. It is said to be monotone decreasing if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A monotone sequence is one that is either monotone increasing or monotone decreasing.

Proposition 5.8 *A monotone increasing sequence either converges to a finite limit or diverges to ∞.*

Similarly, a monotone decreasing sequence either converges to a finite limit or diverges to $-\infty$.

Proof: Let us write $a = \sup a_n$. If $a = \infty$, then it follows immediately that the sequence a_n diverges to ∞. So assume $a < \infty$. By the definition of a supremum, for any $\epsilon > 0$, there is an a_k such that $a_k > a - \epsilon$. Since the sequence is increasing, we have $a_n > a - \epsilon$ for all $n \geq k$. And we of course have $a_n \leq a < a + \epsilon$ for all n. Thus $a = \lim a_n$.

Proof for monotone decreasing sequences is similar. □

Corollary 5.9 *Any bounded monotone sequence is convergent.*

Exercise 5.10 Show that the sequence $a_n = \{(1 + \frac{1}{n})^n\}$ converges.

The limit of the above sequence is denoted by e.

Proposition 5.11 *Every bounded sequence has a convergent subsequence.*

Proof: Let $\{a_n\}$ be a bounded sequence. Let $E = \{x \in \mathbb{R} : x = a_n \text{ for some } n\}$. Let us consider the following two cases now:
Case I. E is finite. In this case, there is an $a \in E$ such that $a = a_n$ for infinitely many n's. Hence the sequence $\{b_n\}$ where $b_n = a$ for all n, is a subsequence of $\{a_n\}$. And clearly $\lim b_n = a$. Thus $\{a_n\}$ has a subsequence converging to a.
Case II. E is infinite. Since the sequence $\{a_n\}$ is bounded, so is the set E. By Bolzano-Weierstrass theorem, it has a limit point, say a. Take $\epsilon_1 = 1$. Choose n_1 such that $a_{n_1} \in (a-\epsilon_1, a+\epsilon_1)\backslash\{a\}$. Next, take $\epsilon_2 = \min\{\frac{1}{2}, |a - a_{n_1}|\}$ and choose n_2 such that $a_{n_2} \in (a-\epsilon_2, a+\epsilon_2)\backslash\{a\}$. Having chosen $n_1, \ldots, n_k$, take $\epsilon_{k+1} = \min\{\frac{1}{k+1}, |a - a_{n_1}|, \ldots, |a - a_{n_k}|\}$ and choose n_{k+1} such that $a_{n_{k+1}} \in (a - \epsilon_{k+1}, a + \epsilon_{k+1})\backslash\{a\}$. The resulting subsequence $\{a_{n_k}\}_k$ converges to a. □

5.3 Cauchy sequences

Let $\{a_n\}$ be a convergent sequence converging to a. Then, given any $\epsilon > 0$, we can choose an $N > 0$ such that $|a_n - a| < \epsilon/2$ for all $n > N$. But then, whenever m and are both greater than N, we have $|a_m - a_n| \leq |a_m - a| + |a_n - a| < \epsilon$. Thus the distance between the m^{th} and n^{th} terms shrinks to zero as both m and n grow. Such sequences are called Cauchy sequences.

Lemma 5.12 *Let $\{a_n\}$ be a cauchy sequence. If a subsequence of $\{a_n\}$ converges, then $\{a_n\}$ also converges and to the same limit.*

Proposition 5.13 *Let $\{a_n\}$ be a Cauchy sequence in $\mathbb{R}$. Then it is convergent.*

Proof: Let us first show that $\{a_n\}$ is bounded. Choose $N > 0$ such that $|a_m - a_n| < 1$ whenever $m, n > N$. Now let $M = \max\{|a_1|, |a_2|, \ldots, |a_N|, |a_{N+1}| + 1\}$. Then for any n, we have $-M < a_n < M$. (for $n \leq N$, it is clear. for $n > N$, it follows from the fact that $|a_n - a_{N+1}| < 1$).

By the previous proposition, $\{a_n\}$ has a subsequence, say $\{b_n\}$ where $b_n = a_{k_n}$ for all n, such that it converges. But if any subsequence of a Cauchy sequence converges, the Cauchy sequence itself also converges and to the same limit. $\square$

Remark 5.14 *In many cases, it is easier to show that a given sequence is a Cauchy sequence than showing that it is convergent, because the latter involves the limit of the sequence.*

Example: $a_n = \sum_n (-1)^n \frac{1}{n}$.

5.4 Limit point of a sequence

Definition 5.15 Let $\{a_n\}$ be a sequence. A point a is called a **limit point** if any neighbourhood of it contains infinitely many terms of the sequence.

Thus if $-\infty < a < \infty$, then a is a limit point if for any $\epsilon > 0$, and any $N > 0$, there exists $n \geq N$ such that $a_n \in N_\epsilon(a)$. The point ∞ is a limit point if for any $N > 0$ and $M > 0$, there exists $n \geq N$ such that $a_n > M$. Similarly $-\infty$ is a limit point if for any $N > 0$ and $M > 0$, there exists $n \geq N$ such that $a_n < -M$.

Proposition 5.16 *Let $\{a_n\}$ be a sequence of real numbers. Then*

1. *a real number a is a limit point of $\{a_n\}$ if and only if there is a subsequence that converges to a.*

2. *∞ (respectively $-\infty$) s a limit point of $\{a_n\}$ if and only if there is a subsequence that diverges to ∞ (respectively $-\infty$).*

Proof: 1. Assume $\{a_n\}$ has a subsequence, say $\{a_{n_k}\}_k$ that converges to a. Take any $\epsilon > 0$. By definition, there is an integer N such that $a - \epsilon < a_{n_k} < a + \epsilon$ for all $k \geq N$. This implies that there are infinitely many terms of the subsequence contained in the interval $(a - \epsilon, a + \epsilon)$. But these are terms of the original sequence. Thus there are infinitely many terms of the sequence $\{a_n\}$ in the interval $(a - \epsilon, a + \epsilon)$. So a is a limit point.

Conversely, assume that a is a limit point of the sequence $\{a_n\}$. We will now produce a subsequence of $\{a_n\}$ that converges to a. Take $\epsilon = 1$. Look at the set $E_1 := \{n \in \mathbb{N} : a_n \in (a - 1, a + 1)\}$. Since the interval $(a - 1, a + 1)$ contains infinitely many terms of the sequence $\{a_n\}$, this is nonempty. Let $n_1 = \min E_1$. Then

$$|a_{n_1} - a| < 1.$$

Next look at $E_2 := \{n \in \mathbb{N} : a_n \in (a - \frac{1}{2}, a + \frac{1}{2})\}$. Again, this is infinite. Hence $E_2 \backslash \{1, 2, \ldots, n_1\}$ is nonempty. Let $n_2 = \min E_2 \backslash \{1, 2, \ldots, n_1\}$. Then $n_2 > n_1$ and

$$|a_{n_2} - a| < \frac{1}{2}.$$

Having chosen $n_1, \ldots, n_k$, choose n_{k+1} to be $\min\{n \in \mathbb{N} : a_n \in (a - \frac{1}{k+1}, a + \frac{1}{k+1}) \backslash \{1, 2, \ldots, n_k\}$. Then $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ and

$$|a_{n_k} - a| < \frac{1}{k} \quad \forall\, k.$$

It now follows that $\lim_{k\to\infty} a_{n_k} = a$.

2. Assume there is a subsequence $\{a_{n_k}\}$ that diverges to ∞. Then given any $M \in \mathbb{R}$, there is an integer N such that $a_{n_k} > M$ for all $k \geq N$. This implies that there are infinitely many terms of $\{a_{n_k}\}$ and hence of the original sequence $\{a_n\}$ that are greater than M. Since this is true for any M, the point ∞ is a limit point of $\{a_n\}$.

Conversely assume that ∞ is a limit point of $\{a_n\}$. Chhose a subsequence $\{a_{n_k}\}$ in the following manner. Let

$$\begin{aligned}
n_1 &= \min\{n \in \mathbb{N} : a_n > 1\}, \\
n_2 &= \min\{n \in \mathbb{N} : n > n_1, a_n > 2\}, \\
\cdots & \quad \cdots \\
n_{k+1} &= \min\{n \in \mathbb{N} : n > n_k, a_n > k+1\}, \\
\cdots & \quad \cdots
\end{aligned}$$

Then the subsequence $\{a_{n_k}\}_k$ diverges to ∞ (**Show this!**).

The proof of the statement involving $-\infty$ is similar. □

Corollary 5.17 *Let $\{a_n\}$ be a sequence and let $\{a_{n_k}\}$ be a subsequence. Then any limit point of $\{a_{n_k}\}$ is also a limit point of the sequence $\{a_n\}$.*

Proof: Exercise. □

Exercise 5.18 Let $\{a_n\}$ be a sequence of real numbers. Let E be the set of its limit points (thus $E \subseteq \bar{\mathbb{R}}$). Show that

1. E is nonempty,
2. both $\sup E$ and $\inf E$ are limit points of $\{a_n\}$ (i. e. points of E)

Exercise 5.19 Let $\{a_n\}$ be a sequence. Show that

1. if $\{a_n\}$ is not bounded above, then ∞ is a limit point of $\{a_n\}$.
2. if $\{a_n\}$ is not bounded below, then $-\infty$ is a limit point of $\{a_n\}$.

Let $\{a_n\}$ be a sequence of real numbers. Define $\limsup a_n$ and $\liminf a_n$ to be the numbers $\sup E$ and $\inf E$ respectively, where E is the set of all limit points of $\{a_n\}$.

By the earlier exercise, lim sup and lim inf of a sequence always exist.

Proposition 5.20 *Let $a \in \mathbb{R}$. Then $\limsup a_n = a$ if and only if for any $\epsilon > 0$, one has $a_n > a - \epsilon$ for infinitely many n's, but $a_n \geq a + \epsilon$ for at most finitely many n's.*

$\limsup a_n = \infty$ if and only if there are infinitely many terms of the sequence in any neighbourhood of ∞.

$\limsup a_n = -\infty$ if and only if the sequence diverges to $-\infty$.

Proof: Assume $\limsup a_n = a$. By defintion, for any $\epsilon > 0$, the interval $(a - \epsilon, a + \epsilon)$ contains infinitely many terms of the sequence. So it is now enough to show that $\{n \in \mathbb{N} : a_n \geq a + \epsilon\}$ is finite. Suppose if possible the above set is infinite. Then one can choose a subsequence $\{a_{n_k}\}_k$ such that $a_{n_k} \geq a + \epsilon$ for all k. Therefore for any limit point ℓ of this subsequence, one must have $\ell \geq a + \epsilon > a = \limsup a_n$. But ℓ has to be limit point of the original sequence $\{a_n\}$ also, in which case one must have $\ell \leq a$. Thus one gets a contradiction.

Conversely, assume that a satisfies the condition stated in the proposition. Take any $\epsilon > 0$. Since infinitely many terms are greater than $a - \epsilon$, but only finitely many are greater than or equal to $a + \epsilon$, there are infinitely many terms in the interval $(a - \epsilon, a + \epsilon)$. Thus a is a limit point. It is enough now to show that if b is a limit point of the sequence, then one must have $b \leq a$. Suppose if possible, b is a limit point and $b > a$. Choose a real c such that $a < c < b$. Since b is a limit point, there must be infinitely many terms that are greater than c, but by the given condition, there are at most finitely many terms greater than c. Thus one gets a contradiction. So one must have $b \leq a$.

Proof of the remaining two parts are left as exercises. □

Proposition 5.21 *$\liminf a_n = a$ if and only if for any $\epsilon > 0$, there are infinitely many terms strictly less than $a + \epsilon$, but at most finitely many terms less than $a - \epsilon$.*

$\liminf a_n = \infty$ if and only if the sequence diverges to ∞.

$\liminf a_n = -\infty$ if and only if there are infinitely many terms of the sequence in any neighbourhood of $-\infty$.

Proof: Similar to the proof of the earlier proposition. □

Proposition 5.22 $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$,
$\liminf(a_n + b_n) \geq \liminf a_n + \liminf b_n$,
If $a_n \leq b_n$ for all n then $\limsup a_n \leq \limsup b_n$ and $\liminf a_n \leq \liminf b_n$.
If $a_n \geq b$ for infinitely many n, then $\limsup a_n \geq b$.
If $a_n \leq b$ for all but finitely many n, then $\limsup a_n \leq b$.

Proof: Exercise. □

Proposition 5.23 *A sequence $\{a_n\}$ converges (or diverges to $\pm\infty$) if and only if* $\limsup a_n = \liminf a_n$ *($= \pm\infty$).*

Proof: Exercise. □

Week 6

5.5 Series

By a **series**, one means an expression of the form $\sum_{n=k}^{\infty} a_n$, where $\{a_n\}$ is a sequence of real numbers. In most situations, the integer k is either 0 or 1 and one writes just $\sum a_n$ instead of $\sum_{n=0}^{\infty} a_n$ or $\sum_{n=1}^{\infty} a_n$. The number $s_n := \sum_{r=k}^{n} a_r$ is called the **n$^{\text{th}}$ partial sum** of the series $\sum a_n$. The series $\sum a_n$ is said to converge if the sequence $\{s_n\}_n$ converges. In such a case, the number $\lim_{n\to\infty} s_n$ is called the **sum of the series** and is denoted by $\sum_{r=k}^{\infty} a_r$. The series $\sum a_n$ is said to diverge to ∞ (respectively $-\infty$) if the sequence $\{s_n\}_n$ diverges to ∞ (respectively $-\infty$). In this case one says that the sum of the series is ∞ (respectively $-\infty$).

Exercise 5.24 Asume that the sums of the two series $\sum a_n$ and $\sum b_n$ are L and M respectively (here L and M are extended reals). Let $c_n = a_n + b_n$ for all $n \in \mathbb{N}$. Show that the sum of the series $\sum c_n$ is $L + M$ whenever $L + M$ is defined.

Lemma 5.25 *Let $\{a_n\}$ be a sequence of real numbers. The series $\sum_{n=k}^{\infty} a_n$ converges if and only if the series $\sum_{n=0}^{\infty} a_n$ converges.*

Proof: Let s_n and t_n denote the n$^{\text{th}}$ partial sums of the two series. Then for $n \geq k$, one has $t_n = s_n + \sum_{r=0}^{k-1} a_r$. Therefore $\lim_{n\to\infty} t_n$ exists if and only if $\lim_{n\to\infty} s_n$ exists, and if the limits exist, then $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n + \sum_{r=0}^{k-1} a_r$. □

Exercise 5.26 Let $\{a_n\}$ be a sequence of real numbers. Show that the series $\sum_{n=k}^{\infty} a_n$ diverges to ∞ (respectively $-\infty$) if and only if the series $\sum_{n=0}^{\infty} a_n$ diverges to ∞ (respectively $-\infty$).

Lemma 5.27 *Let $\sum a_n$ be a convergent series. Then $\lim a_n = 0$.*

Proof: Let s be the sum of the series. Then $\lim_{n\to\infty} s_n = s = \lim_{n\to\infty} s_{n-1}$. Hence $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$. □

The converse is not true. One could for example look at the series $\sum \frac{1}{n}$.

A series $\sum a_n$ is said to **converge absolutely** if the series $\sum |a_n|$ converges. It is said to **converge conditionally** if $\sum a_n$ converges but $\sum |a_n|$ does not.

