Clifford Algebras

1.1 Quadratic forms

Definition 1.1.1 Let V be a vector space over a field $\mathbb{F}$. A map $Q : V \to \mathbb{F}$ obeying the following two conditions is called a *quadratic form* on V:
1) $Q(\alpha v) = \alpha^2 Q(v)$, $v \in V$, $\alpha \in \mathbb{F}$,
2) the map $(u, v) \mapsto B_Q(u, v) \equiv \frac{1}{2}(Q(u+v) - Q(u) - Q(v))$ is bilinear.

Q is called nondegenerate if $B_Q(u, v) = 0$ for all $v \in V$ implies $u = 0$.

Let V be a vector space with a quadratic form Q. The subgroup of $GL(V)$ generated by maps that preserve this quadratic form is called the *orthogonal group* and is denoted by $O(V, Q)$, i.e.

$$O(V, Q) = \{T \in GL(V) : Q \circ T = Q\}.$$

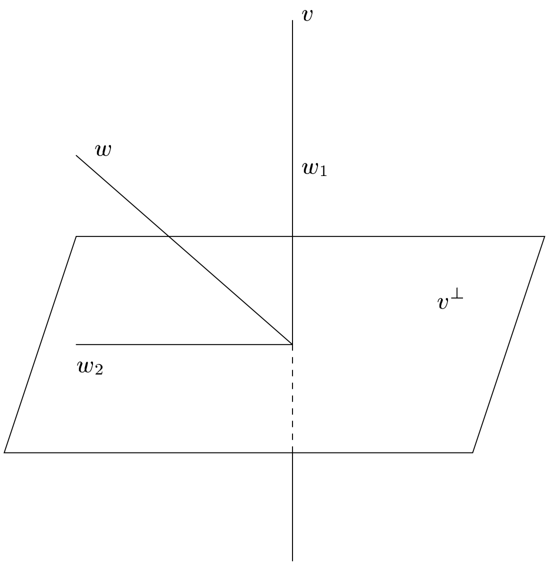
For a finite dimensional vector space V, define the *special orthogonal group* $SO(V, Q)$ by

$$SO(V, Q) = \{T \in GL(V) : Q \circ T = Q, \ \det T = 1\}.$$

Reflection across a hyperplane.

Choose and fix a $v \in V$. Let $v^\perp = \{w \in V : B_Q(v, w) = 0\}$. Let $\rho_v(w)$ denote the reflection of w across the plane $v^\perp$. Clearly $\rho_v(w) = w_2 - w_1$, where w_1 and w_2 are the components of w along v and $v^\perp$ respectively. Now

$$w_1 = \frac{B_Q(v, w)v}{Q(v)}, \quad w_2 = w - w_1.$$



Therefore

$$\rho_v(w) = w - 2Q(v)^{-1}B_Q(v, w)v.$$

Now check that $Q(\rho_v(w)) = Q(w)$, i.e. $\rho_v \in O(V, Q)$. Choosing a basis $\{v_1, \ldots, v_n\}$ such that $v_1 = v$ and $v_i \in v^{\perp}$ for $i \geq 2$, it is easy to see that $\det \rho_v = -1$.

Theorem 1.1.2 (Cartan-Dieudonne) *Let V be of dimension n, and Q be a non-degenerate quadratic form on it. Then*

$$\begin{aligned} O(V, Q) &= \{\rho_{v_1} \cdots \rho_{v_r} : v_i \in V, r \leq n\}, \\ SO(V, Q) &= \{\rho_{v_1} \cdots \rho_{v_r} : v_i \in V, r \leq n, r \textit{ even}\}. \end{aligned}$$

A slightly weaker form (without the restriction $r \leq n$ on the right hand sides) can be proved easily by induction on the dimension of V.

Proposition 1.1.3 *Let V be a vector space of dimension n, equipped with a nondegenerate quadratic form Q. There is a basis $\{e_1, \ldots, e_n\}$ for V such that*

$$\begin{aligned} Q(e_i) &\neq 0 \quad \forall i, \\ B_Q(e_i, e_j) &= 0 \quad \textit{if } i \neq j. \end{aligned}$$

Corollary 1.1.4 *If $\mathbb{F} = \mathbb{C}$, a basis $\{e_1, \ldots, e_n\}$ can be chosen such a way that*

$$\begin{aligned} Q(e_i) &= 1 \quad \forall i, \\ B_Q(e_i, e_j) &= 0 \quad \textit{if } i \neq j. \end{aligned}$$

Corollary 1.1.5 *If $\mathbb{F} = \mathbb{R}$, then the basis $\{e_1, \ldots, e_n\}$ can be so chosen that for some $r \in \{0, 1, \ldots, n\}$,*

$$\begin{aligned} Q(e_i) &= \begin{cases} 1 & \text{for } 1 \le i \le r \\ -1 & \text{if } r < i \le n \end{cases} \\ B_Q(e_i, e_j) &= 0 \text{ for } i \ne j. \end{aligned}$$

Remark 1.1.6 *A field $\mathbb{F}$ is called spin if for any $a \in \mathbb{F}^*$, at least one of the equations $t^2 = a$ and $t^2 = -a$ has a solution in $\mathbb{F}$. The previous corollary is true for any spin field $\mathbb{F}$.*

1.2 Clifford Algebras

Let $\mathcal{T}(V)$ denote the *tensor algebra* over V, that is, $\mathcal{T}(V) = \oplus_{r \ge 0} V^{\otimes r}$. $\mathcal{T}(V)$ is a unital associative algebra with multiplication defined by

$$(u_1 \otimes \ldots \otimes u_r) \cdot (v_1 \otimes \ldots \otimes v_s) = u_1 \otimes \ldots \otimes u_r \otimes v_1 \otimes \ldots \otimes v_s.$$

Let $\mathcal{I}_Q$ denote the ideal generated by $\{v \otimes v + Q(v).I : v \in V\}$. The quotient algebra $\mathcal{T}(V)/\mathcal{I}_Q$ is called the *Clifford algebra* associated to (V, Q), and will be denoted by $C\ell(V, Q)$, or by just $C\ell(V)$ if there is no confusion about Q.

Examples.

1. Taking Q to be identically zero, one gets the exterior algebra $\Lambda^* V$ over V.

2. Take $V = \mathbb{R}$, $Q(x) = x^2$. In this case $C\ell(\mathbb{R}, Q) = \mathbb{C}$.

3. $V = \mathbb{R}$, $Q(x) = -x^2$. $C\ell(\mathbb{R}, Q) = \{\begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{R}\}$. The map $\begin{pmatrix} a & b \\ b & a \end{pmatrix} \mapsto (a + b, a - b)$ is an algebra isomorphism between this and $\mathbb{R} \oplus \mathbb{R}$.

4. $V = \mathbb{R}^2$, $Q(x_1, x_2) = x_1^2 + x_2^2$. $C\ell(V, Q) = \mathbb{H}$.

An equivalent alternative description of $C\ell(V, Q)$ is the following: it is the algebra generated by elements of V obeying the relation $v \cdot v = -Q(v).I$.

It is easy to see that the ideal $\mathcal{I}_Q$ consists of all finite sums of the form

$$\sum u_i \otimes v_i \otimes v_i \otimes w_i + \sum Q(v_i) u_i \otimes w_i,$$

where each u_i comes from some $V^{\otimes r_i}$, and each w_i from some $V^{\otimes s_i}$. It is clear from this description that $V \cap \mathcal{I}_Q = \Phi$. Thus there is a natural embedding $V \hookrightarrow C\ell(V, Q)$.

