Differential Geometry

1.1 Smooth Manifolds

Definition 1.1.1 An *n-dimensional smooth manifold* is a Hausdorff space M with a countable basis satisfying the following conditions:

1. there is an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of M and homeomorphisms $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$
2. $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ is smooth whenever $U_\alpha \cap U_\beta \neq \phi$,
3. the collection $\{U_\alpha, \phi_\alpha\}_{\alpha\in\Lambda}$ is a maximal family satisfying the two properties above.

Each (U_α, ϕ_α) is called a *chart*. A set of charts satisfying properties (i) and (ii) above is called an *atlas*. So the above definition says that an n-dimensional smooth manifold is a Hausdorff space that admits a *maximal* atlas consisting of charts (ϕ_α, U_α), where ϕ_α's are smooth maps from U_α to $\mathbb{R}^n$. The last condition, viz. maximality, is really a technical one—it is enough to check the first two, as any atlas can be enlarged to a unique maximal atlas.

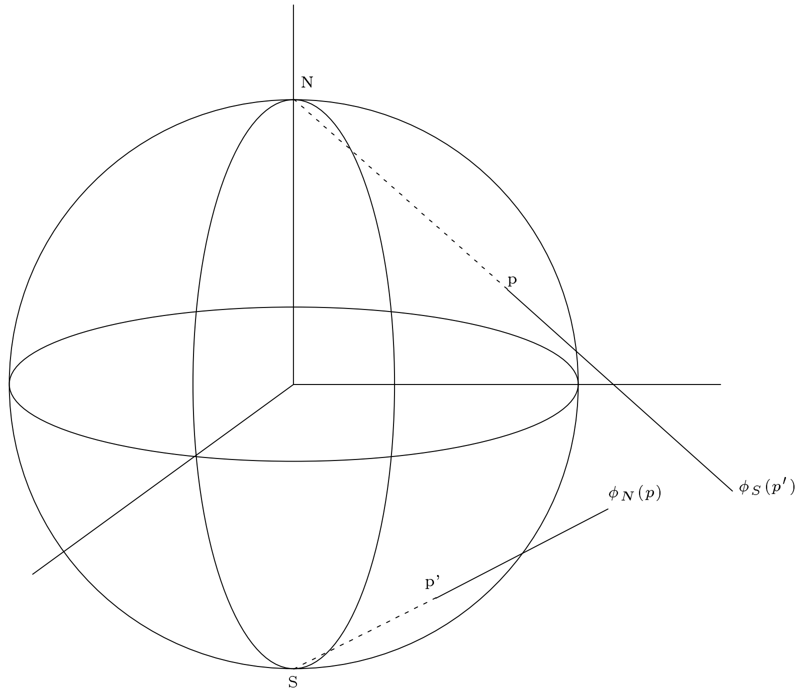
Examples.

1. $\mathbb{R}^n$.
2. $H^2 = \{(x, y) : y > 0\}$.
3. Let M be an open subset of $\mathbb{R}^n$. Then taking $U = M$ and ϕ to be the identity map of U into $\mathbb{R}^n$, we get a smooth manifold structure on M. As a particular example let us take $M = GL(n, \mathbb{R})$. Then M is an open subset of $\mathbb{R}^{n^2}$. Thus $GL(n, \mathbb{R})$ is an n^2-dimensional manifold.

4. $S^n = \{\boldsymbol{x} \in \mathbb{R}^{n+1} : \|\boldsymbol{x}\| = 1\}$. Let $N = e_{n+1}$ be the north pole and $S = -e_{n+1}$ be the south pole. Let $U_S = S^n \backslash \{N\}$ and $U_N = S^n \backslash \{S\}$, and let ϕ_S and ϕ_N be the maps defined as follows:

$$\begin{aligned}
\phi_S(x_1, \ldots, x_{n+1}) &= \left(\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}} \right) \\
\phi_S(x_1, \ldots, x_{n+1}) &= \left(\frac{x_1}{1 + x_{n+1}}, \ldots, \frac{x_n}{1 + x_{n+1}} \right).
\end{aligned}$$

Then $\{(\phi_N, U_N), (\phi_S, U_S)\}$ is an atlas for S^n.



$$\begin{aligned}
\phi_S^{-1}(y_1, \ldots, y_n) &= \left(\frac{2y_1}{1 + \|y\|^2}, \ldots, \frac{2y_n}{1 + \|y\|^2}, \frac{\|y\|^2 - 1}{1 + \|y\|^2} \right) \\
\phi_N^{-1}(y_1, \ldots, y_n) &= \left(\frac{2y_1}{1 + \|y\|^2}, \ldots, \frac{2y_n}{1 + \|y\|^2}, \frac{1 - \|y\|^2}{1 + \|y\|^2} \right) \\
\phi_N \circ \phi_S^{-1}(y_1, \ldots, y_n) &= \phi_N \left(\frac{2y_1}{1 + \|y\|^2}, \ldots, \frac{2y_n}{1 + \|y\|^2}, \frac{\|y\|^2 - 1}{1 + \|y\|^2} \right) \\
&= \left(\frac{y_1}{\|y\|^2}, \ldots, \frac{y_n}{\|y\|^2} \right) \quad \text{if } (y_1, \ldots, y_n) \neq 0. \\
\phi_S \circ \phi_N^{-1}(y_1, \ldots, y_n) &= \phi_S \left(\frac{2y_1}{1 + \|y\|^2}, \ldots, \frac{2y_n}{1 + \|y\|^2}, \frac{1 - \|y\|^2}{1 + \|y\|^2} \right) \\
&= \left(\frac{y_1}{\|y\|^2}, \ldots, \frac{y_n}{\|y\|^2} \right) \quad \text{if } (y_1, \ldots, y_n) \neq 0.
\end{aligned}$$