Lemma 5.28 *If $\sum a_n$ converges absolutely, then it converges.*

Proof: Let s_n denote the $\mathrm{n^{th}}$ partial sum of the series $\sum |a_n|$ and let t_n denote the $\mathrm{n^{th}}$ partial sum of the series $\sum a_n$. Then for any m and n, one has $|t_n - t_m| \leq |s_m - s_n|$. Since the sequence $\{s_n\}$ is Cauchy, it now follows (**how?**) that the sequence $\{t_n\}$ is also Cauchy. □

Proposition 5.29 (Leibneiz's theorem) *Let $\{c_n\}$ be a sequence such that $c_k \downarrow 0$, i.e. $c_k \geq 0$ for all k, c_k's are decreasing, and $\lim c_k = 0$. Then the series $\sum(-1)^n c_n$ converges.*

Proof: It is enough to show that the sequence $s_n = \sum_{k=1}^{n}(-1)^k c_k$ is cauchy. Let $m > n$. Then

$$\begin{aligned} |s_m - s_n| &= |(-1)^{n+1}c_{n+1} + (-1)^{n+2}c_{n+2} + \ldots + (-1)^m c_m| \\ &= |c_{n+1} - c_{n+2} + \ldots + (-1)^{m-n-1}c_m|. \end{aligned}$$

Now the sequence $\{c_n\}$ is decreasing, therefore $c_k - c_{k+1} \geq 0$ for all k. Also, observe that

$$c_{n+1} - c_{n+2} + \ldots + (-1)^{m-n-1}c_m = \begin{cases} (c_{n+1} - c_{n+2}) + \ldots + (c_{m-1} - c_m) & \text{if } m-n \text{ is even,} \\ (c_{n+1} - c_{n+2}) + \ldots + (c_{m-2} - c_{m-1}) + c_m & \text{if } m-n \text{ is odd.} \end{cases}$$

Since the right hand side in nonnegative, we have

$$|c_{n+1} - c_{n+2} + \ldots + (-1)^{m-n-1}c_m| = c_{n+1} - c_{n+2} + \ldots + (-1)^{m-n-1}c_m.$$

Next,

$$c_{n+1} - c_{n+2} + \ldots + (-1)^{m-n-1}c_m = \begin{cases} c_{n+1} - (c_{n+2} - c_{n+3}) - \ldots - c_m & \text{if } m-n \text{ is even,} \\ c_{n+1} - (c_{n+2} - c_{n+3}) - \ldots - (c_{m-1} - c_m) & \text{if } m-n \text{ is odd.} \end{cases}$$

Therefore we have $|s_m - s_n| \leq c_{n+1}$. Since the sequence $\{c_n\}$ converges to 0, it follows that the sequence $\{s_n\}$ is Cauchy. □

5.6 Tests for convergence

Exercise 5.30 Show that the series

$$\sum x^n \quad \begin{cases} \text{converges absolutely} & \text{if } |x| < 1, \\ \text{diverges to } \infty & \text{if } x \geq 1, \\ \text{neither converges nor diverges to } \pm\infty & \text{if } x \leq -1. \end{cases}$$

Theorem 5.31 (Comparison test) *1. If $|a_n| \leq b_n$ for all but finitely many n's and $\sum b_n$ converges, then $\sum a_n$ converges.*

2. If $a_n \geq b_n \geq 0$ for all n, and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Proof: Write $s_n = \sum_{k=1}^{n}|a_k|$, $t_n = \sum_{k=1}^{n} b_k$ and $u_n = \sum_{k=1}^{n} a_k$.

1. By the given condition, there is an integer N such that for all $n \geq N$, one has $|a_n| \leq b_n$. Hence whenever m and n are both greater than or equal to N, then $|s_m - s_n| \leq |t_n - t_m|$. Since the sequence $\{t_n\}$ is cauchy, it follows that $\{s_n\}$ is also cauchy. This means that the series $\sum a_n$ converges absolutely. Hence it converges.

2. By the given condition, we have $u_n \geq t_n$ for all n. Also, since $\{t_n\}$ is increasing, and diverges (to ∞), it follows that given any $M \in \mathbb{R}$, there is an integer N such that $t_n > M$ for all $n \geq N$. By the inequality above, it then follows that $u_n > M$ for all $n \geq N$. Thus the series $\sum a_n$ diverges. □

This is the main test for convergence or divergence of a series. By making explicit choices of the sequence $\{b_n\}$, one can derive various other tests. Indeed all the other tests that we will learn are derived from the above in this manner.

Theorem 5.32 (Root test) *If* $\limsup |a_n|^{1/n} < 1$*, then* $\sum a_n$ *converges absolutely.*
If $\limsup |a_n|^{1/n} > 1$*, then* $\sum a_n$ *does not converge.*

Proof: Write $a = \limsup |a_n|^{1/n}$. If $a < 1$, we can choose an $\epsilon > 0$ such that $a + \epsilon < 1$. Since $a = \limsup |a_n|^{1/n}$, we have $|a_n|^{1/n} \geq a + \epsilon$ for at most finitely many terms. In other words, there is an integer N such that $|a_n|^{1/n} < a + \epsilon$ for all $n \geq N$. But this means $|a_n| < (a+\epsilon)^n$ for all $n \geq N$. Since $0 < a + \epsilon < 1$, it follows that the series $\sum_{n=N}^{\infty} a_n$ converges, which in turn implies that $\sum_{n=1}^{\infty} a_n$ converges.

If $a > 1$, then choose an $\epsilon > 0$ such that $a - \epsilon > 1$. Then $|a_n|^{1/n} > a - \epsilon$ for infinitely many n's. This means $|a_n| > (a-\epsilon)^n > 1$ for infinitely many n's. But then one can not have $\lim_{n\to\infty} a_n = 0$. So the series $\sum a_n$ can not converge. □

Theorem 5.33 (Ratio test) *If* $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$*, then* $\sum a_n$ *converges absolutely.*
If $\left|\frac{a_{n+1}}{a_n}\right| > 1$ *for all but finitely many n's, then* $\sum a_n$ *does not converge.*

Proof: Write $a = \limsup \left|\frac{a_{n+1}}{a_n}\right|$. If $a < 1$, then we can choose an $\epsilon > 0$ such that $a + \epsilon < 1$. Since $a = \limsup \left|\frac{a_{n+1}}{a_n}\right|$, there is an integer N such that $\left|\frac{a_{n+1}}{a_n}\right| < a + \epsilon$ for all $n \geq N$. It then follows that for all $n \geq N$, one has $|a_n| \leq |a_N|(a+\epsilon)^{n-N} = \text{const} \cdot (a+\epsilon)^n$. Hence by comparison test, the series $\sum_{n=N}^{\infty} |a_n|$ converges. Therefore so does the series $\sum_{n=1}^{\infty} |a_n|$.

By the given condition, there is an integer N such that $\left|\frac{a_{n+1}}{a_n}\right| > 1$ for all $n \geq N$. This in particular implies that $|a_n| > |a_N|$ for all $n > N$. Therefore $\lim a_n$ can not be 0. So $\sum a_n$ does not converge. □

Exercise 5.34 Investigate the behaviour of the following series: $\{\sum \frac{x^n}{n!}\}$, $\sum \binom{m+n}{n} x^n$, where $m \in \mathbb{N}$, $\sum n^{\alpha} x^n$.

5.7 Power series

A series of the form $\sum a_n x^n$ is known as a power series. An application of the root test tells us that this series converges absolutely if $\limsup |a_n x^n|^{1/n} < 1$, and does not converge if $\limsup |a_n x^n|^{1/n} > 1$. In other words, the series converges whenever $|x| < \frac{1}{\limsup |a_n|^{1/n}}$, and fails to converge if $|x| > \frac{1}{\limsup |a_n|^{1/n}}$.

Exercise 5.35 Show that the following power series converge for any value of x:

1. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$,
2. $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$,
3. $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$.

Define

$$\begin{aligned}
\exp x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, \\
\cos x &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}, \\
\sin x &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!},
\end{aligned}$$

where the right hand sides are the sums of the corresponding series.

5.8 Product of two series

Proposition 5.36 *Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be two series such that $\sum a_n$ is absolutely convergent and $\sum b_n$ is convergent. Let $\alpha = \sum a_n$ and $\beta = \sum b_n$. Then the product series $\sum c_n$, given by $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, converges and the sum is $\alpha\beta$.*

Proof: Write $A_n = \sum_{k=0}^{n} a_k$, $B_n = \sum_{k=0}^{n} b_k$, and $C_n = \sum_{k=0}^{n} c_k$. Then

$$\begin{aligned}
C_n &= \sum_{k=0}^{n} c_k \\
&= \sum_{k=0}^{n} \sum_{r=0}^{k} a_r b_{k-r} \\
&= \sum_{r=0}^{n} \sum_{k=r}^{n} a_r b_{k-r} \\
&= \sum_{r=0}^{n} \sum_{j=0}^{n-r} a_r b_j \\
&= \sum_{r=0}^{n} a_r B_{n-r} \\
&= \sum_{r=0}^{n} a_r (B_{n-r} - \beta) + \beta A_n.
\end{aligned}$$

Since $\lim_{n\to\infty} A_n = \alpha$, it is now enough to show that $\lim_{n\to\infty} \sum_{r=0}^{n} a_r (B_{n-r} - \beta) = 0$.

Take an $\epsilon > 0$. Let ξ be the sum of the series $\sum |a_n|$. Choose an integer N such that $|B_k - \beta| < \frac{\epsilon}{\xi}$ for $k \geq N$. Then for $n > N$, we have

$$\begin{aligned}
|\sum_{r=0}^{n} a_r(B_{n-r} - \beta)| &\leq \sum_{r=0}^{n-N} |a_r(B_{n-r} - \beta)| + \sum_{r=n-N+1}^{n} |a_r| \cdot |B_{n-r} - \beta| \\
&\leq \sum_{s=N}^{n} |a_{n-s}| \cdot |B_s - \beta| + \sum_{s=0}^{N-1} |a_{n-s}| \cdot |B_s - \beta| \\
&\leq \frac{\epsilon}{\xi} \sum_{s=N}^{n} |a_{n-s}| + \sum_{s=0}^{N-1} |a_{n-s}| \cdot |B_s - \beta| \\
&\leq \frac{\epsilon}{\xi} \cdot \xi + \sum_{s=0}^{N-1} |a_{n-s}| \cdot |B_s - \beta| \\
&\leq \epsilon + \sum_{s=0}^{N-1} |a_{n-s}| \cdot |B_s - \beta|.
\end{aligned}$$

The second term on the right hand side converges to 0 as n tends to ∞. Hence

$$\limsup_{n\to\infty} |\sum_{r=0}^{n} a_r(B_{n-r} - \beta)| \leq \epsilon.$$

Since this is true for any $\epsilon > 0$, it follows that $\limsup_{n\to\infty} |\sum_{r=0}^{n} a_r(B_{n-r} - \beta)| = 0$, which implies that $\lim_{n\to\infty} |\sum_{r=0}^{n} a_r(B_{n-r} - \beta)| = 0$. □

Exercise 5.37 Show that the n$^{\text{th}}$ term of the product of the two series $\sum a_n$ and $\sum b_n$, where $a_n = b_n = (-1)^n \frac{1}{\sqrt{n}}$, does not converge to 0.

The above example illustrates that if both the series $\sum a_n$ and $\sum b_n$ converge conditionally, then the product series need not converge.

Exercise 5.38 Show that

1. $\exp(x + y) = \exp x \exp y$.
2. $\sin(x + y) = \sin x \cos y + \sin y \cos x$,
3. $\sin^2 x + \cos^2 x = 1$.

WEEK 7

6 Limit and continuity

6.1 Limit of a function

Let $E \subseteq \mathbb{R}$, and let $f : E \to \mathbb{R}$ be a function. Let $a \in \mathbb{R}$ be a limit point of E. One says that that **the limit of f as x approaches a (in symbol $x \to a$) exists and is equal to ℓ (in symbols** $\lim_{x\to a} f(x) = \ell$**)** if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - \ell| < \epsilon$ whenever $0 < |x - a| < \delta$ and $x \in E$ (i. e. whenever $x \in (E \backslash \{a\}) \cap (a - \delta, a + \delta)$).

One says that **the limit of f as $x \to a$ is ∞ (and writes** $\lim_{x\to a} f(x) = \infty$**)** if given any $M \in \mathbb{R}$, there exists a $\delta > 0$ such that $f(x) > M$ whenever $x \in (E \backslash \{a\}) \cap (a - \delta, a + \delta)$.

Similarly one writes $\lim_{x\to a} f(x) = -\infty$ if given any $M \in \mathbb{R}$, there exists a $\delta > 0$ such that $f(x) < M$ whenever $x \in (E \backslash \{a\}) \cap (a - \delta, a + \delta)$.

Next assume that the set E is not bounded above. Then one says that **the limit of f as x approaches ∞ (in symbol $x \to \infty$) exists and is equal to ℓ (in symbols** $\lim_{x\to\infty} f(x) = \ell$**)** if for every $\epsilon > 0$, there exists a $N \in \mathbb{R}$ such that $|f(x) - \ell| < \epsilon$ whenever $x \in E \cap [N, \infty)$.

One writes $\lim_{x\to\infty} f(x) = \infty$ if given any $M \in \mathbb{R}$, there exists a $N \in \mathbb{R}$ such that $f(x) > M$ whenever $x \in E \cap [N, \infty)$. One writes $\lim_{x\to\infty} f(x) = -\infty$ if given any $M \in \mathbb{R}$, there exists a $N \in \mathbb{R}$ such that $f(x) < M$ whenever $x \in E \cap [N, \infty)$.

Exercise 6.1 Define the meaning of the following expressions: $\lim_{x\to-\infty} f(x) = \ell$, $\lim_{x\to-\infty} f(x) = \infty$ $\lim_{x\to-\infty} f(x) = -\infty$

Exercise 6.2 Let $c \in \mathbb{R}$ and define f by $f(x) = c$ for all x. Show that $\lim_{x\to a} f(x) = \lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = c$.

Proposition 6.3 *Let f and g be two functions defined on a set E, with* $\lim_{x\to a} f(x) = \ell$ *and* $\lim_{x\to a} g(x) = m$. *Then*

1. $\lim_{x\to a}(f(x) + g(x)) = \ell + m$,
2. $\lim_{x\to a}(f(x)g(x)) = \ell m$,
3. *if $m \neq 0$, then there is a neighbourhod (b, c) containing the point a such that $g(x) \neq 0$ for all $x \in (b, c) \cap (E \backslash \{a\})$, and* $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\ell}{m}$.

Proof: We will prove part 3 here. Parts 1 and 2 will be left as exercises.

Since $\lim_{x\to a} g(x) = m \neq 0$, there is a $\delta_1 > 0$ such that whenever $x \in (a-\delta_1, a+\delta_1)\setminus\{a\}$, one has $|g(x)-m| < \frac{|m|}{2}$. But then $|g(x)| > \frac{|m|}{2} > 0$. Thus $\frac{f(x)}{g(x)}$ is defined on $(a-\delta_1, a+\delta_1)\setminus\{a\}$.