Proposition 1.2.1 *Let $\mathcal{A}$ be a unital associative algebra over $\mathbb{F}$. If $f : V \to \mathcal{A}$ is a linear map satisfying $f(v) \cdot f(v) = -Q(v).I$, then f extends uniquely to an algebra homomorphism from $C\ell(V,Q)$ to $\mathcal{A}$.*

$C\ell(V,Q)$ is the unique associative algebra having this property.

Corollary 1.2.2 *The orthogonal group $O(V,Q)$ extends canonically to a group of automorphisms of $C\ell(V,Q)$.*

Proof: For $T \in O(V,Q)$, take the extension of the map $v \mapsto Tv$ from V to $C\ell(V,Q)$. $\square$

Clifford conjugation.

Define $\alpha \in \operatorname{aut}(C\ell(V,Q))$ to be the extension of the map $v \mapsto -v$ on V. It gives $C\ell(V,Q)$ a $\mathbb{Z}_2$-grading:

$$C\ell(V,Q) = C\ell^0(V,Q) \oplus C\ell^1(V,Q),$$

where $C\ell^i(V,Q) = \{v \in C\ell(V,Q) : \alpha(v) = (-1)^i v\}$. The map $e_{i_1} \dots e_{i_k} \mapsto e_{i_k} \dots e_{i_1}$ on $\mathcal{T}(V)$ descends to an involution $x \mapsto x^t$ on $C\ell(V,Q)$, called the *transpose map*.

The composition of these two maps, namely the map $x \mapsto \alpha(x^t)$ is called the *Clifford conjugation*.

Proposition 1.2.3 *If $V = V_1 \oplus V_2$ is a Q-orthogonal decomposition of V, then there is a natural algebra isomorphism $C\ell(V,Q) \cong C\ell(V_1, Q|_{V_1}) \hat{\otimes} C\ell(V_2, Q|_{V_2})$, where $\hat{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product.*

Notations.

Let π_Q be the canonical projection of $\mathcal{T}(V)$ onto $C\ell(V,Q)$. Let

$$\begin{aligned} \mathcal{T}^r &= \mathcal{T}^r(V) = \oplus_{i \le r} V^{\otimes i} \quad (V^{\otimes 0} := \mathbb{F}), \\ \mathcal{F}^r &= \pi_Q(\mathcal{T}^r), \\ \mathcal{G}^r &= \mathcal{F}^r / \mathcal{F}^{r-1}, \\ \mathcal{G} &= \oplus_{r \ge 0} \mathcal{G}^r. \end{aligned}$$

Suppose $\dim V = n$ and $\{e_1, e_2, \dots\}$ is a basis for V $(1 \le n \le \infty)$. Then

$$V^{\otimes r} = \text{linear span of } \{e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_r} : 1 \le i_1, \dots, i_r \le n\}$$

$$\mathcal{T}^r = \text{linear span of} \left\{ \begin{array}{ll} e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_r} & : \ 1 \leq i_1, \ldots, i_r \leq n \\ e_{j_1} \otimes e_{j_2} \otimes \ldots \otimes e_{j_{r-1}} & : \ 1 \leq j_1, \ldots, j_{r-1} \leq n \\ e_{k_1} \otimes e_{k_2} & : \ 1 \leq k_1, k_2 \leq n \\ e_l & : \ 1 \leq l \leq n \\ I & \end{array} \right\}$$

$$\begin{aligned} \mathcal{F}^r &= \text{linear span of} \left\{ \begin{array}{ll} e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_r} & : \ 1 \leq i_1 < \ldots < i_r \leq n \\ e_{j_1} \otimes e_{j_2} \otimes \ldots \otimes e_{j_{r-1}} & : \ 1 \leq j_1 < \ldots < j_{r-1} \leq n \\ e_{k_1} \otimes e_{k_2} & : \ 1 \leq k_1 < k_2 \leq n \\ e_l & : \ 1 \leq l \leq n \\ I & \end{array} \right\} \\ &= \text{linear span of } \{e_\alpha : \alpha \subseteq \{1, 2, \ldots, n\}\} \\ \mathcal{G}^r &= \text{linear span of } \{e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_r} : 1 \leq i_1 < \ldots < i_r \leq n\} \\ &= \Lambda^r V \end{aligned}$$

It is clear from the above description that $C\ell(V,Q)$, $\mathcal{G}$ and $\Lambda^* V$ are naturally isomorphic as vector spaces. However, as algebras they are not—for example the multiplication in $C\ell(V,Q)$ depends on Q, whereas that in $\Lambda^* V$ does not.

Multiplication in $C\ell(V,Q) = \cup_{r \geq 0} \mathcal{F}^r$ induces a multiplication in $\mathcal{G}$, making it into a graded associative algebra (i.e. $\mathcal{G}^r \mathcal{G}^s \subseteq \mathcal{G}^{r+s}$). It is called the *associated graded algebra*. The vector space isomorphism between $\mathcal{G}$ and $\Lambda^* V$ is actually an algebra isomorphism.

From the description in the previous page, one can easily verify the following lemma.

Lemma 1.2.4 *If $\dim V = n$, then $\dim C\ell(V,Q) = 2^n$.*

Proof: Choose $e_1, \ldots e_n$ such that $Q(e_i) \neq 0$ for all i. It is enough to show that $\{e_\alpha : \alpha \in \mathcal{P}\}$ is independent, where $\mathcal{P}$ denote the power set of $T := \{1, 2, \ldots, n\}$.

Suppose $\sum_{\alpha \in \mathcal{P}} c_\alpha e_\alpha = 0$. Then for any $\beta \in \mathcal{P}$, $\sum_{\alpha \in \mathcal{P}} c_\alpha e_\alpha e_\beta^{-1} = 0$, i.e.

$$c_\beta + c'_{\beta^c} e_T + \sum_{\substack{\alpha \in \mathcal{P}: \\ \alpha \neq \phi, \, \alpha \neq T}} c'_\alpha e_\alpha = 0.$$

Now notice that if $\sum_{\alpha \in S \subseteq \mathcal{P}} d_\alpha e_\alpha = 0$, then for any i, $\sum_{\alpha \in S} d_\alpha (e_i e_\alpha e_i^{-1} + e_\alpha) = 0$, which means $\displaystyle\sum_{\substack{\alpha \in S: \\ e_\alpha e_i = e_i e_\alpha}} d_\alpha e_\alpha = 0$. Applying this for $i = 1, \ldots, n$, we get $\displaystyle\sum_{\substack{\alpha \in S: \\ e_\alpha e_i = e_i e_\alpha \forall i}} d_\alpha e_\alpha = 0$. For $S = \mathcal{P}$, we get $d_\phi e_\phi = 0$, i.e. $d_\phi = 0$. In our present case, we obtain $c_\beta = 0$. $\square$

Some important groups associated with the Clifford algebra.

$$\begin{aligned} C\ell^\times(V,Q) &= \text{group of units in } C\ell(V,Q),\\ P(V,Q) &= \text{subgroup of } C\ell^\times(V,Q) \text{ generated by } \{v\in V : Q(v)\neq 0\},\\ Pin(V,Q) &= \text{subgroup of } C\ell^\times(V,Q) \text{ generated by } \{v\in V : Q(v)=\pm 1\},\\ Spin(V,Q) &= Pin(V,Q)\cap C\ell^0(V,Q). \end{aligned}$$

Examples.