Take any $\epsilon > 0$. We have to choose a $\delta > 0$ such that $\left|\frac{f(x)}{g(x)} - \frac{\ell}{m}\right| < \epsilon$ whenever $0 < |x-a| < \delta$. Now

$$\left|\frac{f(x)}{g(x)} - \frac{\ell}{m}\right| = \left|\frac{mf(x) - \ell g(x)}{mg(x)}\right|.$$

Hence for $0 < |x-a| < \delta_1$, one has

$$\begin{aligned}\left|\frac{f(x)}{g(x)} - \frac{\ell}{m}\right| &\leq \frac{2}{|m|^2}|mf(x) - \ell g(x)| \\ &\leq \frac{2}{|m|^2}(|m||f(x)-\ell| + |\ell||g(x)-m|).\end{aligned}$$

Let $\delta_2 > 0$ and δ_3 be such that $0 < |x-a| < \delta_2$ implies $|f(x)-\ell| < \frac{|m|\epsilon}{4}$ and $0 < |x-a| < \delta_3$ implies $|g(x)-m| < \frac{|m|^2\epsilon}{4|\ell|}$. Let $\delta := \min\{\delta_1, \delta_2, \delta_3\}$. Then for $0 < |x-a| < \delta$ we have $\left|\frac{f(x)}{g(x)} - \frac{\ell}{m}\right| < \epsilon$. □

Exercise 6.4 Prove that the above proposition continues to hold even when a, ℓ and m are allowed to take values in the set of extended reals, provided the right hand sides are defined.

Proposition 6.5 *Let f be a function defined on a set E and let $\lim_{x\to a} f(x) = \ell$. If $f(x) \geq 0$ for all $x \in E$, then $\ell \geq 0$.*

Proof: Exercise. □

Exercise 6.6 Let f and g be two functions defined on a set E, with $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = m$. If $f(x) \leq g(x)$ for all $x \in E$, then show that $\ell \leq m$.

Proposition 6.7 *Let f, g and h be functions defined on a set E, with $\lim_{x\to a} f(x) = \ell = \lim_{x\to a} g(x)$. If $f(x) \leq h(x) \leq g(x)$ for all $x \in E$, then $\lim_{x\to a} h(x) = \ell$.*

Proof:

□

Exercise 6.8 Prove or disprove: Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two functions. Assume that $\lim_{x\to a} g(x) = b$ and $\lim_{x\to b} f(x) = \ell$. Then $\lim_{x\to a} f(g(x)) = \ell$.

6.2 Computation of some limits

Lemma 6.9 $\lim_{x\to 0} \sin x = 0$.

Proof: Let $s_n(x)$ denote the n^{th} partial sum of the series for $\sin x$. Then for $|x| < 1$,

$$\begin{aligned} |s_n(x)| &\leq \sum_{r=0}^{n} \frac{|x|^{2r+1}}{(2r+1)!} \\ &\leq |x| \sum_{r=0}^{n} \frac{1}{(2r+1)!} \\ &\leq |x| \exp(1). \end{aligned}$$

Therefore

$$|\sin x| \leq |x| \exp(1) \quad \text{for } |x| < 1.$$

Now take any $\epsilon > 0$. Choose $\delta = \min\{\frac{\epsilon}{\exp(1)}, 1\}$. Then for $|x| < \delta$, one has $|\sin x| < \epsilon$. □

Lemma 6.10 $\lim_{x \to 0} \cos x = 1$.

Proof: Let $s_n(x)$ denote the n^{th} partial sum of the series for $\cos x$. Then for $|x| < 1$,

$$\begin{aligned} |s_n(x) - 1| &\leq \sum_{r=1}^{n} \frac{|x|^{2r}}{(2r)!} \\ &\leq |x|^2 \sum_{r=1}^{n} \frac{1}{(2r)!} \\ &\leq |x|^2 \exp(1). \end{aligned}$$

Therefore

$$|\cos x - 1| \leq |x|^2 \exp(1) \quad \text{for } |x| < 1.$$

Now take any $\epsilon > 0$. Choose $\delta = \min\{\frac{\epsilon}{\exp(1)}, 1\}$. Then for $|x| < \delta$, one has $|\cos x - 1| < \epsilon$. □

Lemma 6.11 $\lim_{x \to 0} \left| \frac{\sin x}{x} - 1 \right| = 0$.

Proof: As before, let $s_n(x)$ denote the n^{th} partial sum of the series for $\sin x$. Then for $0 < |x| < 1$,

$$\begin{aligned} \left|\frac{s_n(x)}{x} - 1\right| &= \sum_{r=1}^{n} (-1)^r \frac{|x|^{2r}}{(2r+1)!} \\ &\leq |x|^2 \sum_{r=1}^{n} \frac{1}{(2r+1)!} \\ &\leq |x|^2 \exp(1). \end{aligned}$$

Therefore

$$\left|\frac{\sin x}{x} - 1\right| \leq |x|^2 \exp(1) \quad \text{for } 0 < |x| < 1.$$

Hence the result. □

Exercise 6.12 Show that

1. $\lim_{x\to 0} \exp x = 1$.
2. $\lim_{x\to 0} \left|\frac{\exp x - 1 - x}{x}\right| = 0$.
3. $\lim_{x\to 0} \left|\frac{\exp x}{x^n}\right| = \infty$.

6.3 Limits via limits of sequences

Proposition 6.13 *Let f be a function defined on a set E. Then $\lim_{x\to a} f(x) = \ell$ if and only if for any sequence $\{x_n\}$ in E (with $x_n \neq a$ for all n) converging to a, one has $\lim_{n\to\infty} f(x_n) = \ell$.*

Proof: Observe that a is a limit point of the set E and hence there will always exist sequences with terms in E and different from a that converge to a.

Assume $\lim_{x\to a} f(x) = \ell$. Let $\{x_n\}$ be a sequence in E converging to a with none of the terms equal to a. Let $\epsilon > 0$ be given. Since $\lim_{x\to a} f(x) = \ell$, there exists a $\delta > 0$ such that $|f(x) - \ell| < \epsilon$ for $0 < |x - a| < \delta$. Since $\lim_{n\to\infty} x_n = a$, corresponding to this δ, there exists an integer N such that $|x_n - a| < \delta$ if $n \geq N$. Since none of the terms of the sequence $\{x_n\}$ is a, it follows that $0 < |x_n - a| < \delta$ for $n \geq N$. Therefore whenever $n \geq N$, one has $|f(x_n) - \ell| < \epsilon$. Thus $\lim_{n\to\infty} f(x_n) = \ell$.

Now let us prove the converse. Assume $\lim_{x\to a} f(x) \neq \ell$. This means that there is an $\epsilon > 0$ such that for every $\delta > 0$, there is an $x \in E$ with $0 < |x - a| < \delta$ and $|f(x) - \ell| \geq \epsilon$. In particular, for each $n \in \mathbb{Z}_+$, there is an x_n with $0 < |x_n - a| < \frac{1}{n}$ and $|f(x_n) - \ell| \geq \epsilon$. But then $\{x_n\}$ is a sequence with terms from E different from a, and $\lim_{n\to\infty} x_n = a$, but $|f(x_n) - \ell| \geq \epsilon$ for all n, so that $\lim_{n\to\infty} f(x_n) \neq \ell$. □

Exercise 6.14 State and prove analogous statements where the quantities a and ℓ are allowed to be $\pm\infty$.

6.4 Continuity

A function f defined on a set E is said to be **continuous at a point** $a \in E$ if $\lim_{x\to a} f(x) = f(a)$. It is said to be continuous everywhere (or just continuous) if it is continuous at every point $a \in E$.

Proposition 6.15 *A function f defined on a set E is continuous at a point $a \in E$ if and only if for any sequence $\{x_n\}$ in E converging to a, one has $\lim_{n\to\infty} f(x_n) = f(a)$.*

Proof: If for any sequence $\{x_n\}$ in E converging to a, one has $\lim_{n\to\infty} f(x_n) = f(a)$, then it follows from the earlier result that $\lim_{x\to a} f(x) = f(a)$, i. e. f is continuous at a.

Now to prove the converse, assume that f is continuous at a. Take a sequence $\{x_n\}$ such that $x_n \in E$ for all n and $\lim_{n\to\infty} x_n = a$. Let $\epsilon > 0$ be given. We want to show that there is an integer N such that for $n \geq N$, one has $|f(x_n) - f(a)| < \epsilon$. Since $\lim_{x\to a} f(x) = f(a)$, there is a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$. Again, since $\lim_{n\to\infty} x_n = a$, corresponding to this δ, there is an integer N such that for $n \geq N$, we have $|x_n - a| < \delta$. Combining these two observations, it follows that for $n \geq N$, we have $|f(x_n) - fa)| < \epsilon$. □

Exercise 6.16 Show that a constant function and the function $x \mapsto x$ are continuous everywhere.

Proposition 6.17 *Let f and g be two functions defined in some neighbourhood of a point $a \in \mathbb{R}$. Assume f and g are continuous at a. Then $f + g$ and fg are continuous at a.*

If, moreover, $g(a) \neq 0$, then f/g is continuous at a.

Proof: The proof follows from the corresponding properties of limits. □

Proposition 6.18 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions. Assume that g is continuous at a and f is continuous at $g(a)$. Then $f \circ g$ is continuous at a.*

Proof: Take an $\epsilon > 0$. Since f is continuous at $g(a)$, there is a $\delta' > 0$ such that $|f(y) - f(g(a))| < \epsilon$ whenever $|y - g(a)| < \delta'$. Again, since g is continuous at a, given this δ', there is a $\delta > 0$ such that $|g(x) - g(a)| < \delta'$ whenever $|x - a| < \delta$. Therefore for $|x - a| < \delta$, we have $|f(g(x)) - f(g(a))| < \epsilon$. □

Exercise 6.19 Show that a polynomial function is continuous everywhere.

Exercise 6.20 Show that the function exp is continuous at the point 0. Now use the equality $\exp(x + y) = \exp(x)\exp(y)$ for all x and y to prove that the exponential function is continuous everywhere.

Exercise 6.21 Show that the function sin is continuous at the point 0. Now use the equality $\sin(x + y) = \sin x \cos y + \cos x \sin y$ for all x and y to prove that the sine function is continuous everywhere.

Arguing along similar lines, show that the cosine function is also continuous everywhere.

Proposition 6.22 *Let $p \geq 0$. Then the function $x \mapsto x^p$ is continuous on $[0, \infty)$.*

If $p < 0$, then it is continuous on $(0, \infty)$.

Proof: It is enough to prove continuity at the point $x = 0$ and $x = 1$ (**why?**), i. e. it is enough to show that $\lim_{x\to 0+} x^p = 0$ and $\lim_{x\to 0}(1 + x)^p = 1$. Let us first prove the second equality. If $p \in \mathbb{N}$, then it follows by using the binomial expansion of $(1 + x)^p$. Since $p = [p] + (p - [p])$, it is now enough to prove the above for $0 \leq p < 1$. So assume $0 \leq p < 1$. Then for $x > 0$, $1 < (1 + x)^p < 1 + x$ and for $-1 < x < 0$, one has $1 + x < (1 + x)^p < 1$. Combining these, one gets $1 - |x| < (1 + x)^p < 1 + |x|$ whenever $x > -1$. Therefore it follows that $\lim_{x\to 0}(1 + x)^p = 1$.

Let us now show that $\lim_{x\to 0+} x^p = 0$. Take $\epsilon > 0$. Choose $\delta = \epsilon^{\frac{1}{p}}$. Then for $0 < x < \delta$, one has $0 < x^p < \delta^p = \epsilon$.

The case $p < 0$ now follows from the observation that $x^p = \frac{1}{x^{-p}}$. □

Proposition 6.23 *Let $p > 0$. Then the function $x \mapsto p^x$ is continuous.*

Proof: It is enough to prove the result for $p > 1$. Let $a \in \mathbb{R}$. Then $|p^x - p^a| = p^a \cdot |p^{x-a} - 1|$. Therefore it is enough to prove continuity at $x = 0$.

Take any $\epsilon > 0$. Choose an $\epsilon' > 0$ such that $1 - \epsilon < \frac{1}{1+\epsilon'}$ and $\epsilon' < \epsilon$. Since $\lim_{n\to\infty} p^{1/n} = 1$, it follows that there is an integer N such that $|p^{1/N} - 1| < \epsilon'$. Since $p > 1$, one in fact has $1 < p^{1/N} < 1 + \epsilon'$. Hence for any $x \in (0, \frac{1}{N})$, one has $1 < p^x < p^{1/N} < 1 + \epsilon'$. Consequently, for $x \in (-\frac{1}{N}, 0)$, we have $\frac{1}{1+\epsilon'} < p^x < 1$. Thus whenever $|x| < \frac{1}{N}$, we have $p^x \in (1 - \epsilon, 1 + \epsilon)$. □

Exercise 6.24 Show that the function f given by

$$f(x) = \begin{cases} \exp(-\frac{1}{x^2}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous

6.5 Properties of continuous functions

Proposition 6.25 *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, and assume that $f(x) = 0$ for all $x \in \mathbb{Q}$. Then $f(x) = 0$ for all $x \in \mathbb{R}$.*

Proof: Take an $x \in \mathbb{R}$. Since the closure of $\mathbb{Q}$ is $\mathbb{R}$, there is a sequence $\{x_n\}$ such that $x_n \in \mathbb{Q}$ for all n and $\lim_{n\to\infty} x_n = x$. Since f is continuous at x, we have $f(x) = \lim_{n\to\infty} f(x_n)$. But $f(x_n) = 0$ for all n. Hence $f(x) = 0$. □

Exercise 6.26 Let f and g be two continuous functions such that $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Show that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Exercise 6.27 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Let E be a subset of $\mathbb{R}$ such that $\bar{E} = \mathbb{R}$. If $f(x) = 0$ for all $x \in E$, then show that $f(x) = 0$ for all $x \in \mathbb{R}$.

Proposition 6.28 *Let f be a function defined on a set E. Assume f is continuous at a point $a \in E$ and $f(a) \neq 0$. Then there is a neighbourhood (b, c) around a such that the sign of $f(x)$ for every $x \in (b, c) \cap E$ is same as that of $f(a)$.*

Proof: Take $\epsilon = \frac{|f(a)|}{2}$. Since $f(a) \neq 0$, ϵ is positive. Using continuity of f at a, there is a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$ and $x \in E$. Thus if $f(a) > 0$ then for all $x \in (a - \delta, a + \delta) \cap E$, one has $\frac{f(a)}{2} < f(x) < \frac{3f(a)}{2}$, and if $f(a) < 0$, then for all $x \in (a - \delta, a + \delta) \cap E$, one has $\frac{3f(a)}{2} < f(x) < \frac{f(a)}{2}$. □

Theorem 6.29 *Let f be continuous on an interval $[a, b]$ and let $f(a)$ and $f(b)$ be of opposite signs. Then there is a point $c \in (a, b)$ such that $f(c) = 0$.*

Proof: Assume $f(a) < 0$ and $f(b) > 0$. Let $E = \{x \in [a, b] : f(x) > 0\}$. Since $f(b) > 0$, this is nonempty. Let $c := \inf E$.

First, observe that by the sign-preserving property, there is a $\delta > 0$ such that $f(x) > 0$ for all $x \in (b - \delta, b]$, so that $c \leq b - \delta < b$. Similarly, there is a $\delta' > 0$ such that $f(x) < 0$ for all $x \in [a, a + \delta')$. Hence for any $y \in E$, we have $y \geq a + \delta'$. Therefore $c \geq a + \delta' > a$. Thus $c \in (a, b)$.