1. $V=\mathbb{R}$, $Q(x)=x^2$: In this case, $C\ell(V,Q)$ is $\mathbb{R}\oplus\mathbb{R}e_1$ with multiplication rule:

$$(s\oplus te_1)(s'\oplus t'e_1) = (ss'-tt')\oplus(st'+s't)e_1,$$

which is nothing but $\mathbb{C}$. We have $i(V)=i\mathbb{R}$, $i(V^\times)=i\mathbb{R}^*$, $C\ell^0=\mathbb{R}$, α is just the complex conjugation, and

$$\begin{aligned} C\ell^\times &= \mathbb{C}\backslash\{0\},\\ P &= \mathbb{R}^*\cup i\mathbb{R}^*,\\ Pin &= \{1,-1,i,-i\},\\ Spin &= \{1,-1\}. \end{aligned}$$

2. $V=\mathbb{R}$, $Q(x)=-x^2$:
$C\ell(V,Q)=\left\{\left(\begin{smallmatrix} a & b\\ b & a\end{smallmatrix}\right) : a,b\in\mathbb{R}\right\}\cong\mathbb{R}\oplus\mathbb{R}$ $\left(\left(\begin{smallmatrix} a & b\\ b & a\end{smallmatrix}\right)\mapsto a+b\oplus a-b\right)$.

$$\begin{aligned} i(V) &= \left\{\begin{pmatrix} 0 & b\\ b & 0\end{pmatrix} : b\in\mathbb{R}\right\}\cong\{(b,-b) : b\in\mathbb{R}\},\\ C\ell^0 &= \left\{\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix} : a\in\mathbb{R}\right\}\cong\{(a,a) : a\in\mathbb{R}\}\\ C\ell^\times &= \left\{\begin{pmatrix} a & b\\ b & a\end{pmatrix} : a^2\neq b^2\right\}\cong\{(x,y) : xy\neq 0\},\\ P &= \left\{\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix},\begin{pmatrix} 0 & b\\ b & 0\end{pmatrix} : 0\neq a,b\in\mathbb{R}\right\}\cong\{(x,x),(y,-y) : 0\neq x,y\in\mathbb{R}\},\\ Pin &= \left\{\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\begin{pmatrix} -1 & 0\\ 0 & -1\end{pmatrix},\begin{pmatrix} 0 & -1\\ -1 & 0\end{pmatrix}\right\}\cong\{(1,1),(1,-1),(-1,-1),(-1,1)\},\\ Spin &= \left\{\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix} -1 & 0\\ 0 & -1\end{pmatrix}\right\}\cong\{(1,1),(-1,-1)\}. \end{aligned}$$

3. $V = \mathbb{R}^2$, $Q(x_1, x_2) = x_1^2 + x_2^2$.

$$\begin{aligned}
C\ell(V,Q) &= \mathbb{H} \quad (a + be_1 + ce_2 + de_1e_2 \leftrightarrow (a+id) + (b+ic)j)\,),\\
i(V^\times) &= \{zj : z \in \mathbb{C}\backslash\{0\}\},\\
C\ell^\times &= \mathbb{H}\backslash\{0\},\\
C\ell^0 &= \{a + ib : a, b \in \mathbb{R}\},\\
P &= \{a + bi, aj + bk : 0 \neq a, b \in \mathbb{R}\} = \{z, zj : z \in \mathbb{C}\backslash\{0\}\},\\
Pin &= \{a + bi, aj + bk : a^2 + b^2 = 1, a, b \in \mathbb{R}\} = \{z, zj : z \in \mathbb{C}, |z| = 1\},\\
Spin &= \{a + bi : a^2 + b^2 = 1, a, b \in \mathbb{R}\} = \{z : z \in \mathbb{C}\backslash\{0\}\}.
\end{aligned}$$

4. $V = \mathbb{R}^3$, $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$.
Any element of $C\ell_{3,0}$ is of the form

$$x_0+x_1e_1+x_2e_2+x_3e_3+x_{12}e_1e_2+x_{13}e_1e_3+x_{23}e_2e_3+x_{123}e_1e_2e_3, \quad x_i, x_{ij}, x_{ijk} \in \mathbb{R}.$$

Write this in the form $h_1 + h_2e_3$, where

$$\begin{aligned}
h_1 &= x_0 + x_1e_1 + x_2e_2 + x_{12}e_1e_2 &&= (x_0 + ix_{12}) + (x_1 + ix_2)j,\\
h_2 &= x_3 + x_{13}e_1 + x_{23}e_2 + x_{123}e_1e_2 &&= (x_3 + ix_{123}) + (x_{13} + ix_{23})j.
\end{aligned}$$

Check that
(a) $(h_1 + h_2e_3)(h_1' + h_2'e_3) = (h_1h_1' - h_2\alpha h_2') + (h_1h_2' + h_2\alpha h_1)e_3$, ($\alpha$ being the grading automorphism in $\mathbb{H} \cong C\ell_{2,0}$)
(b) the map

$$\phi : h_1 + h_2e_3 \mapsto \begin{pmatrix} h_1 & h_2 \\ -\alpha(h_2) & \alpha(h_1) \end{pmatrix} \in M_2(\mathbb{H})$$

gives an injective algebra homomorphism from $C\ell_{3,0}$ to $M_2(\mathbb{H})$,
(c) the map

$$\begin{pmatrix} h_1 & h_2 \\ -\alpha(h_2) & \alpha(h_1) \end{pmatrix} \mapsto (h_1 + h_2i) \oplus (h_1 - h_2i)$$

is an algebra isomorphism between $\phi(C\ell_{3,0})$ and $\mathbb{H} \oplus \mathbb{H}$.

From now on, we will identify $C\ell_{3,0}$ and $\phi(C\ell_{3,0})$. Since

$$\alpha \begin{pmatrix} h_1 & h_2 \\ -\alpha(h_2) & \alpha(h_1) \end{pmatrix} = \begin{pmatrix} \alpha(h_1) & -\alpha(h_2) \\ h_2 & h_1 \end{pmatrix},$$

we have

$$C\ell_{3,0}^0 = \left\{ \begin{pmatrix} h_1 & h_2 \\ -\alpha(h_2) & \alpha(h_1) \end{pmatrix} : h_1 = \alpha(h_1), h_2 = -\alpha(h_2) \right\}$$

$$
\begin{aligned}
&= \left\{ \begin{pmatrix} z & wj \\ wj & z \end{pmatrix} : z, w \in \mathbb{C} \right\} \\
&= \{ z - (iw)j \oplus z + (iw)j : z, w \in \mathbb{C} \}, \\
i(V) &= \left\{ \begin{pmatrix} wj & t \\ -t & -wj \end{pmatrix} : w \in \mathbb{C}, t \in \mathbb{R} \right\}, \\
i(V^{\times}) &= \left\{ \begin{pmatrix} wj & t \\ -t & -wj \end{pmatrix} : w \in \mathbb{C}, t \in \mathbb{R}, |w|^2 + t^2 \neq 0 \right\}, \\
P(V,Q) &= \text{group generated by } i(V^{\times}), \\
Pin(V,Q) &= \text{group generated by } \left\{ \begin{pmatrix} wj & t \\ -t & -wj \end{pmatrix} : w \in \mathbb{C}, t \in \mathbb{R}, |w|^2 + t^2 = 1 \right\}.
\end{aligned}
$$