Since $c = \inf E$, there is a sequence $\{c_n\}$ converging to c such that $c_n \in E$ for all E. By continuity of f at c, it follws that $f(c) \geq 0$. If $f(c) > 0$, by the sign-preserving property, there is a $\delta > 0$ such that $f(x) > 0$ for all $x \in (c - \delta, c + \delta) \cap [a, b]$. Since $c \in (a, b)$, this in particular implies that there is a $c' < c$ such that $c' \in [a, b] \cap E$. But then c can not be $\inf E$. Therefore we must have $f(c) = 0$. □

Theorem 6.30 *Let f be continuous on a closed bounded interval $[a, b]$. Given any $\epsilon > 0$, there is a partition*

$$x_0 = a < x_1 < \ldots < x_n = b$$

of $[a, b]$ such that

$$\text{over each subinterval of the form } [a_k, a_{k+1}], \text{ one has } \sup_{x,y} |f(x) - f(y)| < \epsilon. \tag{6.2}$$

Proof: Assume that the conclusion of the theorem is not true. Then there exists an $\epsilon > 0$ for which there is no partition satisfying the condition stated above. Now apply an argument similar to the one used in the proof of the Bolzano-Weierstrass theorem to produce two sequences $\{a_n\}$ and $\{b_n\}$ such that

$$a_0 = a, \quad b_0 = b, \quad a_n \leq a_{n+1}, \quad b_{n+1} \leq b_n, \quad b_{n+1} - a_{n+1} = \frac{a_n - b_n}{2},$$

and for each n, $[a_n, b_n]$ does not admit any partition that satisfies the condition stated in the theorem. It follows that the sequences a_n and b_n both converge to a common point c. Use continuity of f at c to get a $\delta > 0$ such that $|f(x) - f(c)| < \frac{\epsilon}{3}$ for $x \in (c - \delta, c + \delta)$. Now choose n large enough so that $|a_n - c| < \delta$ and $|b_n - c| < \delta$. Then $[a_n, b_n] \subseteq (c - \delta, c + \delta)$. Hence for any $x \in [a_n, b_n]$, we have $|f(x) - f(c)| < \frac{\epsilon}{3}$. Therefore $\sup\{|f(x) - f(y)| : x, y \in [a_n, b_n]\} \leq \frac{2\epsilon}{3} < \epsilon$. This contradicts the choice of the a_n's and b_n's. □

Corollary 6.31 *A continuous function on a closed bounded interval is bounded.*

Proof: Get a partition $x_0 = a < x_1 < \ldots < x_n = b$ of the interval $[a, b]$ on which f is continuous such that one has $\sup\{|f(x) - f(y)| : x, y \in [x_k, x_{k+1}]\} < 1$ for each k. Write $m = \min\{f(x_i) : 1 \leq i \leq n\}$ and $M = \max\{f(x_i) : 1 \leq i \leq n\}$. Then it follows that $m - 1 \leq f(x) \leq M + 1$ for all $x \in [a, b]$. □

Proposition 6.32 *Let f be continuous on a closed bounded interval $[a, b]$. Then there are points $c, d \in [a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$.*

Proof: Let $E := \{f(x) : x \in [a, b]\}$, $m := \inf E$ and $M := \sup E$. Then $m \leq f(x) \leq M$ for all $x \in [a, b]$. We will prove the existence of the point c with $f(c) = m$. Proof of existence of d with $f(d) = M$ is similar and is left as an exercise.

There is a sequence with terms in E that converges to m. The sequence will be of the form $\{f(x_n)\}$, where each x_n comes from $[a, b]$. Since the sequence $\{x_n\}$ is bounded, it has a convergent subsequence, say $\{x_{n_k}\}_k$. Write $c = \lim_{k\to\infty} x_{n_k}$. Then clearly $c \in [a, b]$ and since f is continuous, $\lim_{k\to\infty} f(x_{n_k}) = f(c)$. Bt we know that the sequence $\{f(x_n)\}$ converges to m. So we must have $f(c) = m$. □

WEEK 8

7 Some important functions

7.1 Trigonometric functions

1. Show that the functions sin and cos are continuous everywhere.

2. Let $Z := \{x \in \mathbb{R}_+ : \cos x = 0\}$. Show that this is nonempty by showing that $\cos 2 < 0$. Let $\xi := \inf Z$. Show that $\cos \xi = 0$. Show that $\cos x > 0$ for $0 \le x \le 1$. Hence conclude that $1 < \xi < 2$.

3. Prove that

$$\cos 2n\xi = (-1)^n, \quad \sin 2n\xi = 0 \quad \text{for all } n \in \mathbb{Z}.$$
$$\cos(2\xi + x) = -\sin x, \quad \sin(2\xi + x) = -\cos x.$$
$$\cos(4\xi + x) = \cos x, \quad \sin(4\xi + x) = \sin x.$$

4. Show that

$$\cos x \begin{cases} > 0 & \text{for } 0 \le x < \xi, \\ = 0 & \text{for } x = \xi, \\ < 0 & \text{for } \xi < x < 3\xi, \\ = 0 & \text{for } x = 3\xi, \\ > 0 & \text{for } 3\xi < x \le 4\xi. \end{cases}$$

5. Since $\sin^2 x + \cos^2 x = 1$ for all x, one has $\sin^2 \xi = 1$. Hence $\sin \xi$ is either $+1$ or -1. Suppose if possible $\sin \xi = -1$. Then $\cos(\xi+x) = \cos \xi \cos x - \sin \xi \sin x = \sin x$. Therefore whenever $0 < x < 2\xi$, one must have $\sin x < 0$. But it is easy to see that $\sin 1 > 0$. Sine $1 < \xi$, this is impossible. Hence we must have $\sin \xi = 1$. It now follows from the equality $\cos(\xi + x) = \sin x$ and the properties of the cosine function that

$$\sin x \begin{cases} = 0 & \text{if } x = 0, \\ > 0 & \text{for } 0 < x < 2\xi, \\ = 0 & \text{for } x = 2\xi, \\ < 0 & \text{for } 2\xi < x < 4\xi, \\ = 0 & \text{for } x = 4\xi. \end{cases}$$

6. Show that

$$\cos x \begin{cases} < 1 & \text{for } 0 < x < 4\xi, \\ > -1 & \text{for } \begin{cases} 0 \le x < 2\xi, \\ 2\xi < x \le 4\xi \end{cases} \end{cases}$$

$$\sin x \begin{cases} < 1 & \text{for } \begin{cases} 0 \le x < \xi, \\ \xi < x \le 4\xi \end{cases} \\ > -1 & \text{for } \begin{cases} 0 \le x < 3\xi, \\ 3\xi < x \le 4\xi \end{cases} \end{cases},$$

(Since $|\sin x| > 0$ for $x \in (0, 4\xi) \setminus \{2\xi\}$, one must have $|\cos x| < 1$ for these values of x)

7. Use the above parts to show that

$$\cos x \begin{cases} = 1 & \text{for } x = 0, \\ \in (0,1) & \text{for } 0 \le x < \xi, \\ = 0 & \text{for } x = \xi, \\ \in (-1,0) & \text{for } \xi < x < 2\xi, \\ = -1 & \text{for } x = 2\xi, \\ \in (-1,0) & \text{for } 2\xi < x < 3\xi, \\ = 0 & \text{for } x = 3\xi, \\ \in (0,1) & \text{for } 3\xi < x < 4\xi. \end{cases}$$

$$\sin x \begin{cases} = 0 & \text{for } x = 0, \\ \in (0,1) & \text{for } 0 \le x < \xi, \\ = 1 & \text{for } x = \xi, \\ \in (0,1) & \text{for } \xi < x < 2\xi, \\ = 0 & \text{for } x = 2\xi, \\ \in (-1,0) & \text{for } 2\xi < x < 3\xi, \\ = -1 & \text{for } x = 3\xi, \\ \in (-1,0) & \text{for } 3\xi < x < 4\xi. \end{cases}$$

Standard notation for the number 2ξ in mathematics is π.

Exercise 7.1 Show that for $0 < |x| < 1$, one has $\cos x < \frac{\sin x}{x} < 1$.

7.2 Exponential function

Show that

1. Write $e := \exp(1)$. Show that $e^n = \exp(n)$ for all $n \in \mathbb{Z}$. Conclude that $e^r = \exp(r)$ for any $r \in \mathbb{Q}$.

   Now use continuity of the two functions $x \mapsto e^x$ and $x \mapsto \exp(x)$ to prove that $e^x = \exp(x)$ for all $x \in \mathbb{R}$.

2. $\lim_{x\to\infty} \exp x = \infty$.

3. $\exp x > 0$ for $x \geq 0$. Conclude that $\exp x > 0$ for all $x \in \mathbb{R}$.

4. $\lim_{x\to-\infty} \exp x = 0$.

5. $\exp x < \exp y$ whenever $x < y$.

Thus exp is a bijection from $\mathbb{R}$ onto $(0, \infty)$. The inverse of this function, from $(0, \infty)$ to $\mathbb{R}$, is called the **logarithm** function, and is denoted by log.
Show that

1. $\log(xy) = \log x + \log y$ for all $x, y \in (0, \infty)$,

2. if $0 < x < y$ then $\log x < \log y$,

3. $\lim_{x\to 0+} \log x = -\infty$,

4. $\lim_{x\to\infty} \log x = \infty$.

5. Let $p > 0$. Then $p^x = \exp(x \log p)$ for all $x \in \mathbb{R}$.

Week 9

8 Differentiability

8.1 Derivatives

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined on some neighbourhood of a point a. If the limit $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists and is finite, then we say the function f is **differentiable** at the point a. The number $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ is then called the **derivative** of the function f at the point a and is denoted by $f'(a)$. If a function f is differentiable at every point x in a set E, one says that f is differentiable on E. If a function f is differentiable on $\mathbb{R}$, one says that f is differentiable.

If a function f is defined on an interval of the form $[a, b)$ and the limit $\lim_{x\to a+} \frac{f(x)-f(a)}{x-a}$ exists and is finite, then this number is called the **right derivative** of f at a. Similarly if f is defined on an interval of the form $(c, a]$ and the limit $\lim_{x\to a-} \frac{f(x)-f(a)}{x-a}$ exists and is finite, then it is called the **left derivative** of f at a. Thus a function is differentiable at a point if and only if the right and the left derivatives at that point exist and are equal.

Exercise 8.1 Let $n \in \mathbb{N}$ and f be the function given by $f(x) = x^n$. Show that

$$f'(a) = \begin{cases} na^{n-1} & \text{if } n > 0, \\ 0 & \text{if } n = 0 \end{cases}$$

for any $a \in \mathbb{R}$.

Proposition 8.2 *If a function f is differentiable at a point a, then it is continuous at a.*

Proof: Since the limit $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ is finite, there is a $\delta > 0$ such that $|\frac{f(x)-f(a)}{x-a}|$ is bounded (by, say, M) on $(a-\delta, a+\delta)\backslash\{a\}$. Hence for any $x \in (a-\delta, a+\delta)$, we have $|f(x) - f(a)| \le M|x-a|$. Given any $\epsilon > 0$ now, choose $\delta' = \min\{\delta, \epsilon/M\}$. Then for $x \in (a-\delta', a+\delta')$, one has $|f(x) - f(a)| \le M|x-a| < M\delta' \le \epsilon$. □

A function that is continuous at a point need not be differentiable there. Look at the function $x \mapsto |x|$ for example. It is continuous but not differentiable at 0.

Proposition 8.3 *Let f and g be two functions, both differentiable at a point a. Then $f \pm g$ and fg are also differentiable at a, and one has*

$$(f \pm g)'(a) = f'(a) \pm g'(a), \qquad (fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

If $g(a) \neq 0$, then $\frac{f}{g}$ is also differentiable at a, and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Proof: Exercise. □

Proposition 8.4 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions such that f is differentiable at a and g is differentiable at $f(a)$. Then the composition $g \circ f$ is differentiable at a and $(g \circ f)'(a) = g'(f(a))f'(a)$.*

Proof: Since g is differentiable at $f(a)$, there is an $\epsilon > 0$ such that g is defined over $(f(a) - \epsilon, f(a) + \epsilon)$. Since f is differentiable at a, in particular, it is defined in some neighbourhood of a and is also continuous at a. Therefore one can get hold of a $\delta > 0$ such that f is defined over $(a - \delta, a + \delta)$ and for any $x \in (a - \delta, a + \delta)$, one has $f(x) \in (f(a) - \epsilon, f(a) + \epsilon)$. Thus $g \circ f$ is defined over $(a - \delta, a + \delta)$.

Next, define a function h on $(f(a) - \epsilon, f(a) + \epsilon)$ as follows:

$$h(y) = \begin{cases} \frac{g(y) - g(f(a))}{y - f(a)} & \text{if } y \neq f(a), \\ g'(f(a)) & \text{if } y = f(a). \end{cases}$$

The function h is then continuous on $(f(a) - \epsilon, f(a) + \epsilon)$, and one has

$$\frac{g(f(x)) - g(f(a))}{x - a} = h(f(x)) \cdot \frac{f(x) - f(a)}{x - a}$$

for all $x \in (a - \delta, a + \delta)$. Hence

$$\begin{aligned} (g \circ f)'(a) &= \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} \\ &= h(f(a))f'(a) \\ &= g'(f(a))f'(a). \end{aligned}$$

□

Proposition 8.5 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that f has a local maximum at a point c, and that f is differentiable at c. Then $f'(c) = 0$.*

Proof: Assume if possible that $f'(c) > 0$. Then there is a $\delta > 0$ such that for $0 < |x - c| < \delta$, one has $|\frac{f(x)-f(c)}{x-c} - f'(c)| < \frac{1}{2}f'(c)$. Thus for $x \in (c, c+\delta)$, one has $\frac{f(x)-f(c)}{x-c} > f'(c) - \frac{1}{2}f'(c) = \frac{1}{2}f'(c) > 0$. Therefore $f(x) > f(c)$ for all $x \in (c, c+\delta)$. This contradicts the fact that f has a local maximum at c.

A similar argument shows that one can not have $f'(c) < 0$. Hence $f'(c) = 0$. □

Exercise 8.6 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that f has a local minimum at a point c, and that f is differentiable at c. Then show that $f'(c) = 0$.

8.2 Mean value theorem and applications

Theorem 8.7 (Rolle's theorem) *Let $f : [a, b] \to \mathbb{R}$ be continuous. Assume that $f'(x)$ exists for $x \in (a, b)$ and $f(a) = f(b)$. Then there is a $c \in (a, b)$ such that $f'(c) = 0$.*

Proof: Let $m = \inf\{f(x) : x \in [a, b]\}$ and $M = \sup\{f(x) : x \in [a, b]\}$. If $m = M$, then f has to be a constant function. Therefore for any $c \in (a, b)$, $f'(c) = 0$. So let us now look at the case $m \neq M$. Since f is continuous, there are points $c, d \in [a, b]$ such that $f(c) = m$ and $f(d) = M$. Since $m \neq M$, at least one of m and M has to be different from $f(a) = f(b)$. Assume $m \neq f(a)$. Then $c \neq a$ and $c \neq b$. So $c \in (a, b)$. Since $f'(c)$ exists, we must have $f'(c) = 0$. Similarly if $M \neq f(a)$, then $d \in (a, b)$ and $f'(d) = 0$. □

Theorem 8.8 (Mean value theorem) *Let $f : [a, b] \to \mathbb{R}$ be continuous. Assume that $f'(x)$ exists for $x \in (a, b)$. Then there is a $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.*

Proof: Apply the previous result to the function $g(x) = (x-a)(f(b)-f(a))-(b-a)(f(x)-f(a))$. □

Theorem 8.9 (Generalized mean value theorem) *Let $f, g : [a, b] \to \mathbb{R}$ be continuous. Assume that both are differentiable on (a, b). Then there is a $c \in (a, b)$ such that $g'(c)(f(b) - f(a)) = f'(c)(b - a)$.*

Proof: Apply Rolle's theorem to the function $g(x) = (g(x) - g(a))(f(b) - f(a)) - (g(b) - g(a))(f(x) - f(a))$. □

Maxima/minima of a function.