An element $\left(\begin{smallmatrix} z & wj \\ wj & z \end{smallmatrix}\right)$ is invertible if and only if $|z|^2 + |w|^2 \neq 0$. Therefore $C\ell^{\times} \cap C\ell_3^0 = \left\{ \left(\begin{smallmatrix} z & wj \\ wj & z \end{smallmatrix}\right) : |z|^2 + |w|^2 \neq 0 \right\}$. It is easy to see that

$$
\begin{aligned}
P \cap C\ell^0 &\subseteq C\ell^{\times} \cap C\ell_3^0, \\
Spin_3 \equiv Pin \cap C\ell^0 &\subseteq \left\{ \begin{pmatrix} z & wj \\ wj & z \end{pmatrix} : |z|^2 + |w|^2 = 1 \right\} \\
&= \{ z + wj \oplus z - wj : |z|^2 + |w|^2 = 1 \}.
\end{aligned}
$$

The following equalities show that both the above inclusions are actually equalities:

$$
\begin{aligned}
\begin{pmatrix} z & wj \\ wj & z \end{pmatrix} &= \begin{cases} \left(\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & z^{-1}wj \\ z^{-1}wj & 1 \end{smallmatrix}\right) & \text{if } z \neq 0 \\ \left(\begin{smallmatrix} wj & 0 \\ 0 & -wj \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) & \text{if } z = 0 \end{cases} \\
\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} &= \begin{pmatrix} -zj & 0 \\ 0 & zj \end{pmatrix} \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}, \\
\begin{pmatrix} 1 & wj \\ wj & 1 \end{pmatrix} &= \begin{pmatrix} wj & -1 \\ 1 & -wj \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\end{aligned}
$$

Finally the map

$$
\psi : \begin{pmatrix} z & wj \\ wj & z \end{pmatrix} \mapsto \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}
$$

is a group isomorphism between $P \cap C\ell^0$ and $\left\{ \left(\begin{smallmatrix} z & w \\ -\bar{w} & \bar{z} \end{smallmatrix}\right) \in GL_2(\mathbb{C}) : |z|^2 + |w|^2 \neq 0 \right\}$, and $\psi(Spin) = SU(2)$. (If we identify $C\ell_3^0$ with $\mathbb{C} \oplus \mathbb{C}$ by $\left(\begin{smallmatrix} z & wj \\ wj & z \end{smallmatrix}\right) \mapsto z \oplus w$, then ψ is just the action by left multiplication).

Proposition 1.2.5 *As algebras over $\mathbb{R}$, we have the following isomorphisms:*

1. $M_n(\mathbb{R}) \otimes M_m(\mathbb{R}) \cong M_{mn}(\mathbb{R})$,

2. $M_n(\mathbb{R}) \otimes \mathbb{C} \cong M_n(\mathbb{C})$,
3. $M_n(\mathbb{R}) \otimes \mathbb{H} \cong M_n(\mathbb{H})$,
4. $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$,
5. $\mathbb{C} \otimes \mathbb{H} \cong M_2(\mathbb{C})$,
6. $\mathbb{H} \otimes \mathbb{H} \cong M_4(\mathbb{R})$.

Proof: Parts 1, 2 and 3 are easy. For part 4, use the map

$$\begin{aligned} 1 \oplus 0 &\mapsto \frac{1}{2}(1 \otimes 1 + i \otimes i), & i \oplus 0 &\mapsto \frac{1}{2}(i \otimes 1 - 1 \otimes i), \\ 0 \oplus 1 &\mapsto \frac{1}{2}(1 \otimes 1 - i \otimes i), & 0 \oplus i &\mapsto \frac{1}{2}(i \otimes 1 + 1 \otimes i). \end{aligned}$$

For part 5, view $\mathbb{H}$ as a left $\mathbb{C}$-module. Observe that for $z \in \mathbb{C}$ and $h \in \mathbb{H}$, $x \mapsto zx\bar{h}$ is a linear map. Now define

$$\phi : \mathbb{C} \otimes \mathbb{H} \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{H}, \mathbb{H}) \cong M_2(\mathbb{C})$$

by the prescription $\phi_{z,h}(x) = zx\bar{h}$. Check that ϕ is one-one and linear. Therefore by dimensionality argument, it is onto as well.

For part 6, view $\mathbb{H}$ as a vector space over $\mathbb{R}$, and define

$$\phi : \mathbb{H} \otimes \mathbb{H} \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{H}, \mathbb{H}) \cong M_4(\mathbb{R})$$

by $\phi_{h_1,h_2}(x) = h_1 x \bar{h}_2$. Rest is similar. □

1.3 Structure of Clifford algebras

Theorem 1.3.1 *Let n, r and s be nonnegative integers. Then*

1. $C\ell_{n,0} \otimes C\ell_{0,2} \cong C\ell_{0,n+2}$,
2. $C\ell_{0,n} \otimes C\ell_{2,0} \cong C\ell_{n+2,0}$,
3. $C\ell_{r,s} \otimes C\ell_{1,1} \cong C\ell_{r+1,s+1}$.

Proof: Write $r + s = n$. Notice that the dimension of the left hand side equals that of the right hand side in each case. Denote by $(f_1, \ldots, f_{n+2})$ a basis for $\mathbb{R}^{n+2}$. Choose elements $e_1, \ldots, e_n$ in the first term of the left hand side satisfying

$$e_i^2 = -1 \quad \forall i, \qquad e_ie_j + e_je_i = 0 \quad \forall i \neq j \qquad \text{in part 1,}$$

$$e_i^2 = 1 \quad \forall i, \qquad e_ie_j + e_je_i = 0 \quad \forall i \neq j \qquad \text{in part 2,}$$

$$e_i^2 = -1 \quad \text{for } 1 \leq i \leq r; e_i^2 = 1 \quad \text{for } r+1 \leq i \leq n; \qquad e_ie_j + e_je_i = 0 \quad \forall i \neq j \qquad \text{in part 3,}$$

and similarly choose e_{n+1} and e_{n+2} in the second term such that

$$e_{n+1}^2 = 1 = e_{n+2}^2, \qquad e_{n+1}e_{n+2} + e_{n+2}e_{n+1} = 0 \qquad \text{in part 1,}$$

$$e_{n+1}^2 = -1 = e_{n+2}^2, \qquad e_{n+1}e_{n+2} + e_{n+2}e_{n+1} = 0 \qquad \text{in part 2,}$$

$$e_{n+1}^2 = -1 = -e_{n+2}^2, \qquad e_{n+1}e_{n+2} + e_{n+2}e_{n+1} = 0 \qquad \text{in part 3.}$$

Now define a linear map from $\mathbb{R}^{n+2}$ to the left hand side by the prescription:

$$\psi : f_i \mapsto \begin{cases} e_i \otimes e_{n+1}e_{n+2} & \text{if } 1 \leq i \leq n, \\ 1 \otimes e_i & \text{if } n < i \leq n+2. \end{cases}$$

This preserves the quadratic form in each case, so that the map extends to a homomorphism from the right hand side to the left hand side. Also, ψ is injective. Dimensions of the LHS and the RHS being equal, ψ actually is an isomorphism. □

We have already seen that

$$C\ell_{1,0} = \mathbb{C}, \quad C\ell_{0,1} = \mathbb{R} \oplus \mathbb{R}, \quad C\ell_{2,0} = \mathbb{H}.$$

Similarly one can show that

$$C\ell_{0,2} = M_2(\mathbb{R}), \quad C\ell_{1,1} = M_2(\mathbb{R}).$$

Using the previous theorem, we have the following table:

	1	2	3	4	5	6	7	8
$C\ell_{n,0}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$	$M_8(\mathbb{R})$	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$	$M_{16}(\mathbb{R})$
$C\ell_{0,n}$	$\mathbb{R} \oplus \mathbb{R}$	$M_2(\mathbb{R})$	$M_2(\mathbb{C})$	$M_2(\mathbb{H})$	$M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$	$M_4(\mathbb{H})$	$M_8(\mathbb{C})$	$M_{16}(\mathbb{R})$

The following theorem will now tell us how to fill up all the columns on the right.