Theorem 8.10 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Assume that $f'(x) \geq 0$ for $x \in (a, b)$ and $f'(x) \leq 0$ for $x \in (b, c)$. Assume also that f is continuous at b. Then $f(x) \leq f(b)$ for all $x \in (a, c)$.*

Proof: Take any $x \in (a, b)$. Use the mean value theorem over the interval $[x, b]$ to get a $s \in (x, b) \subseteq (a, b)$ such that $f(b) - f(x) = f'(s)(b - x)$. Since $x < b$ and $0 \leq f'(s)$, we have $f(x) \leq f(b)$. Next take any $y \in (b, c)$. This time use mean value theorem over the interval $[b, y]$ to get a point $t \in (b, y) \subseteq (b, c)$ such that $f(y) - f(b) = f'(t)(y - b)$. Since $b < y$ and $f'(t) \leq 0$, one has $f(y) \leq f(b)$. □

L'Hopital's rule. The next two theorems are very useful for evaluating limits in many situations and go by the name L'Hopital's rule.

Theorem 8.11 *Let f and g be two functions from (a, b) to $\mathbb{R}$ such that*

1. *they are differentiable on (a, b),*
2. *$g'(x) \neq 0$ for $x \in (a, b)$,*
3. *$\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L$.*
4. *$\lim_{x \to a+} f(x) = 0 = \lim_{x \to a+} g(x)$.*

Then $\lim_{x \to a+} \frac{f(x)}{g(x)} = L$.

Proof: First let us look at the case when $L \in \mathbb{R}$. Since $g'(x) \neq 0$ for $x \in (a, b)$, it follows from the mean value theorem that $g(x) \neq g(y)$ whenever $x \neq y$. Since $\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L$, given $\epsilon > 0$, there there is a $\delta > 0$ such that $|\frac{f'(x)}{g'(x)} - L| < \frac{\epsilon}{2}$ for $a < x < a + \delta$. Take any $x \in (a, a + \delta)$. Then by the generalized mean value theorem, for any $y \in (a, a + \delta)$, there is a z between x and y and hence in $(a, a + \delta)$ such that $f'(z)(g(x) - g(y)) = g'(z)(f(x) - f(y))$. Since $g'(x) \neq 0$ for $x \in (a, b)$, it follows from the mean value theorem that $g(x) \neq g(y)$ whenever $x \neq y$. Thus we have $\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(z)}{g'(z)}$, where $z \in (a, a + \delta)$. Hence $|\frac{f(x) - f(y)}{g(x) - g(y)} - L| < \frac{\epsilon}{2}$. Next observe that $g(x) \neq 0$ (see exercise 8.12 below). Therefore taking limit as $y \to a+$ and using condition 4, we get $|\frac{f(x)}{g(x)} - L| \leq \frac{\epsilon}{2} < \epsilon$. Thus $\lim_{x \to a+} \frac{f(x)}{g(x)} = L$.

The cases $L = \pm\infty$ are similar and are left as exercises. □

Exercise 8.12 Show that under the assumtions 1, 2 and 4 above, $g(x) \neq 0$ for all $x \in (a, b)$.

Theorem 8.13 *Let f and g be two functions from (a, b) to $\mathbb{R}$ such that*

1. *they are differentiable on (a, b),*
2. *$g'(x) \neq 0$ for $x \in (a, b)$,*
3. *$\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L$.*
4. *$\lim_{x \to a+} g(x) = \infty$.*

Then $\lim_{x\to a+} \frac{f(x)}{g(x)} = L$.

Proof: Let us first consider the case when $L \in \mathbb{R}$. Since $g'(x) \neq 0$ for $x \in (a, b)$, it follows that $g(x) \neq g(y)$ for $x \neq y$ in (a, b). Also, by condition 4, there is a point $c \in (a, b)$ such that $g(x) > 0$ for all $x \in (a, c)$. Now, first choose a $\delta > 0$ such that $\left|\frac{f'(x)}{g'(x)} - L\right| < \frac{\epsilon}{2}$ and $g(x) > 0$ for $x \in (a, a + \delta)$. Next, take any $y \in (a, a + \frac{\delta}{2})$. Then by generalized mean value theorem, we have

$$\left|\frac{f(y) - f(a + \frac{\delta}{2})}{g(y) - g(a + \frac{\delta}{2})} - L\right| < \frac{\epsilon}{2},$$

i. e.

$$L - \frac{\epsilon}{2} < \frac{f(y) - f(a + \frac{\delta}{2})}{g(y) - g(a + \frac{\delta}{2})}$$

Now choose a $\xi > 0$ such that $\xi < \frac{\delta}{2}$ and $g(x) > g(a + \frac{\delta}{2})$ for $x \in (a, a + \xi)$. Then we have, for $y \in (a, a + \xi)$,

$$L - \frac{\epsilon}{2} - (L - \frac{\epsilon}{2})\frac{g(a + \frac{\delta}{2})}{g(y)} < \frac{f(y) - f(a + \frac{\delta}{2})}{g(y)} < L + \frac{\epsilon}{2} - (L + \frac{\epsilon}{2})\frac{g(a + \frac{\delta}{2})}{g(y)} < L + \frac{\epsilon}{2}.$$

Next, choose a $\xi' > 0$ such that $\xi' \leq \xi$ and the following two inequalities hold:

$$\left|L - \frac{\epsilon}{2}\right| \frac{g(a + \frac{\delta}{2})}{g(y)} < \frac{\epsilon}{4}, \qquad \left|\frac{f(a + \frac{\delta}{2})}{g(y)}\right| < \frac{\epsilon}{4}.$$

Then, for $y \in (a, a + \xi')$, we will have $L - \epsilon < \frac{f(y)}{g(y)} < L + \epsilon$.

Proofs for the cases $L = \pm\infty$ are left as exercises. □

Taylor's theorem.

Theorem 8.14 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function, n be a positive integer and let α and β be two real numbers. Assume that $f^{(n)}(x)$ exists for all $x \in (\alpha, \beta)$ and $f^{(n-1)}$ is continuous over $[\alpha, \beta]$. Then for any two points $a, b \in [\alpha, \beta]$, there is a point c between a and b such that*

$$f(b) = f(a) + f'(a)(b - a) + f''(a)\frac{(b - a)^2}{2!} + \cdots + f^{(n-1)}(a)\frac{(b - a)^{n-1}}{(n - 1)!} + f^{(n)}(c)\frac{(b - a)^n}{n!}.$$

Proof: Define two functions g and h as follows:

$$g(x) = f(x) + f'(x)(b - x) + \cdots + f^{(n-1)}(x)\frac{(b - x)^{n-1}}{(n - 1)!}, \qquad h(x) = (b - x)^n$$

and apply the generalized mean value theorem. □

The following exercise outlines another proof of the above result.

Exercise 8.15 Write $M = f(b) - f(a) - f'(a)(b-a) - f''(a)\frac{(b-a)^2}{2!} - \cdots - f^{(n-1)}(a)\frac{(b-a)^{n-1}}{(n-1)!}$. Show that $M = f^{(n)}(c)\frac{(b-a)^n}{n!}$ for some c by completing the following steps. Let $P(t) = f(a) + f'(a)(t-a) + \cdots + f^{(n-1)}(a)\frac{(t-a)^{n-1}}{(n-1)!}$ and $g(t) = f(t) - P(t) - \frac{M}{(b-a)^n}(t-a)^n$. Show that

1. $g(a) = 0 = g(b)$ and $g^{(k)}(a) = 0$ for $1 \le k \le n-1$.
2. there is some c between a and b such that $g^{(n)}(c) = 0$.
3. $g^{(n)}(t) = f^{(n)}(t) - \frac{n!M}{(b-a)^n}$.

Week 10

8.3 Problems

1. Check whether the following functions are differentiable:

$$|x \sin x|, \qquad |x \cos x|, \qquad |x|^3.$$

2. Assume $a_1, \ldots, a_n$ be reals such that $a_1 + \cdots + a_n = 0$. Show that the equation $a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} = 0$ has a real root in $(0, 1)$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and $\lim_{x\to\infty} f'(x) = 0$. Define $a_n = f(n+1) - f(n)$. Show that the sequence $\{a_n\}$ converges to 0.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f'(x)$ exists for all $x \neq 0$ and $\lim_{x\to 0} f'(x) = 1$. Show that f is differentiable at 0 and $f'(0) = 1$.

5. Let $f : (0, \infty) \to \mathbb{R}$ be differentiable, and $|f'(x)| < 1$ for $x \in (0, \infty)$. Define $a_n = f(\frac{1}{n})$. Show that the sequence $\{a_n\}$ converges.

6. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that it is differentiable on $(0, \infty)$, $f(0) = 0$ and f' is increasing on $(0, \infty)$. Show that the function $x \mapsto \frac{f(x)}{x}$ is increasing over $(0, \infty)$.

7. Define

$$f(x) = \begin{cases} \exp(-\frac{1}{x^2}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

   Show that

   (a) f is continuous,

   (b) $f'(x)$ exists for all x, and $f'(0) = 0$,

   (c) for any $n \in \mathbb{Z}_+$,

$$f^{(n)}(x) = \begin{cases} P_n(\frac{1}{x}) \exp(-\frac{1}{x^2}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

   where P_n is some polynomial.

8. Let
$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
Show that f is differentiable everywhere but f' is not continuous at 0.

9. Prove that

   (a) $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

   (b) if $0 < x < y$ and $n \in \mathbb{Z}_+$, then $nx^{n-1}(y-x) < y^n - x^n < ny^{n-1}(y-x)$.

9 Integration

9.1 Riemann integrals

Let $[a, b]$ be a closed bounded interval. Recall that a finite subset P of $[a, b]$ containing both the endpoints a and b is called a **partition** of $[a, b]$. Suppose $P = \{x_0, x_1, \ldots, x_n\}$ and $a = x_0 < x_1 < \cdots < x_n = b$ (we would often express this by saying that 'let $P : x_0 < x_1 < \cdots < x_n$ be a partition of $[a, b]$'). Then the intervals $[x_i, x_{i+1}]$ are called subintervals coming from the partition P. Let P and Q be two partitions of the same interval $[a, b]$. Then P is said to be **finer** than Q if $Q \subseteq P$. If P is finer than Q, then any subinterval from Q can be written as a finite union of subintervals from P. Let P be a partition of $[a, b]$. The number $\max\{x_{i+1} - x_i : 0 \leq i \leq n-1\}$ is called the **length** of the partition P and is denoted by $|P|$.

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $P : x_0 < x_1 < \cdots < x_n$ be a partition of $[a, b]$. Define two numbers $U(f, [a, b], P)$ and $L(f, [a, b], P)$ as follows:

$$U(f, [a, b], P) = \sum_{i=0}^{n-1} \Big(\sup\{f(x) : x \in [x_i, x_{i+1}]\}\Big)(x_{i+1} - x_i),$$

$$L(f, [a, b], P) = \sum_{i=0}^{n-1} \Big(\inf\{f(x) : x \in [x_i, x_{i+1}]\}\Big)(x_{i+1} - x_i).$$

When there is no ambiguity about the interval $[a, b]$ in question, we will just write $U(f, P)$ and $L(f, P)$ respectively for the above quantities. It is clear from the definitions that $L(f, P) \leq U(f, P)$ for any partition P of $[a, b]$.

Lemma 9.1 *Let P_1 and P_2 be two partitions of $[a, b]$, and assume P_1 is finer than P_2. Then*

$$U(f, P_1) \leq U(f, P_2), \qquad L(f, P_1) \geq L(f, P_2).$$

Lemma 9.2 *Let P_1 and P_2 be any two partitions of $[a, b]$. Then*

$$L(f, P_1) \leq U(f, P_2).$$

Proof: Let $P = P_1 \cup P_2$. Then P is finer than both P_1 and P_2. Hence $L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2)$. □

Next, let us define the **upper integral** of f over the interval $[a, b]$ to be $\inf\{U(f, [a, b], P) : P \text{ is a partition of } [a, b]\}$. We will denote this number by $U(f, [a, b])$ or, if there is no confusion about the interval $[a, b]$, by just $U(f)$. Similarly define the **lower integral** of f over $[a, b]$ by $\sup\{L(f, [a, b], P) : P \text{ is a partition of } [a, b]\}$ and denote it by $L(f, a, b])$ or by simply $L(f)$.

Exercise 9.3 Show that $L(f) \leq U(f)$.

Exercise 9.4 Define a function $f : [0, 1] \to \mathbb{R}$ as follows

$$f(x) = \begin{cases} 1 & \text{if x is rational,} \\ 0 & \text{if x is irrational.} \end{cases}$$

Compute $U(f, [0, 1])$ and $L(f, [0, 1])$.

Exercise 9.5 Let $f : [0, 1] \to \mathbb{R}$ be the function given by $f(x) = x$. Compute $U(f, [0, 1])$ and $L(f, [0, 1])$.

We say that a function f is **Riemann integrable** over the interval $[a, b]$ if $U(f, [a, b]) = L(f, a, b])$. The common value is denoted by $\int_a^b f(x)dx$ and is referred to as the (**Riemann**) **integral** of f over $[a, b]$.

Exercise 9.6 Let $f : [a, b] \to \mathbb{R}$ be a bounded function with $f(x) \geq 0$ for all x. Show that both $U(f)$ and $L(f)$ are nonnegative.

Use this to show that if f and g are two integrable functions over $[a, b]$ such that $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

Proposition 9.7 *Let $f : [a, b] \to \mathbb{R}$ be bounded. The integral $\int_a^b f(x)dx$ exists if and only if for every $\epsilon > 0$, there is a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$.*

Proof: First assume that f is integrable, i. e. $U(f) = L(f)$. Take an $\epsilon > 0$. Then there exist partitions P_1 and P_2 such that $U(f, P_1) < U(f) + \frac{\epsilon}{2}$ and $L(f, P_2) > L(f) - \frac{\epsilon}{2}$. Let $P = P_1 \cup P_2$. Then P is finer than both P_1 and P_2. Hence $U(f, P) \leq U(f, P_1)$ and $L(f, P) \geq L(f, P_2)$. Combining this with our earlier observation, we get $U(f, P) - L(f, P) \leq U(f, P_1) - L(f, P_2) < U(f) - L(f) + \epsilon = \epsilon$.

Now to prove the converse, assume that given any $\epsilon > 0$, one can choose a partition P with $U(f, P) - L(f, P) < \epsilon$. Since for any partition P, $U(f) \leq U(f, P)$ and $L(f) \geq L(f, P)$, we have $U(f) - L(f) \leq U(f, P) - L(f, P)$. Hence for any $\epsilon > 0$, we have $U(f) - L(f) \leq \epsilon$. So we must have $U(f) - L(f) \leq 0$, which implies that $U(f) = L(f)$, i. e. f is integrable. □

Exercise 9.8 Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $P : x_0 = a < x_1 < \cdots < x_n = b$ be a partition of $[a, b]$. Show that

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=0}^{n-1} \sup\{f(x) - f(y) : x, y \in [x_i, x_{i+1}]\}(x_{i+1} - x_i) \\ &= \sum_{i=0}^{n-1} \sup\{|f(x) - f(y)| : x, y \in [x_i, x_{i+1}]\}(x_{i+1} - x_i). \end{aligned}$$

Proposition 9.9 *Let f and g be two functions on $[a,b]$. Assume that both are integrable. Let $c \in \mathbb{R}$. Then the functions cf and $f \pm g$ are also integrable over $[a,b]$.*

Proof: The case $c = 0$ is trivial because $U(cf) = L(cf) = 0$ in that case. So assume $c \neq 0$. Let P be a partition of $[a,b]$. Then by the exercise above,

$$U(cf,P) - L(cf,P) = \sum_{i=0}^{n-1} \sup\{|cf(x) - cf(y)| : x,y \in [x_i, x_{i+1}]\}(x_{i+1} - x_i) = |c|(U(f,P) - L(f,P)).$$

So given any $\epsilon > 0$, choose a partition P such that $U(f,P) - L(f,P) < \frac{\epsilon}{c}$. Then one has $U(cf,P) - L(cf,P) < \epsilon$.

Next, observe that for any partition P, $U(f+g,P) \leq U(f,P) + U(g,P)$ and $L(f+g,P) \geq L(f,P) + L(g,P)$. Hence $U(f+g,P) - L(f+g,P) \leq U(f,P) - L(f,P) + U(g,P) - L(g,P)$. Now given $\epsilon > 0$, choose partitions P_1 and P_2 such that $U(f,P_1) - L(f,P_1) < \frac{\epsilon}{2}$ and $U(g,P_2) - L(g,P_2) < \frac{\epsilon}{2}$. Let $P = P_1 \cup P_2$. Then

$$\begin{aligned} U(f+g,P) - L(f+g,P) &\leq U(f,P) - L(f,P) + U(g,P) - L(g,P) \\ &\leq U(f,P_1) - L(f,P_1) + U(g,P_2) - L(g,P_2) \\ &< \epsilon. \end{aligned}$$

□

Proposition 9.10 *Let $f : [a,b] \to \mathbb{R}$ be integrable. Then the functions $|f|$ and f^2 are integrable.*

Proof: Let $P : x_0 = a < x_1 < \cdots < x_n = b$ be a partition of $[a,b]$. Then

$$U(|f|,P) - L(|f|,P) = \sum_{i=0}^{n-1} \sup\{|f(x)| - |f(y)| : x,y \in [x_i, x_{i+1}]\}(x_{i+1} - x_i).$$

Since $|s| - |t| \leq |s - t|$ for any $s,t \in \mathbb{R}$, one gets

$$\begin{aligned} U(|f|,P) - L(|f|,P) &\leq \sum_{i=0}^{n-1} \sup\{|f(x) - f(y)| : x,y \in [x_i, x_{i+1}]\}(x_{i+1} - x_i) \\ &= \sum_{i=0}^{n-1} \sup\{f(x) - f(y) : x,y \in [x_i, x_{i+1}]\}(x_{i+1} - x_i) \\ &= U(f,P) - L(f,P). \end{aligned}$$

Hence integrability of f implies that of $|f|$.

Next,

$$U(f^2,P) - L(f^2,P) = \sum_{i=0}^{n-1} \sup\{(f(x))^2 - (f(y))^2 : x,y \in [x_i, x_{i+1}]\}(x_{i+1} - x_i).$$

Now

$$\begin{aligned}(f(x))^2 - (f(y))^2 &= (|f(x)| + |f(y)|)(|f(x)| - |f(y)|) \\ &\leq (|f(x)| + |f(y)|)|f(x) - f(y)|.\end{aligned}$$

Let M be a positive real such that $|f(x)| \leq M$ for $x \in [a,b]$. Then we have

$$(f(x))^2 - (f(y))^2 \leq (|f(x)| + |f(y)|)|f(x) - f(y)| \leq 2M|f(x) - f(y)|.$$

Hence

$$\begin{aligned}U(f^2,P) - L(f^2,P) &\leq 2M\sum_{i=0}^{n-1} \sup\{|f(x) - f(y)| : x, y \in [x_i, x_{i+1}]\}(x_{i+1} - x_i) \\ &= 2M(U(f,P) - L(f,P)).\end{aligned}$$

Thus, given $\epsilon > 0$, choose a partition P such that $U(f,P) - L(f,P) < \frac{\epsilon}{2M}$. Then it follows that $U(f^2,P) - L(f^2,P) < \epsilon$. □

Exercise 9.11 Show that if f and g are two integrable functions on $[a,b]$, then the product fg is also integrable.

Theorem 9.12 *Let $f : [a,b] \to \mathbb{R}$ be monotone. Then f is integrable over $[a,b]$.*

Proof: Let us first look at the case when f is increasing. Then for any interval of the form $[s,t] \subseteq [a,b]$, we have

$$\sup\{f(x) : x \in [s,t]\} = f(t), \qquad \inf\{f(x) : x \in [s,t]\} = f(s).$$

Hence for any partition $P : x_0 = a < x_1 < \cdots < x_n = b$ of $[a,b]$,

$$U(f,P) - L(f,P) = \sum_{i=0}^{n-1}(f(x_{i+1}) - f(x_i))(x_{i+1} - x_i).$$

Therefore if we take x_i to be $a + i\frac{b-a}{n}$, so that the differences $x_{i+1} - x_i$ are all equal to $\frac{1}{n}$, then

$$U(f,P) - L(f,P) = \frac{1}{n}\sum_{i=0}^{n-1}(f(x_{i+1}) - f(x_i)) = \frac{f(b) - f(a)}{n}.$$

Now given $\epsilon > 0$, choose n large enough so that $\frac{f(b)-f(a)}{n} < \epsilon$. Then $U(f,P) - L(f,P) < \epsilon$.

If f is decreasing and integrable, then $-f$ is increasing and integrable. So the result in this case follows from the earlier case. □

Theorem 9.13 *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then f is integrable over $[a,b]$.*

Proof: Take any $\epsilon > 0$. Since f is continuous over $[a,b]$, there is a partition $P : x_0 = a < x_1 < \cdots < x_n = b$ such that for every i,

$$\sup\{|f(x) - f(y)| : x, y \in [x_i, x_{i+1}]\} < \frac{\epsilon}{b-a}.$$

But then

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=0}^{n-1}\Big(\sup\{|f(x) - f(y)| : x, y \in [x_i, x_{i+1}]\}\Big)(x_{i+1} - x_i) \\ &< \frac{\epsilon}{b-a}\sum_{i=0}^{n-1}(x_{i+1} - x_i) \\ &= \epsilon. \end{aligned}$$

□

Exercise 9.14 Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Let $c \in (a,b)$, and suppose P_1 and P_2 are two partitions of $[a,c]$ and $[c,b]$ respectively. Let $P = P_1 \cup P_2$. Show that P is a partition of $[a,b]$, and

$$U(f,P) = U(f,P_1) + U(f,P_2), \qquad L(f,P) = L(f,P_1) + L(f,P_2).$$

Theorem 9.15 *Let $f : [a,b] \to \mathbb{R}$ be bounded. Assume that f has only finitely many discontinuities on $[a,b]$. Then f is integrable.*

Proof: Take an $\epsilon > 0$. We want to produce a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$. Let $d_1, d_2, \ldots, d_k$ be the discontinuity points of f, indexed in such a way that $d_1 < d_2 < \cdots < d_k$. Let $M = \sup\{?? : x \in [a,b]\}$. Let $\xi > 0$ be a positive number such that the intervals $[a, d_1 + \xi], [d_2 - \xi, d_2 + \xi], \ldots, [d_k - \xi, b]$ are all disjoint. Then for any i, $1 \le i \le k$, we have $\sup\{f(x) - f(y) : x, y \in [d_i - \xi, d_i + \xi] \cap [a,b]\} \le 2M$. Now, first take $\xi = \frac{\epsilon}{8Mk}$. Observe that f is integrable over the intervals

$$[a, d_1 - \xi], [d_1 + \xi, d_2 - \xi], [d_2 + \xi, d_3 - \xi], \ldots, [d_{k-1} + \xi, d_k - \xi], [d_k + \xi, b]$$

(omit the first interval if $d_1 - \xi \le a$ and omit the last if $d_k + \xi \ge b$). So These intervals will admit partitions $P_0, P_1, \ldots, P_k$ respectively such that for each i, one has $U(f,P_i) - L(f,P_i) < \frac{\epsilon}{2k+2}$. Finally, let $P = P_0 \cup \ldots \cup P_k \cup \{a,b\}$. Then

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=0}^{k}\Big(U(f,P_i) - L(f,P_i)\Big) \\ &\quad + \sum_{i=1}^{k}\sup\{f(x) - f(y) : x, y \in [d_i - \xi, d_i + \xi] \cap [a,b]\}(d_i + \xi - d_i - \xi) \\ &\le (k+1)\frac{\epsilon}{2k+2} + 2M\sum_{i=1}^{k}2\xi \\ &< \frac{\epsilon}{2} + 4Mk\frac{\epsilon}{8Mk} \\ &= \epsilon. \end{aligned}$$

□

Week 11

Exercise 9.16 Let f be integrable over $[a, b]$. Let $g : [a, b] \to \mathbb{R}$ be another function such that $f(x) = g(x)$ for all but one point in $[a, b]$. Show that g is integrable over $[a, b]$ and $\int_a^b g = \int_a^b f$.

Theorem 9.17 (Lebesgue) *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable over $[a, b]$ if and only if the set of discontinuity points of f in $[a, b]$ is of measure zero.*

Proposition 9.18 *Let f be integrable over $[a, b]$. Then*

$$\Big|\int_a^b f(x)dx\Big| \le \int_a^b |f(x)|dx.$$

Proof: We have already seen that if f is integrable over $[a, b]$, then so is $|f|$. Next, observe that for any partition P, we have $\int_a^b f \le U(f, P) \le U(|f|, P)$. Hence taking infimum over P, one gets the required inequality. □

Proposition 9.19 *Let f be integrable over $[a, b]$ and $[c, d]$ be a subinterval of $[a, b]$. Then f is integrable over $[c, d]$.*

Proof: Take $\epsilon > 0$. Choose a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. Let $P' := P \cup \{c, d\}$ and let P_1, P_2 and P_3 be the restrictions of P' to $[a, c]$, $[c, d]$ and $[d, b]$ respectively. Then

$$U(f, P') = U(f, P_1) + U(f, P_2) + U(f, P_3), \qquad L(f, P') = L(f, P_1) + L(f, P_2) + L(f, P_3).$$

Therefore

$$U(f, P_2) - L(f, P_2) \le U(f, P') - L(f, P'),$$

and since P' is finer than P, $U(f, P') - L(f, P') \le U(f, P) - L(f, P)$. Hence

$$U(f, P') - L(f, P') < \epsilon.$$

□

Proposition 9.20 *Let f be a function on $[a,b]$ and let $c \in (a,b)$. Then f is integrable over $[a,b]$ if and only if f is integrable over both $[a,c]$ and $[c,b]$. Moreover, in such a case, one has*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof: If f is integrable over $[a,b]$, then by the previous result it is integrable over both $[a,c]$ and $[c,b]$. To prove the converse, assume that f is integrable over $[a,c]$ and $[c,b]$. Take an $\epsilon > 0$. Choose partitions P_1 of $[a,c]$ and P_2 of $[c,b]$ such that

$$U(f,P_1) - L(f,P_1) < \frac{\epsilon}{2}, \qquad U(f,P_2) - L(f,P_2) < \frac{\epsilon}{2}.$$

Now take P to be the partition $P_1 \cup P_2$ of $[a,b]$. Then

$$U(f,P) - L(f,P) = U(f,P_1) - L(f,P_1) + U(f,P_2) - L(f,P_2) < \epsilon,$$

so that f is integrable over $[a,b]$.

Now assume that f is integrable over $[a,b]$ (and hence in $[a,c]$ and on $[c,b]$). Take any partition P_1 of $[a,c]$ and P_2 of $[c,b]$. Then $P := P_1 \cup P_2$ is a partition of $[a,b]$. Now $U(f,P_1) + U(f,P_2) = U(f,P) \geq \int_a^b f$. Hence taking infimum ver P_1 and P_2, we get $\int_a^c f + \int_c^b f \geq \int_a^b f$. Again, $L(f,P_1) + L(f,P_2) = L(f,P) \leq \int_a^b f$. This time taking supremum over P_1 and P_2, one gets $\int_a^c f + \int_c^b f \leq \int_a^b f$. □

Proposition 9.21 *Let $f : [a,b] \to \mathbb{R}$ be integrable. Then given any $\epsilon > 0$, there is a $\delta > 0$ such that if P is any partition with $|P| < \delta$, and $c_i \in [x_i, x_{i+1}]$, then*

$$|\sum_{i=0}^{n-1} f(c_i)(x_{i+1} - x_i) - \int_a^b f(x)dx| < \epsilon.$$

Proof: Write $I = \int_a^b f(x)dx$. Take an $\epsilon > 0$.Then there is a partition $P_1 : a = y_0 < y_1 \cdots < y_m = b$ such that $U(f,P_1) < I + \frac{\epsilon}{2}$. Let n be the number of elements in P_1, and M be $\sup\{|f(x)| : x \in [a,b]\}$. Now let $\delta_1 = \frac{\epsilon}{2Mn}$ and let $P : a = x_0 < x_1 < \cdots < x_n = b$ be a partition with $|P| < \delta_1$. We will show that $U(f,P) < I + \epsilon$. Break the sum in the expression for $U(f,P)$ into two parts as follows:

$$\begin{aligned} U(f,P) &= \sum_{i=0}^{n-1} \sup_{x \in [x_i, x_{i+1}]} f(x)\Big(x_{i+1} - x_i\Big) \\ &= \sum_{i : P_1 \cap (x_i, x_{i+1}) = \phi} \sup_{x \in [x_i, x_{i+1}]} f(x)\Big(x_{i+1} - x_i\Big) + \sum_{i : P_1 \cap (x_i, x_{i+1}) \neq \phi} \sup_{x \in [x_i, x_{i+1}]} f(x)\Big(x_{i+1} - x_i\Big) \\ &= \sum_{j=0}^{m-1} \sum_{i : [x_i, x_{i+1}] \subseteq [y_j, y_{j+1}]} \sup_{x \in [x_i, x_{i+1}]} f(x)\Big(x_{i+1} - x_i\Big) \\ &\quad + \sum_{i : P_1 \cap (x_i, x_{i+1}) \neq \phi} \sup_{x \in [x_i, x_{i+1}]} f(x)\Big(x_{i+1} - x_i\Big). \end{aligned}$$

Denote the first term by A and the second term by B. Then

$$A \leq \sum_{j=0}^{m-1} \sup_{x \in [y_j, y_{j+1}]} f(x)\Big(y_{j+1} - y_j\Big) = U(f, P_1) < I + \frac{\epsilon}{2},$$

and

$$B \leq \sum_{i:P_1 \cap (x_i, x_{i+1})] \neq \phi} M(x_{i+1} - x_i) \leq M \cdot |P| \cdot n < \frac{\epsilon}{2}.$$

Thus $U(f, P) < I + \epsilon$.