Theorem 1.3.2 *For all $n \geq 0$, one has the following periodicity isomorphisms:*

$$\begin{aligned} C\ell_{n+8,0} &\cong C\ell_{n,0} \otimes C\ell_{8,0}, \\ C\ell_{0,n+8} &\cong C\ell_{0,n} \otimes C\ell_{0,8}. \end{aligned}$$

Proof: Using theorem 1.3.1, we get

$$\begin{aligned} C\ell_{n+8,0} &\cong C\ell_{0,n+6} \otimes C\ell_{2,0} \\ &\cong C\ell_{n+4,0} \otimes C\ell_{0,2} \otimes C\ell_{2,0} \\ &\cong C\ell_{0,n+2} \otimes C\ell_{2,0} \otimes C\ell_{0,2} \otimes C\ell_{2,0} \\ &\cong C\ell_{n,0} \otimes C\ell_{0,2} \otimes C\ell_{2,0} \otimes C\ell_{0,2} \otimes C\ell_{2,0} \\ &\cong C\ell_{n,0} \otimes C\ell_{8,0}. \end{aligned}$$

Similarly,

$$\begin{aligned} C\ell_{0,n+8} &\cong C\ell_{n+6,0} \otimes C\ell_{0,2} \\ &\cong C\ell_{0,n+4} \otimes C\ell_{2,0} \otimes C\ell_{0,2} \\ &\cong C\ell_{n+2,0} \otimes C\ell_{0,2} \otimes C\ell_{2,0} \otimes C\ell_{0,2} \\ &\cong C\ell_{0,n} \otimes C\ell_{2,0} \otimes C\ell_{0,2} \otimes C\ell_{2,0} \otimes C\ell_{0,2} \\ &\cong C\ell_{0,n} \otimes C\ell_{0,8}. \end{aligned}$$

□

We can now go back to theorem 1.3.1 again and use part 3 to get the following table for $C\ell_{r,s}$'s.

1.4 Complex Clifford algebras

Complexification of real vector spaces and algebras.

Let V be a real vector space, and let J be a real linear operator on V such that $J^2 = -I$. then V can be made into a complex vector space by the following prescription:

$$(s + it) \cdot u := su + tJu, \quad u \in V, s, t \in \mathbb{R}.$$

A real linear operator T will be complex linear if and only if $TJ = JT$.

Suppose now that $\mathcal{A}$ is an algebra over $\mathbb{R}$. The space $\mathcal{A}_{\mathbb{C}} := \{(a_1, a_2) : a_1, a_2 \in \mathcal{A}\}$ is then a real algebra equipped with the following operations:

$$\begin{aligned} (a_1, a_2) + (b_1, b_2) &= (a_1 + b_1, a_2 + b_2), \\ (a_1, a_2) \cdot (b_1, b_2) &= (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1), \\ s(a_1, a_2) &= (sa_1, sa_2). \end{aligned}$$

Define a map $J : \mathcal{A}_{\mathbb{C}} \to \mathcal{A}_{\mathbb{C}}$ by $J(a_1, a_2) = (a_2, -a_1)$. Then $J^2 = -I$, and the prescription

$$(s + it)x = sx + tJx$$

makes $\mathcal{A}_{\mathbb{C}}$ into a complex algebra, called the *complexification* of $\mathcal{A}$.

Exercise. If $\mathcal{A}^1$ and $\mathcal{A}^2$ are two real algebras, then $\mathcal{A}^1_{\mathbb{C}} \otimes \mathcal{A}^2_{\mathbb{C}} \cong (\mathcal{A}^1 \otimes_{\mathbb{R}} \mathcal{A}^2)_{\mathbb{C}}$ $((a_1, b_1) \otimes (a_2, b_2) \mapsto (a_1 \otimes a_2 - b_1 \otimes b_2, a_1 \otimes b_2 + b_1 \otimes a_2)$ is the required isomorphism).

Complexification of a quadratic form.

Let V be a real vector space with quadratic form Q. Complexification of V is the space $V_{\mathbb{C}} := \{(v_1, v_2) : v_i \in V\}$ along with the operations

$$\begin{aligned} (u_1, u_2) + (v_1, v_2) &= (u_1 + v_1, u_2 + v_2), \\ (s + it)(u_1, u_2) &= (su_1 - tu_2, su_2 + tu_1). \end{aligned}$$

Define a bilinear form on $V_{\mathbb{C}}$ by

$$B((u_1, u_2), (v_1, v_2)) = B_Q(u_1, v_1) - B_Q(u_2, v_2) + i(B_Q(u_1, v_2). + B_Q(u_2, v_1))$$

The resulting quadratic form $Q_{\mathbb{C}}$ is called the complexification of Q.

Lemma 1.4.1 *As algebras over $\mathbb{C}$, we have $C\ell(V, Q)_{\mathbb{C}} \cong C\ell(V_{\mathbb{C}}, Q_{\mathbb{C}})$.*

It follows from the above observations that

$$\mathbb{C}\ell_n \cong C\ell_{r,s} \otimes_{\mathbb{R}} \mathbb{C} \quad \text{whenever } r + s = n.$$

Therefore

$$\begin{aligned} \mathbb{C}\ell_{n+2} &\cong (C\ell_{n+2})_{\mathbb{C}} \\ &\cong (C\ell_{0,n} \otimes C\ell_{2,0})_{\mathbb{C}} \\ &\cong \mathbb{C}\ell_n \otimes \mathbb{C}\ell_2. \end{aligned}$$

Now from the table of $C\ell_{r,s}$,

$$C\ell_{0,0} = \mathbb{R}, \quad C\ell_{1,0} = \mathbb{C}, \quad C\ell_{2,0} = \mathbb{H},$$

and consequently

$$\mathbb{C}\ell_0 = \mathbb{C}, \quad \mathbb{C}\ell_1 = \mathbb{C} \oplus \mathbb{C}, \quad \mathbb{C}\ell_2 = \mathbb{H} \otimes \mathbb{C} \cong M_2(\mathbb{C}).$$

From the periodicity isomorphism above, one can now compute all the $\mathbb{C}\ell_n$'s.

	1	2	3	4	5	6	7	8
$\mathbb{C}\ell_n$	$\mathbb{C} \oplus \mathbb{C}$	$M_2(\mathbb{C})$	$M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$	$M_4(\mathbb{C})$	$M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$	$M_8(\mathbb{C})$	$M_8(\mathbb{C}) \oplus M_8(\mathbb{C})$	$M_{16}(\mathbb{C})$

1.5 Adjoint representation

Define for $\phi \in C\ell^\times$, $\mathrm{Ad}_\phi x = \phi x \phi^{-1}$, $x \in C\ell(V, Q)$. This gives a representation $\mathrm{Ad} : C\ell^\times \to \mathrm{aut}\,(C\ell)$, called the *adjoint representation.* Take an element $v \in V$ with $Q(v) \neq 0$. Then for any $w \in V$, $\mathrm{Ad}_v(w) = -\rho_v(w)$. Since any element of $P(V, Q)$ is of the form $v_1 v_2 \dots v_r$, where the v_i's are from V and $Q(v_i) \neq 0$ for all i. Therefore $\mathrm{Ad}\,(P(V, Q))$ consists of elements of the form $(-1)^r \rho_{v_1} \dots \rho_{v_r}$, so that Ad gives a homomorphism of $P(V, Q)$ into $O(V, Q)$, which means in particular that $\mathrm{Ad}_\phi(V) = V$ for all $\phi \in P(V, Q)$.