Applying this now to the function $-f$, one gets a $\delta_2 > 0$ such that whenever P is a partition with $|P| < \delta_2$, then $U(-f, P) < -I + \epsilon$. This, along with the observation that $U(-f, P) = -L(f, P)$, imply that $L(f, P) < I - \epsilon$. Now take $\delta = \min\{\delta_1, \delta_2\}$. Then for partitions with $|P| < \delta$, we have

$$I - \epsilon < L(f, P) \leq U(f, P) < I + \epsilon.$$

Since $L(f, P) \leq \sum_i f(c_i)(x_{i+1} - x_i) \leq U(f, P)$, the result follows. □

Exercise 9.22 Let $f : [a, b] \to \mathbb{R}$ be integrable. Show that

$$\int_a^b f = \lim_{n \to \infty} \frac{b-a}{n} \sum_{k=0}^{n-1} f(a + k\frac{b-a}{n}).$$

Theorem 9.23 (First mean value theorem) *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there is a $c \in [a, b]$ such that $\int_a^b f(x)dx = f(c)(b - a)$.*

Proof: Let $m = \inf\{f(x) : x \in [a, b]\}$ and $M = \sup\{f(x) : x \in [a, b]\}$. Since f is continuous on $[a, b]$, there exist points s and t in $[a, b]$ such that $f(s) = m$ and $f(t) = M$. Also observe that since $m \leq f(x) \leq M$ for all $x \in [a, b]$, we have $m(b - a) \leq \int_a^b f \leq M(b - a)$, i. e.

$$m \leq \frac{1}{b-a} \int_a^b f \leq M.$$

Therefore by intermediate value theorem, there is some point c between s and t (and hence in the interval $[a, b]$) such that $f(c) = \frac{1}{b-a} \int_a^b f$. □

Theorem 9.24 (First fundamental theorem) *Let $f : [a, b] \to \mathbb{R}$ be integrable. Define $F(x) = \int_a^x f(t)dt$ for $x \in [a, b]$. If f is continuous at a point $c \in [a, b]$, then F is differentiable at c and $F'(c) = f(c)$.*

Proof: Take any $\epsilon > 0$. Using continuity of f at c, choose a $\delta > 0$ such that $|t - c| < \delta$ implies $|f(t) - f(c)| < \frac{\epsilon}{2}$. Then for $0 < |h| < \delta$, we have

$$\begin{aligned} \Big|\frac{F(c+h) - F(c)}{h} - f(c)\Big| &= \frac{1}{|h|}\Big|\int_c^{c+h} (f(t) - f(c))dt\Big| \\ &\leq \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

□

Corollary 9.25 (Change of variable) *Let f and g be two functions. Assume that g is differentiable on some interval containing $[a, b]$ and g' is continuous on $[a, b]$. Assume also that f is continuous on $g([a, b])$. Then*

$$\int_a^b f(g(t))g'(t)dt = \int_{g(a)}^{g(b)} f(t)dt.$$

Proof: Define three functions F, G and H as follows:

$$F(x) = \int_{g(a)}^{x} f(t)dt, \qquad G(x) = \int_a^x f(g(t))g'(t)dt, \qquad H(x) = F \circ g(x).$$

Then $H'(x) - G'(x) = F'(g(x))g'(x) - f(g(x))g'(x) = f(g(x))g'(x) - f(g(x))g'(x) = 0$ for all $x \in [a, b]$. Therefore $H(x) - G(x)$ is constant over $[a, b]$. Since $H(a) = F(g(a)) = 0$ and $G(a) = 0$, we have $G(x) = H(x)$ for all $x \in [a, b]$. Hence $G(b) = H(b)$, which is what we needed to prove. □

Theorem 9.26 (Second fundamental theorem) *Let f be differentiable on some interval containing $[a, b]$. Assume f' is integrable over $[a, b]$. Then $\int_a^b f'(x)dx = f(b) - f(a)$.*

Proof: Take any partition $P : a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$. For each i, $0 \le i \le n - 1$, by the mean value theorem for derivatives, there is a $c_i \in [x_i, x_{i+1}]$ such that $f(x_{i+1}) - f(x_i) = f'(c_i)(x_{i+1} - x_i)$. Hence

$$\sum_{i=0}^{n-1} f'(c_i)(x_{i+1} - x_i) = \sum_{i=0}^{n-1} f(x_{i+1}) - f(x_i) = f(b) - f(a).$$

But we also have $L(f', P) \le \sum_{i=0}^{n-1} f'(c_i)(x_{i+1} - x_i) \le U(f', P)$. Thus for any partition P of $[a, b]$, we have $L(f', P) \le f(b) - f(a) \le U(f', P)$. Since f' is integrable over $[a, b]$, we must have $\int_a^b f' = f(b) - f(a)$. □

Corollary 9.27 (integration by parts) *Let f and g be two functions, both differentiable on some interval containing $[a, b]$ and assume that $f'g$ and fg' are both integrable over $[a, b]$. Then*

$$\int_a^b f(x)g'(x)dx + \int_a^b f'(x)g(x)dx = f(b)g(b) - f(a)g(a).$$

Proof: Apply the second fundamental theorem to the function $\phi(x) := f(x)g(x)$. □

Theorem 9.28 (Weighted mean value theorem) *Let $f, g : [a, b] \to \mathbb{R}$ be continuous. Assume that g does not change sign over $[a, b]$. Then there is a $c \in [a, b]$ such that*

$$\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx.$$

Proof: Since g does not change sign, either $g(x) \geq 0$ for all $x \in [a,b]$ or $g(x) \leq 0$ for all $x \in [a,b]$. Let us first assume that $g(x) \geq 0$ for all $x \in [a,b]$. Let $m = \inf\{f(x) : x \in [a,b]\}$ and $M = \sup\{f(x) : x \in [a,b]\}$. Since f is continuous, there are points s and t in $[a,b]$ such that $f(s) = m$ and $f(t) = M$. Now for all $x \in [a,b]$, $m \leq f(x) \leq M$. Hence $mg(x) \leq f(x)g(x) \leq Mg(x)$ for $x \in [a,b]$. Integrating over $[a,b]$, we get

$$m\int_a^b g \leq \int_a^b fg \leq M\int_a^b g.$$

Now if $\int_a^b g = 0$, then $\int_a^b fg$ is also 0 and hence for any choice of $c \in [a,b]$, the required equality holds. So assume $\int_a^b g > 0$. Then

$$m \leq \frac{\int_a^b fg}{\int_a^b g} \leq M.$$

Since f is continuous, there is a point c between s and t (hence in $[a,b]$) such that $f(c) = \frac{\int_a^b fg}{\int_a^b g}$.

Proof for the other case when $g(x) \leq 0$ for all $x \in [a,b]$ is similar and is left as an exercise.
□

Theorem 9.29 (second mean value theorem) *Let g be continuous and f be diferentiable over some interval containing $[a,b]$. Assume that f' does not change sign and is integrable on $[a,b]$. Then there is some $c \in [a,b]$ such that*

$$\int_a^b f(x)g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx.$$

Proof: Define $G(x) = \int_a^x g(t)dt$ for $x \in [a,b]$. Since g is continuous, $G'(x) = g(x)$ for all $x \in [a,b]$. Hence the integral on the left hand side can be written as $\int_a^b f(x)G'(x)dx$. Since f' does not change sign, f is monotone and hence integrable. Apply integration by parts formula to get

$$\int_a^b f(x)G'(x)dx = f(b)G(b) - f(a)G(a) - \int_a^b G(x)f'(x)dx.$$

Next apply the weighted mean value theorem to the integral on the right hand side above to get a point $c \in [a,b]$ such that

$$\int_a^b G(x)f'(x)dx = G(c)\int_a^b f'(x)dx.$$

By the second fundamental theorem, $\int_a^b f'(x)dx = f(b) - f(a)$. Hence combining these observations, one gets

$$\begin{aligned}
\int_a^b f(x)G'(x)dx &= f(b)G(b) - f(a)G(a) - G(c)(f(b) - f(a)) \\
&= f(a)\Big(G(c) - G(a)\Big) + f(b)\Big(G(b) - G(c)\Big) \\
&= f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx.
\end{aligned}$$

□

9.2 Improper Riemann integrals

Let $a \in \mathbb{R}$. A function f is said to be **improper Riemann integrable** over $[a,\infty)$ if for any $b \geq a$, f is integrable over $[a,b]$, and if the limit $\lim_{b\to\infty}\int_a^b f$ exists and is finite. The integral of f over $[a,\infty)$ in such a case is defined to be the above limit. One also uses the phrase '**the integral $\int_a^\infty f$ converges**' to mean that f is improper Riemann integrable over $[a,\infty)$.

Similarly a function f is improper Riemann integrable over an interval $(-\infty,b]$ if it is integrable over all intervals of the form $[a,b]$ for $a \leq b$ and if the limit $\lim_{a\to-\infty}\int_a^b f$ exists and is finite. The limit is then denoted by $\int_{-\infty}^b f$ and is called the integral of f over $(-\infty,b]$.

Exercise 9.30 Let $a \in \mathbb{R}$ and assume that f is improper Riemann integrable over both $(-\infty,a]$ and $[a,\infty)$. Show that f is improper Riemann integrable over $(-\infty,b]$ and $[b,\infty)$ for any $b \in \mathbb{R}$ and

$$\int_{-\infty}^a f + \int_a^\infty f = \int_{-\infty}^b f + \int_b^\infty f.$$

If there is an $a \in \mathbb{R}$ such that f is improper Riemann integrable over both $(-\infty,a]$ and $[a,\infty)$, then one says that f is improper Riemann integrable over $(-\infty,\infty)$ (or $\int_{-\infty}^\infty f$ converges) and one defines the integral $\int_{-\infty}^\infty f$ to be the sum $\int_{-\infty}^a f + \int_a^\infty f$.

The type of improper integrals defined above are called **improper integrals of the first kind**.

Exercise 9.31 Let $f : [a,\infty) \to \mathbb{R}$. Show that $\int_a^\infty f$ converges if and only if for every $\epsilon > 0$, there is an $M \in \mathbb{R}$ such that whenever $b, c > M$, one has $\left|\int_b^c f\right| < \epsilon$.

Proposition 9.32 *Let $f : [a,\infty) \to \mathbb{R}$ be nonnegative and integrable over $[a,b]$ for any $b \geq a$. Assume that there is an $M > 0$ such that $\int_a^b f \leq M$ for all $b \geq a$. Then $\int_a^\infty f$ converges.*

Proposition 9.33 (Comparison test) *Let $f, g : [a,\infty) \to \mathbb{R}$ be two functions integrable over any closed bounded subinterval and assume that $0 \leq f(x) \leq g(x)$ for all x. If $\int_a^\infty g$ converges, then $\int_a^\infty f$ also converges and if $\int_a^\infty f$ diverges, then $\int_a^\infty g$ also diverges.*

Corollary 9.34 *If $\int_a^\infty |f|$ converges, then $\int_a^\infty f$ also converges.*

Exercise 9.35 Let $f : [a,b] \to \mathbb{R}$ be bounded. Assume that f is integrable on $[c,b]$ for any $c \in (a,b]$. Show that f is integrable on $[a,b]$ and

$$\int_a^b f = \lim_{c\to a+}\int_c^b f.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We will call a point d **a point of infinite discontinuity** for f if there are points $a < d < b$ such that $\sup\{|f(x)| : x \in [a,b]\} = \infty$, but for any $\epsilon > 0$, $\sup\{|f(x)| : x \in [a,b]\backslash(d-\epsilon, d+\epsilon)\} < \infty$.

Exercise 9.36 Suppose d is a point of infinite discontinuity of a function f. Show that there is a sequence $\{x_n\}$ converging to d such that the sequence $\{|f(x_n)|\}$ diverges to ∞.

Let $f : [a, b] \to \mathbb{R}$ be a function such that a is a point of infinite discontinuity of f but f is integrable on $[c, b]$ for any $c \in (a, b]$. If the limit $\lim_{c \to a+} \int_c^b f$ exists and is finite, then one says that f is improper Riemann integrable over $[a, b]$ and the integral $\int_a^b f$ is defined to be the above limit. Similarly if b is a point of infinite discontinuity of f in $[a, b]$, f is integrable over $[a, c]$ for all $c \in [a, b)$ and if the limit $\lim_{c \to b-} \int_a^c f$ exists and is finite, then one says that f is improper Riemann integrable over $[a, b]$ and the integral $\int_a^b f$ is defined to be this limit.

Week 12–13

10 Sequences and series of functions

Let E be a subset of $\mathbb{R}$. A sequence of functions on E is a map from $\mathbb{N}$ to the set of functions from E to $\mathbb{R}$. If n is mapped to the function f_n for each $n \in \mathbb{N}$, then we just say that we have a sequence of functions $\{f_n\}$. One says that a sequence of functions $\{f_n\}$ **converges pointwise** to a function f if for each $x \in E$, one has $\lim_{n\to\infty} f_n(x) = f(x)$.

Examples.

1. Let $\{c_n\}$ be a sequence of reals. Define

$$f_n(x) = c_n, \quad x \in \mathbb{R}.$$

   If $\{c_n\}$ converges to c, then $\{f_n\}$ converges pointwise to the constant function $x \mapsto c$.

2. Let $f_n(x) = x^n$ for $x \in [0,1]$. Then $\{f_n\}$ converges pointwise to the function f on $[0,1]$ given by

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

3. Let

$$f_n(x) = \frac{x^{2n}}{1 + x^{2n}}, \quad x \in \mathbb{R}.$$

   The sequence f_n converges pointwise to the function f given by

$$f(x) = \begin{cases} 0 & \text{if } -1 < x < 1, \\ \frac{1}{2} & \text{if } x = \pm 1, \\ 1 & \text{if } |x| > 1. \end{cases}$$

4. Let

$$f_n(x) = |x|^{1+\frac{1}{n}}, \quad x \in (-1,1).$$

   The sequence f_n converges pointwise to the function $x \mapsto |x|$ on $(-1,1)$.

5. Let
$$f_n(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 - \frac{1}{n}, \\ n^2(x - 1 + \frac{1}{n}) & \text{if } 1 - \frac{1}{n} \le x < 1, \\ 0 & \text{if } x = 1. \end{cases}$$
The sequence f_n converges pointwise to the constant function $x \mapsto 0$.

6. Let $\mathbb{Q} \cap [0,1] = \{r_1, r_2, \ldots\}$. Define f_n to be the indicator function of the set $\{r_1, r_2, \ldots, r_n\}$. The sequence f_n converges pointwise to the function f on $[0,1]$ given by
$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

7. Let
$$f_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}, \quad x \in \mathbb{R}.$$
The sequence f_n converges pointwise to the function $x \mapsto \exp x$.

8. Let
$$f_n(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}, \quad x \in \mathbb{R}.$$
The sequence f_n converges pointwise to the function $x \mapsto \cos x$.

9. Let
$$f_n(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad x \in \mathbb{R}.$$
The sequence f_n converges pointwise to the function $x \mapsto \sin x$.

10. Let
$$f_n(x) = \frac{\sin nx}{\sqrt{n}}, \quad x \in \mathbb{R}.$$
The sequence f_n converges pointwise to the constant function $x \mapsto 0$.

Suppose we have a sequence of functions $\{f_n\}$, defined on some commom domain $E \subseteq \mathbb{R}$, that converges to a function f pointwise. We will now try to see what properties of the functions f_n carry over to the limit function f. More specifically, the questions we will try to answer are:

1. if each f_n is continuous, is f also continuous?

2. if each f_n is differentiable, is f also differentiable? and whether or not the sequence $\{f_n'\}$ converge?

3. if each f_n is integrable over some fixed interval $[a,b]$, is f also integrable over $[a,b]$? If yes, can one relate the number $\int_a^b f$ with the numbers $\int_a^b f_n$?

A quick look at the examples above will tell us that the answer to all these questions are negative (look at examples 2 and 3 for question 1, examples 4 and 10 for question 2 and examples 5 and 6 for question 3). Therefore what we do next is change the notion of convergence a little bit and try to answer the above questions for this new convergence.

A sequence of functions f_n defined on a set E is said to **converge to a function** f **uniformly** if the sequence $\{d_n\}$ given by $d_n := \sup_{x\in E} |f_n(x) - f(x)|$ converges to 0.

Exercise 10.1 Show that a sequence of functions f_n defined on a set E converges uniformly to a function f if and only if given any $\epsilon > 0$, there exists a natural number N such that whenever $n \geq N$, one has $|f_n(x)-f(x)| < \epsilon$ for all $x \in E$.