Twisted adjoint representation.

$\widetilde{\mathrm{Ad}}_\phi x := \alpha(\phi) x \phi^{-1}$. Clearly for $v \in V$, $\widetilde{\mathrm{Ad}}_v = \rho_v$. Thus this again gives a homomorphism from $P(V, Q)$ into $O(V, Q)$ and as before, $\widetilde{\mathrm{Ad}}_\phi V = V$ for all $\phi \in P(V, Q)$. Define

$$\widetilde{P}(V, Q) = \{\phi \in C\ell^\times : \widetilde{\mathrm{Ad}}_\phi V = V\}.$$

$\widetilde{\mathrm{Ad}}$ gives a homomorphisms from $\widetilde{P}(V, Q)$ to $GL(V)$. Let us now have a look at the restriction of $\widetilde{\mathrm{Ad}}$ to *Pin* and *Spin* groups. Since $\widetilde{\mathrm{Ad}}_v = \rho_v$ for v with $Q(v) \neq 0$, and since $\rho_{tv} = \rho_v$ for $0 \neq t \in \mathbb{F}$, it follows that if $\mathbb{F}$ is a spin field then $\widetilde{\mathrm{Ad}}$ maps *Pin* onto $O(V, Q)$ and *Spin* onto $SO(V, Q)$. In fact one has the following theorem.

Theorem 1.5.1 *If V is finite dimensional, with a nondegenerate quadratic form Q, over a spin field $\mathbb{F}$, then there are exact sequences*

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{K} & \to & Spin(V, Q) & \overset{\widetilde{Ad}}{\to} & SO(V, Q) & \to & I \\ 0 & \to & \mathbb{K} & \to & Pin(V, Q) & \overset{\widetilde{Ad}}{\to} & O(V, Q) & \to & I \end{array}$$

where

$$\mathbb{K} = \begin{cases} \mathbb{Z}_2 & \textit{if } \sqrt{-1} \notin \mathbb{F}, \\ \{\pm 1, \pm\sqrt{-1}\} & \textit{otherwise.} \end{cases}$$

Before we go to the proof, we need the following.

Proposition 1.5.2 *Let V and Q be as in the previous theorem. Then the kernel of the homomorphism*

$$\widetilde{Ad} : \widetilde{P}(V,Q) \to GL(V)$$

is $\mathbb{F}^* \equiv \mathbb{F}\backslash\{0\}$.

Proof: Take a $\phi \in \ker \widetilde{\mathrm{Ad}}$. Write $\phi = \phi_0 + \phi_1$, where $\phi_0 \in C\ell^0$, $\phi_1 \in C\ell^1$. Choose a basis $\{e_1, \ldots, e_n\}$ for V such that $Q(e_i) \neq 0$, $B_Q(e_i, e_j) = 0$ for $i \neq j$. Now use the facts that $\phi_0 v = v\phi_0$ and $\phi_1 v = -v\phi_1$ for all $v \in V$ to show that neither of ϕ_0 or ϕ_1 involve $\{e_1, \ldots, e_n\}$. Therefore ϕ_0 is a scalar and $\phi_1 = 0$. $\square$

The norm map. Define $N : C\ell(V,Q) \to C\ell(V,Q)$ by $N(\phi) = \phi \cdot \alpha(\phi^t)$. This is called the norm map, and one has (1) $N(v) = Q(v)$ for $v \in V$, and (2) N restricted to $\widetilde{P}(V,Q)$ is a homomorphism into $\mathbb{F}^*$.

Proof of the theorem: Clearly $\widetilde{\mathrm{Ad}}(Pin) = O(V,Q)$ and $\widetilde{\mathrm{Ad}}(Spin) = SO(V,Q)$. Suppose now that $x \in \ker \widetilde{\mathrm{Ad}}|_{Spin}$. Choose an orthogonal basis $\{e_1, \ldots, e_n\}$ for V. Write x in the form $x_0 + e_1 x_1$ where x_0 and x_1 are polynomials in $e_2, \ldots, e_n$. Since $\alpha(x) = x$, we get

$$\alpha(x_0) = x_0, \quad \alpha(x_1) = -x_1.$$

Now $\widetilde{\mathrm{Ad}}_x e_1 = e_1$ implies $xe_1 = e_1 x$, i.e. $x_0 e_1 + e_1 x_1 e_1 = e_1 x_0 + e_1^2 x_1 = x_0 e_1 - e_1 x_1 e_1$, so that $e_1 x_1 e_1 = 0$. Hence $x_1 = 0$. Thus x does not involve e_1. In the same way, x does not involve any of the e_i's, which means $x \in \mathbb{F}$.

On the other hand, $x \in Pin$. So $x = v_1 \ldots v_r$, $v_i \in V$, $Q(v_i) = \pm 1$. Therefore

$$\begin{aligned} x^2 = x \cdot \bar{x} &= N(x) \\ &= v_1 \ldots v_r(-v_r) \ldots (-v_1) \\ &= (-1)^r v_1 \ldots v_r v_r \ldots v_1 \\ &= Q(v_1) \ldots Q(v_r) = \pm 1. \end{aligned}$$

$\square$

Theorem 1.5.3 *For all (r,s), there are short exact sequences*

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}_2 & \to & Spin_{r,s} & \to & SO_{r,s} & \to & I \\ 0 & \to & \mathbb{Z}_2 & \to & Pin_{r,s} & \to & O_{r,s} & \to & I. \end{array}$$

For $(r,s) \neq (1,1)$, the double coverings are nontrivial over each component of $O_{r,s}$.

Proof: Existence of the exact sequences follow from the earlier theorem. To show that the covering is nontrivial, we will exhibit a path connecting -1 and 1 in $Spin_{r,s}$. Choose $e_1, e_2 \in \mathbb{R}^{r+s}$ such that

$$Q(e_1) = Q(e_2) = \pm 1, \quad B_Q(e_1, e_2) = 0.$$

Now define

$$\gamma(t) = (e_1 \cos t + e_2 \sin t)(e_2 \sin t - e_1 \cos t).$$

Then $\{\gamma(0), \gamma(\pi/2)\} = \{-1, 1\}$. □

1.6 Structure of Clifford modules

Definition 1.6.1 *Let $\mathbb{K}$ be a field containing $\mathbb{F}$. A $\mathbb{K}$-representation of the Clifford algebra $C\ell(V,Q)$ is an algebra homomorphism*

$$\rho : C\ell(V,Q) \to Hom_{\mathbb{K}}(W,W)$$

where W is a finite dimensional vector space over $\mathbb{K}$. W is called a $C\ell(V,Q)$-module over $\mathbb{K}$.

Lemma 1.6.2 *Let $\mathbb{F}$ be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Then*

1. *the natural representation of $M_n(\mathbb{F})$ on $\mathbb{F}^n$ is the only irreducible real representation of $M_n(\mathbb{F})$,*

2. *$M_n(\mathbb{F}) \oplus M_n(\mathbb{F})$ has exactly two inequivalent irreducible real representations which are just the irreducible representations of the first copy and the second copy.*

From the table of $C\ell_{r,s}$, it is now clear that if $\nu_{r,s}$ denote the number of inequivalent irreducible representations of $C\ell_{r,s}$, then

$$\nu_{r,s} = \begin{cases} 2 & \text{if } r - s \equiv 3 \bmod 4, \\ 1 & \text{otherwise.} \end{cases} \tag{1.6.1}$$

Similarly, from the table of $\mathbb{C}\ell_n$, we have

$$\nu_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd.} \end{cases} \tag{1.6.2}$$

Let $V = \mathbb{R}^{r+s}$ be equipped with the quadratic form $Q(x_1, \ldots, x_{r+s}) = \sum_1^r x_i^2 - \sum_{r+1}^{r+s} x_i^2$. Choose an orientation for $\mathbb{R}^{r+s}$. Let $\{e_1, \ldots, e_{r+s}\}$ be a positively oriented Q-orthonormal basis. Define

$$\omega = e_1 \cdots e_{r+s}.$$

ω is independent of the choice of the basis. It is called the *oriented volume element.* It plays a very important role in determining the irreducible representations.

Proposition 1.6.3 *Write $n = r + s$. then*

$$\omega^2 = (-1)^{\frac{n(n+1)}{2}+s}, \tag{1.6.3}$$

$$v\omega = (-1)^{n-1}\omega v \quad \forall v \in \mathbb{R}^n. \tag{1.6.4}$$

Proposition 1.6.4 *Suppose $\omega^2 = 1$. Write $\pi^+ = \frac{1}{2}(1+\omega)$, $\pi^- = \frac{1}{2}(1-\omega)$. Then we have*

$$\pi^+ + \pi^- = 1, \quad (\pi^+)^2 = \pi^+, \quad (\pi^-)^2 = \pi^-, \quad \pi^+\pi^- = 0 = \pi^-\pi^+.$$

Proposition 1.6.5 *Assume $\omega^2 = 1$ and $r + s$ is odd. Then $C\ell_{r,s}$ decomposes as a direct sum of isomorphic subalgebras*

$$C\ell_{r,s} = C\ell_{r,s}^+ \oplus C\ell_{r,s}^-,$$

where $C\ell_{r,s}^\pm = \pi^\pm \cdot C\ell_{r,s} = C\ell_{r,s} \cdot \pi^\pm$. Moreover $\alpha(C\ell_{r,s}^\pm) = C\ell_{r,s}^\mp$.

Corollary 1.6.6 *If $n \equiv 3 \bmod 4$, then $C\ell_n$ admits a decomposition*

$$C\ell_n = C\ell_n^+ \oplus C\ell_n^-,$$

where $C\ell_n^\pm = \pi^\pm C\ell_n$, $\alpha(C\ell_n^\pm) = C\ell_n^\mp$.

Proposition 1.6.7 *Assume $n \equiv 3 \bmod 4$. Let $\rho : C\ell_n \to Hom_{\mathbb{R}}(W, W)$ be an irreducible module. Then $\rho(\omega) = I$ or $-I$. Both the possibilities occur, and the corresponding representations constitute* all *(two in this case) the irreducible representations of $C\ell_n$.*

Proof: $I = \rho(\omega^2) = \rho(\omega)^2$. Write $W^{\pm} = \ker(\rho(\omega) \pm I)$. Then $W = W^+ \oplus W^-$, and since in this case ω is in the centre, each of W^+ and W^- are $C\ell_n$-invariant. Therefore $W = W^+$ or $W = W^-$. The equality $\rho\alpha(\omega) = -\rho(\omega)$ shows that both possibilities do occur. That these are equivalent is obvious. Equation (1.6.2) now tells us these are all ireducible representations. □

Proposition 1.6.8 *Assume $\omega^2 = 1$ and $r+s$ is even. Then any real $C\ell_{r,s}$-module W admits a decomposition*

$$W = W^+ \oplus W^-,$$

where $W^{\pm} = \pi^{\pm}W$ are the eigenspaces of W.

For any $e \in \mathbb{R}^{r+s}$ with $Q(e) \neq 0$, module multiplication by e gives an isomorphism between W^+ and W^-.

Proposition 1.6.9

$$\begin{aligned} C\ell_{r+s} &\cong C\ell^0_{r+1,s}, \\ C\ell_n &\cong C\ell^0_n. \end{aligned}$$

Proposition 1.6.10 *Suppose $n \equiv 0 \bmod 4$. Let $\rho : C\ell_n \to Hom_{\mathbb{R}}(W,W)$ be a real irreducible representation. The subspace $W^{\pm} = \rho(\pi^{\pm})W$ are invariant under $C\ell^0_n$. Under the isomorphism $C\ell^0_n \cong C\ell_{n-1}$, these spaces correspond to the two real irreducible representations of $C\ell_{n-1}$.*

Proof: Follows from the fact that ω commutes with the elements of $C\ell^0_n$. □

We have seen that for $n \equiv 3 \bmod 4$, there are two inequivalent irreducible representations of $C\ell_n$. Call them ρ_1 and ρ_2. Both of them give rise to representations of $Spin_n$ when restricted there. We have also seen that

$$\rho_1\alpha(x) = \rho_2(x) \quad \forall\, x.$$

Therefore $\rho_1|_{Spin_n} = \rho_2|_{Spin_n}$. We are now ready for the following definition.

Definition 1.6.11 The *real spinor representation* of $Spin_n$ is the restriction

$$\Delta_n : Spin_n \to GL(S)$$

of an irreducible real representation of $C\ell_n$ on S.

Theorem 1.6.12 *1. Δ_n is irreducible if $n \equiv 3$, 5, 6, or 7 mod 8.*

2. Δ_n is a direct sum of two equivalent irreducible representations if $n \equiv 1$ or 2 mod 8.

3. Δ_{4m} is a direct sum of two inequivalent irreducible represtations Δ_{4m}^+ and Δ_{4m}^-.

Proof: Observe that (a) any representation of $Spin_n$ extends uniquely to a representation of $C\ell_n^0$, (b) a representation of $Spin_n$ is irreducible if and only if the corresponding representation of $C\ell_n^0$ is so, and (c) two representations of $Spin_n$ are equivalent if and only if the corresponding representations of $C\ell_n^0$ are so.

Now from the table, check that for $n \equiv 0 \bmod 4$, restriction of an irreducible representation of $C\ell_n$ to $C\ell_n^0 \cong C\ell_{n-1}$ is a direct sum of the two inequivalent irreducible representations of $C\ell_{n-1}$, which gives part (3). For $n \equiv 3$, 5, 6 or 7 mod 8, restriction of an irreducible representation of $C\ell_n$ to $C\ell_{n-1} \cong C\ell_n^0$ is still irreducible. So we get part (1). For $n \equiv 1$ or 2 mod 8, restriction of an irreducible representation of $C\ell_n$ to $C\ell_{n-1}^0$ is of dimension 2^{4k+1} or 2^{4k+2} respectively (where n is $8k+1$ or $8k+2$); whereas dimension of the only irreducible representation of $C\ell_{n-1}$ in these cases are 2^{4k} and 2^{4k+1} respectively. Therefore the restriction must be a direct sum of two equivalent irreducible representations. $\square$

Example. Δ_3. $C\ell_{3,0} = \mathbb{H} \oplus \mathbb{H}$. Two irreducible representations ρ_1 and ρ_2 are given by

$$\begin{aligned}\rho_r\Big((a_1 + b_1 i + c_1 j + d_1 k) \oplus (a_2 + b_2 i + c_2 j + d_2 k)\Big)(x_0 + x_1 i + x_2 j + x_3 k) \\ = (a_r + b_r i + c_r j + d_r k)(x_0 + x_1 i + x_2 j + x_3 k),\end{aligned}$$

that is,

$$\rho_r((a_1+b_1i+c_1j+d_1k)\oplus(a_2+b_2i+c_2j+d_2k)) = \begin{pmatrix} a_r & -b_r & -c_r & -d_r \\ b_r & a_r & -c_r & d_r \\ c_r & d_r & a_r & -b_r \\ d_r & -c_r & b_r & a_r \end{pmatrix}, \quad r = 1, 2.$$

$$Spin_{3,0} = \{(a + bi + cj + dk) \oplus (a + bi - cj - dk) : a^2 + b^2 + c^2 + d^2 = 1\}.$$