Proposition 10.2 *Let f_n be a sequence of functions defined on a set E. Assume that f_n converges uniformly to a function f. Then f_n converges to f pointwise.*

A sequence of functions $\{f_n\}$ defined on a set E is said to be **uniformly Cauchy** if for any given $\epsilon > 0$, there is a natural number N such that whenever $m, n \geq N$, one has $\sup_{x\in E} |f_n(x) - f_m(x)| < \epsilon$.

Exercise 10.3 Let f_n be a sequence of functions that converge uniformly to a function f on a set E. Show that $\{f_n\}$ is uniformly Cauchy.

Next, we will show that just like in the case of a sequence of real numbers, the converse of the above statement is also true.

Exercise 10.4 Let $\{f_n\}$ be uniformly Cauchy over some set E. Let $c \in E$. Show that the sequence $\{f_n(c)\}$ is Cauchy.

It follows from the above exercise that if $\{f_n\}$ is uniformly Cauchy over E, then for any $c \in E$, the sequence $\{f_n(c)\}$ converges. Define a function f on E by the prescription $f(x) = \lim_{n\to\infty} f_n(x)$. Then the sequence f_n converges to the function f pointwise. The next result tells us that the sequence $\{f_n\}$ actually converges uniformly.

Proposition 10.5 *Let $\{f_n\}$ be uniformly Cauchy. Then f_n converges to some function f uniformly.*

Proof: Define f by

$$f(x) = \lim_{n\to\infty} f_n(x), \quad x \in E.$$

We have seen already that $\{f_n\}$ converges to f pointwise. Next, we will show that $\{f_n\}$ converges to this f uniformly. Take an $\epsilon > 0$. Choose an N such that $\sup_{x\in E} |f_n(x)-f_m(x)| < \frac{\epsilon}{2}$ whenever $m, n \geq N$. Take an $n \geq N$. Then for any $x \in E$ and $m \geq N$, we have

$$|f_n(x) - f_m(x)| < \frac{\epsilon}{2}.$$

Taking limit as $m \to \infty$ in the above inequality, one gets

$$|f_n(x) - f(x)| \le \frac{\epsilon}{2}.$$

Since this is true for all $x \in E$, we have $\sup_{x \in E} |f_n(x) - f(x)| \le \frac{\epsilon}{2} < \epsilon$. □

Now let us go back to the first of the three questions that we wanted to answer.

Proposition 10.6 *Let f_n be a sequence of functions defined on an interval $[a, b]$. Assume that the sequence converges uniformly to a function f and each f_n is continuous at a point $c \in [a, b]$. Then f is also continuous at c.*

Proof: We have to show that given any $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - c| < \delta$ (and $x \in [a, b]$) implies $|f(x) - f(c)| < \epsilon$. Now for any n,

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_n(x) + f_n(x) - f_n(c) + f_n(c) - f(c)| \\ &\le |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)|. \end{aligned}$$

Since the sequence $\{f_n\}$ converges to f uniformly, we can choose an N such that $\sup_x |f_n(x) - f(x)| < \frac{\epsilon}{3}$ for $n \ge N$. Now use the earlier inequality for $n = N$ to get

$$|f(x) - f(c)| \le 2\frac{\epsilon}{3} + |f_N(x) - f_N(c)|.$$

Use continuity of f_N at c to get a $\delta > 0$ such that $|x - c| < \delta$ (and $x \in [a, b]$) implies $|f_N(x) - f_N(c)| < \frac{\epsilon}{3}$. □

Essentially the same proof works for the following slightly more general statement.

Proposition 10.7 *Let f_n be a sequence of functions defined on an interval $[a, b]$. Assume that f_n converges to f uniformly. Let $c \in [a, b]$. Asume that the limits $a_n := \lim_{x \to c} f_n(x)$ exist and the sequence $\{a_n\}$ converges to a real a. Then $\lim_{x \to c} f(x) = a$.*

Proof: Use the inequality

$$\begin{aligned} |f(x) - a| &= |f(x) - f_n(x) + f_n(x) - a_n + a_n - a| \\ &\le |f(x) - f_n(x)| + |f_n(x) - a_n| + |a_n - a|. \end{aligned}$$

□

Next we turn to integrability.

Proposition 10.8 *Let f_n be a sequence of functions that converge uniformly to a function f on an interval $[a, b]$. Assume that each f_n is integrable on $[a, b]$. Then f is also integrable on $[a, b]$ and*

$$\int_a^b f = \lim_{n \to \infty} \int_a^b f_n.$$

Proof: First let us show that f is integrable over $[a,b]$. Take an $\epsilon > 0$. Choose an N such that $\sup_x |f_n(x) - f(x)| < \frac{\epsilon}{4(b-a)}$ for $n \geq N$. Next, use integrability of f_N to get a partition P of $[a,b]$ such that $U(f_N, P) - L(f_N, P) < \frac{\epsilon}{2}$. Since for all $x \in [a,b]$, we have $|f_N(x) - f(x)| < \frac{\epsilon}{4(b-a)}$, it follows that $U(f,P) \leq U(f_N, P) + \frac{\epsilon}{4}$ and $L(f,P) \geq L(f_N, P) - \frac{\epsilon}{4}$. Hence

$$U(f,P) - L(f,P) \leq U(f_N, P) - L(f_N, P) + \frac{\epsilon}{2} < \epsilon.$$

Thus f is integrable over $[a,b]$.

Let N be as above. Then for $n \geq N$, we have

$$f(x) - \frac{\epsilon}{4(b-a)} < f_n(x) < f(x) + \frac{\epsilon}{4(b-a)}.$$

Hence by integrating, we get

$$\int_a^b f - \frac{\epsilon}{4} \leq \int_a^b f_n \leq \int_a^b f + \frac{\epsilon}{4},$$

i. e. $\left|\int_a^b f - \int_a^b f_n\right| \leq \frac{\epsilon}{4} < \epsilon.$ □

Exercise 10.9 Let f_n be a sequence of functions that converge uniformly to a function f on the interval $[0,1]$. Assume that f_n is integrable on $[0, 1-\frac{1}{n}]$ for each n, and f is bounded on $[0,1]$. Show that f is integrable on $[0,1]$ and

$$\int_0^1 f = \lim_{n\to\infty} \int_0^{1-\frac{1}{n}} f_n.$$

Differentiability continues to be ill-behaved, even with uniform convergence, as the following exercises illustrate.

Exercise 10.10 Let $f_n(x) = \frac{\sin nx}{\sqrt{n}}$, $x \in \mathbb{R}$. Show that $\{f_n\}$ converges uniformly to the zero function, each f_n is differentiable, but the sequence $\{f_n'\}$ does not converge even pointwise.

Exercise 10.11 Let $f_n(x) = |x|^{1+\frac{1}{n}}$, $x \in (-1,1)$. Show that the sequence $\{f_n\}$ converges uniformly to the function f defined by $f(x) := |x|$. Show that each f_n is differentiable throughout $(-1,1)$, but f is not.

Proposition 10.12 *Let f_n be a sequence of functions that converge uniformly to a function f on an interval (a,b). Assume that each is differentiable on (a,b) and the sequence f_n' converges uniformly to some function g. Then f is differentiable on (a,b) and $f' = g$.*

Proof: Let $c \in (a,b)$. We want to show that f is differentiable at c and $f'(c) = g(c)$. Define a sequence of functions h_n on (a,b) as follows:

$$h_n(x) = \begin{cases} \frac{f_n(x) - f_n(c)}{x-c} & \text{if } x \neq c, \\ f_n'(c) & \text{if } x = c. \end{cases}$$

Since f_n converges to f and f'_n converges to g, it follows that the sequence h_n converges pointwise to the function h given by

$$h(x) = \begin{cases} \frac{f(x)-f(c)}{x-c} & \text{if } x \neq c, \\ g(c) & \text{if } x = c. \end{cases}$$

Next we will show that the sequence $\{h_n\}$ in uniformly Cauchy, from which it will then follow that h_n converges to h uniformly. Also observe that each h_n is continuous at c. Therefore it will then follow that the limit function h is also continuous at c. But this means $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = g(c)$, which is precisely what we want.

Take an $\epsilon > 0$. Look at the difference $|h_n(x) - h_m(x)|$. For $x = c$, we have

$$|h_n(c) - h_m(c)| = |f'_n(c) - f'_m(c)| \leq \sup_x |f'_n(x) - f'_m(x)|.$$

For $x \neq c$,

$$|h_n(x) - h_m(x)| = \left|\frac{f_n(x) - f_m(x) - f_n(c) + f_m(c)}{x - c}\right|$$

Now applying mean value theorem (to the function $f_n - f_m$), we get some d between x and c such that $\frac{f_n(x)-f_m(x)-f_n(c)+f_m(c)}{x-c} = f'_n(d) - f'_m(d)$. Therefore $|h_n(x) - h_m(x)| \leq \sup_x |f'_n(x) - f'_m(x)|$. Thus combining the two cases, we get $\sup_x |h_n(x) - h_m(x)| \leq \sup_x |f'_n(x) - f'_m(x)|$. Since the sequence $\{f'_n\}$ converges uniformly to g, it is uniformly Cauchy. Hence there is an N such that for $m, n \geq N$, $\sup_x |f'_n(x) - f'_m(x)| < \epsilon$. Hence for $m, n \geq N$, we get $\sup_x |h_n(x) - h_m(x)| < \epsilon$. □

The interval (a, b) plays no special role in the above result. It can be replaced by any other set, possibly unbounded. When restricted to a bounded set, however, one could weaken the assumptions by making use of the mean value theorem, which means one gets a slightly stronger result.

Corollary 10.13 *Let f_n be a sequence of functions defined on an interval (a, b). Assume that each is differentiable on (a, b) and the sequence f'_n converges uniformly to some function g. Assume also that there is a point $c \in (a, b)$ such that the sequence $\{f_n(c)\}$ converges. Then there is a function f on (a, b) such that f_n converges to f uniformly on (a, b), f is differentiable on (a, b) and $f' = g$.*

Proof: If the sequence $\{f_n\}$ converges uniformly to some function f, then by the previous result, we are through. So it is enough to prove that the sequence $\{f_n\}$ is uniformly Cauchy. Let $x \in (a, b)$. Then

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f_m(x) - f_n(c) + f_m(c) + f_n(c) - f_m(c)| \\ &\leq |f_n(x) - f_m(x) - f_n(c) + f_m(c)| + |f_n(c) - f_m(c)| \end{aligned}$$

Applying mean value theorem to the function $f_n - f_m$, we get $f_n(x) - f_m(x) - f_n(c) + f_m(c) = f'_n(d) - f'_m(d)$ for some d between x and c (and hence in (a, b)). Therefore $|f_n(x) - f_m(x)| \leq \sup_y |f'_n(y) - f'_m(y)| + |f_n(c) - f_m(c)|$. Thus

$$\sup_x |f_n(x) - f_m(x)| \leq \sup_y |f'_n(y) - f'_m(y)| + |f_n(c) - f_m(c)|.$$

Since the sequence $\{f_n(c)\}$ converges, it is Cauchy. So given $\epsilon > 0$, there is an N_1 such that $|f_n(c) - f_m(c)| < \frac{\epsilon}{2}$ for $m, n \geq N_1$. Similarly, since the sequence $\{f_n\}$ is uniformly convergent, and hence uniformly Cauchy, there is an N_2 such that $\sup_y |f'_n(y) - f'_m(y)| < \frac{\epsilon}{2}$ for $m, n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then for $m, n \geq N$, we have $\sup_x |f_n(x) - f_m(x)| < \epsilon$. □

10.1 Double sequences

A double sequence of real numbers is a map from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{R}$.

Proposition 10.14 *Assume that* $\{a_{mn}\}_n$ *converges to* b_m *uniformly in* m *and* $\lim_{m\to\infty} b_m = b$. *Then* $\lim_{m,n\to\infty} a_{mn} = b$.

Proposition 10.15 *Assume that* $\lim_{m,n\to\infty} a_{mn} = b$ *and* $\lim_{n\to\infty} a_{mn} = b_m$ *for each* m. *Then* $\lim_{m\to\infty} b_m = b$.

Existence of $\lim_{m,n\to\infty} a_{mn}$ does not imply the existence of either $\lim_{m\to\infty} a_{mn}$ or $\lim_{n\to\infty} a_{mn}$, as the following example illustrates.

WEEK 14-15

11 Multivariable calculus

Functions from $\mathbb{R}^m$ to $\mathbb{R}^n$.

differences and similarities between $\mathbb{R}$ and $\mathbb{R}^m$ for $m > 1$:

Week ???

12 Misc

Let $x > 0$. Then $\lim \left(1+\frac{x}{n}\right)^n$ exists and is finite. Call this number e_x. We will now show that the limit $\lim \left(1-\frac{x}{n}\right)^n$ also exists and is equal to $\frac{1}{e_x}$.

$$\begin{aligned}\left(1+\frac{x}{n}\right)^n\left(1-\frac{x}{n}\right)^n &= \left(1-\frac{x^2}{n^2}\right)^n \\ &= 1+\sum_{r=1}^{n}\binom{n}{r}(-1)^r\frac{x^{2r}}{n^{2r}}.\end{aligned}$$

Hence

$$\begin{aligned}\left|\left(1+\frac{x}{n}\right)^n\left(1-\frac{x}{n}\right)^n-1\right| &\le \sum_{r=1}^{n}\binom{n}{r}\frac{x^{2r}}{n^{2r}} \\ &\le \frac{1}{n}\sum_{r=1}^{n}\binom{n}{r}\frac{x^{2r}}{n^{r}} \\ &\le \frac{1}{n}\sum_{r=1}^{n}\frac{x^{2r}}{r!} \\ &\le \frac{1}{n}\exp(x^2).\end{aligned}$$

Therefore $\lim \left(1+\frac{x}{n}\right)^n\left(1-\frac{x}{n}\right)^n = 1$. This, combined with the observation that $\lim \frac{1}{(1+\frac{x}{n})^n} = \frac{1}{e_x}$, gives us $\lim \left(1-\frac{x}{n}\right)^n = \frac{1}{e_x}$.

Thus the limit $\lim \left(1+\frac{x}{n}\right)^n$ exists and is finite for all real values of x. Let us denote the limit by e_x for every x, so that $e_{-x} = \frac{1}{e_x}$ for all x.

Next we show that $e_x e_y = e_{x+y}$ whenever $x \ge 0$ and $y \ge 0$. Observe that

$$\begin{aligned}\left(1+\frac{x}{n}\right)^n\left(1+\frac{y}{n}\right)^n &= \left(1+\frac{x+y}{n}+\frac{xy}{n^2}\right)^n \\ &= \left(1+\frac{x+y}{n}\right)^n+\sum_{r=1}^{n}\binom{n}{r}\frac{(xy)^{2r}}{n^{2r}}.\end{aligned}$$

Therefore

$$\begin{aligned} 0 &\leq \left(1+\frac{x}{n}\right)^n \left(1+\frac{y}{n}\right)^n - \left(1+\frac{x+y}{n}\right)^n \\ &\leq \frac{1}{n}\sum_{r=1}^{n}\frac{(xy)^{2r}}{r!} \\ &\leq \frac{1}{n}\exp(xy) \end{aligned}$$

Hence

$$\lim_{n\to\infty}\left(1+\frac{x+y}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \lim_{n\to\infty}\left(1+\frac{y}{n}\right)^n,$$

i. e. $e_{x+y} = e_x e_y$.

Exercise 12.1 Let $x, y > 0$ and $x - y > 0$. Show that $e_x e_{-y} = e_{x-y}$ (Use the equality $e_{x-y}e_y = e_x$) and $e_{-x}e_y = e_{y-x}$

Now prove that $e_x e_y = e_{x+y}$ for all x and y.

12.1 Proof of irrationality of π