Observe that if T denotes $\begin{pmatrix} o & I \\ I & 0 \end{pmatrix}$, then $T\rho_1(x)T^{-1} = \rho_2(x)$ for all $x \in Spin_{3,0}$. Also, $J = \begin{pmatrix} o & -I \\ I & 0 \end{pmatrix} \in \rho(Spin_3)'$ and $J^2 = -I$. If we put a complex structure on $\mathbb{R}^4$ using the map J, then

$$\rho((a+bi+cj+dk)\oplus(a+bi-cj-dk)) = \begin{pmatrix} a+ci & -(b-di) \\ b+di & a-ci \end{pmatrix}.$$

Remark 1.6.13 1. Spin representations are complex for $n \equiv 2$, 3, 4, 5 or 6 mod 8, quaternionic for $n \equiv 3$, 4 or 5 mod 8.

2. Any complex representation of $C\ell_{r,s}$ extends to a representation of $\mathbb{C}\ell_{r+s}$.

Proposition 1.6.14 *Let $\rho : C\ell_n \to Hom_{\mathbb{R}}(W,W)$ be a real representation of $C\ell_n$. Then there exists an innner product on W such that*

$$\langle e\cdot w, e\cdot w'\rangle = \langle w, w'\rangle \quad \forall\, w, w' \in W,\ e \in \mathbb{R}^n \text{ with } \|e\| = 1.$$

Moreover, one has $\rho(x)^* = \rho(\alpha(x^t))$ for all $x \in C\ell_n$.

Proof: Choose an orthonormal basis $\{e-1,\dots,e_n\}$ in $\mathbb{R}^n \subseteq C\ell_n$. Then the group Γ_n generated by $\{e-1,\dots,e_n,-1,1\}$ is called the *Clifford group*. So

$$\Gamma_n = \{\pm 1, \pm e_i, \pm e_{i_1}e_{i_2}, \dots, \pm e_1 e - 2\cdots e_n : 1 \le i \le n, 1 \le i_1 < i_2 < \ldots \le n\},$$

and $|\Gamma_n| = 2^{n+1} < \infty$. Notice that any representation of $C\ell_n$ arises from a representation of Γ_n carrying $-I$ to I, and vice versa. Denote by the same symbol ρ, the corresponding representation Γ_n on W. Now take any inner product $\langle\cdot,\cdot\rangle_0$ on W. Define

$$\langle w, w'\rangle := \sum_{g\in\Gamma_n} \langle \rho(g)w, \rho(g)w'\rangle_0, \quad w, w' \in W.$$

Then it is easy to check that $\langle e_i \cdot w, e_i \cdot w'\rangle = \langle w, w'\rangle$ for all $w, w' \in W$, and $\langle e_i\cdot w, e_j \cdot w\rangle = -\langle w, e_i e_j w\rangle = \langle w, e_j e_i w\rangle = -\langle e_j \cdot w, e_i\cdot w\rangle$ so that $\langle e_i w, e_j w\rangle = 0$ for $i \neq j$. So for any $v = \sum v_i e_i \in \mathbb{R}^n$ with $\sum v_i^2 = 1$, one has

$$\langle v\cdot w, v\cdot w\rangle = \sum v_i^2 \langle e_i w, e_i w\rangle + \sum_{i\neq j} v_i v_j \langle e_i w, e_j w\rangle = \langle w, w\rangle.$$

Next, since $\langle e_i \cdot w, w'\rangle = -\langle w, e_i\cdot w'\rangle$, we have $\rho(e_i)^* = \rho(-e_i)$. By linearity, $\rho(v)^* = \rho(-v)$ for all $v \in \mathbb{R}^n$. Therefore for any element x of the form $v_1 \dots v_r$,

$$\begin{aligned} \rho(x)^* &= \rho(v_r)^* \dots \rho(v_1)^* \\ &= \rho(-v_r)\dots\rho(-v_1) \\ &= \rho((-v_r)\dots(-v_1)) \\ &= \rho(\alpha(x^t)). \end{aligned}$$

By linearity, the same thing holds for all $x \in C\ell_n$. $\square$

Suppose ρ is complex (i.e. if there exists $J \in \rho(C\ell_n)'$ such that $J^2 = -I$). Define

$$\langle w, w' \rangle_0 := \langle w, w' \rangle + \langle Jw, Jw' \rangle + i(\langle Jw, w' \rangle - \langle w, Jw' \rangle).$$

This gives a sesquilinear form such that multiplication by unit vectors are unitary. Thus whenever the spin representation Δ_n is complex, it can be made unitary.

For a complex Clifford algebra $\mathbb{C}\ell_n$, define the complex volume element

$$\omega_{\mathbb{C}} = i^{[(n+1)/2]} e_1 e_2 \dots e_n.$$

Check that $\omega_{\mathbb{C}} v = (-1)^{n+1} v \omega_{\mathbb{C}}$ for all $v \in \mathbb{C}^n$, and $\omega_{\mathbb{C}}^2 = I$. As in the real case, define $\mathbb{C}\ell_n^{\pm} = (I \pm \omega_{\mathbb{C}})\mathbb{C}\ell_n$.

Proposition 1.6.15 *Assume n is odd. Then for any irreducible representation $\rho : \mathbb{C}\ell_n \to Hom_{\mathbb{R}}(W, W)$, $\rho(\omega_{\mathbb{C}})$ can be I or $-I$. Both are attained and the corresponding representations are inequivalent and are* all *the irreducible representations of $\mathbb{C}\ell_n$.*

If n is even, then W has a decomposition $W^+ \oplus W^-$ such that $W^{\pm}$ are invariant under $\mathbb{C}\ell_n^0 \cong \mathbb{C}\ell_{n-1}$ and gives the two inequivalent irreducible representations of $\mathbb{C}\ell_{n-1}$.

1.7 Application to geometry

Theorem 1.7.1 *Let n be a positive integer. Write n in the form*

$$n = 2^{4a+b}(2c+1), \quad 0 \le b \le 3.$$

Then there exist at least $8a + 2^b - 1$ linearly independent vector fields on S^n.

Proof: First verify that $\mathbb{R}^{n+1}$ is a $C\ell_m$-module for $m = 8a + 2^b - 1$. Next choose an inner product on $\mathbb{R}^{n+1}$ such that the Clifford multiplication by unit vectors is orthogonal. Choose a basis $x_1, \dots, x_m$ for $\mathbb{R}^m$, and define vector fields on $\mathbb{R}^{n+1}$ by

$$X_i(u) = x_i u, \quad i = 1, \dots, m.$$

We have seen that the linear transformation $u \mapsto x \cdot u$ is skew-symmetric. So $\langle X_i(u), u \rangle = 0$, i.e. $X_i(u) \in T_u(S^n)$.

To show independence, observe that $\sum c_i X_i(u) = 0$ implies $(\sum c_i x_i) \cdot u = 0$, which on multiplication on the left by $\sum c_i x_i$ gives $-\| \sum c_i x_i \|^2 = 0$, i.e. $c_i = 0$ for all i. $\square$