$\phi_N(U_N \cap U_S) = \mathbb{R}^n \backslash \{0\} = \phi_S(U_N \cap U_S)$

5. Real projective plane $P^n\mathbb{R}$. It is the space of lines in $\mathbb{R}^{n+1}$ passing through the origin, that is, $\mathbb{R}^{n+1}/\sim$, where $(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n)$ if for some $\lambda \in \mathbb{R}\backslash\{0\}$, one has $y_i = \lambda x_i$ for all i. A family of coordinate charts for this space is given by

$$\begin{aligned} V_i &= \{[x_1, \ldots, x_{n+1}] : x_i \neq 0\}, \quad i = 1, \ldots, n+1, \\ \phi_i([x_1, \ldots, x_{n+1}]) &= \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right). \end{aligned}$$

$P^n(\mathbb{R})$ can be identified with the sphere S^n if each point of S^n is identified with its antipodal point. Viewed this way, we have another atlas consisting of the following charts:

$$\begin{aligned} U_i^+ &= \{(x_1, \ldots, x_n) \in S^n : x_i > 0\} \\ U_i^- &= \{(x_1, \ldots, x_n) \in S^n : x_i < 0\} \\ \phi_i^{\pm}(x_1, \ldots, x_n) &= (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n). \end{aligned}$$

Exercise. Let X be a topological space. A functional structure on X is a function F_X defined on the collection of all open sets U in X such that
1. $F_X(U)$ is a subalgebra of $C(U)$,
2. $F_X(U)$ contains all constant functions,
3. $V \subseteq U$ and $f \in F_X(U) \Rightarrow f|_V \in F_X(V)$,
4. if $U = \cup_\alpha U_\alpha$ and $f|_{U_\alpha} \in F_X(U_\alpha)$ for all α, then $f \in F_X(U)$.

The pair (X, F_X) is called a *functionally structured space.*

Following are some examples of functionally structured spaces: $X = \mathbb{R}^n$ with
(i) $F_X(U) = C^0(U)$,
(ii) $F_X(U) = C^k(U)$,
(iii) $F_X(U) = C^\infty(U)$,
(iv) $F_X(U) = C^\omega(U)$.

A *morphism* from a functionally structured space (X, F_X) to another space (Y, F_Y) is a continuous map $\phi : X \to Y$ such that for every open set U in Y, the composition $f \mapsto f \circ \phi$ carries $F_Y(U)$ into $F_X(\phi^{-1}(U))$. An *isomorphism* is a morphism ϕ such that the map is bijective and ϕ^{-1} is a morphism from (Y, F_Y) to (X, F_X).

An *n-dimensional smooth manifold* is a second countable functionally structured Hausdorff space (M, F) which is locally isomorphic to $(\mathbb{R}^n, C^\infty)$. That is, each point of M has a neighbourhood U such that for some open set V in $\mathbb{R}^n$, (U, F_U) is isomorphic to (V, C_V^∞).

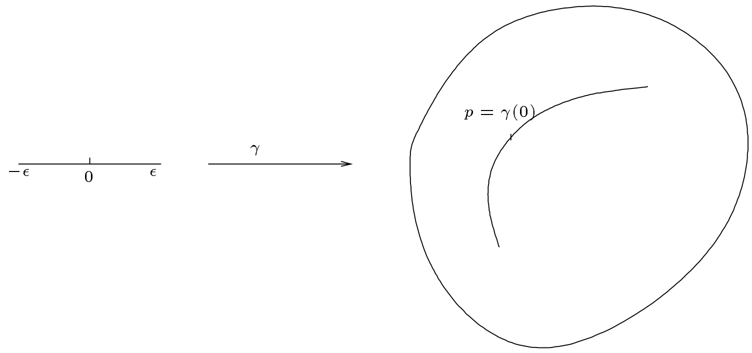
If (X, F_X) is a given functionally structured space and $U \subseteq X$ is open, then let $F_U(V) := F_X(V)$ for open sets $V \subseteq U$. Then (U, F_U) becomes functionally structured.

Show that the above definition of a smooth manifold is equivalent to the definition given earlier.

Smooth functions.

Let M and N be two smooth manifolds, of dimensions m and n respectively. Let p be a point in M. A function $f : M \to N$ is said to be *smooth* at p if for any chart (ϕ, U) around p in M and (ψ, V) around $f(p)$ in N, the map $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth at $\phi(p)$. The space of functions $f : M \to \mathbb{R}$ that are smooth at the point p will be denoted by $C^\infty(p)$. Its easy to check that $C^\infty(p)$ is a vector space.

A *curve* on M is a continuous map from a closed interval $[a, b]$ in $\mathbb{R}$ to M. If this function is smooth (resp. piecewise smooth), we say it is a smooth (resp. piecewise smooth) curve. By *a smooth curve passing through a point* p, we will mean a curve γ defined on some neighbourhood of 0, say, $(-\epsilon, \epsilon)$ to M such that $\gamma(0) = p$.



Any smooth curve γ passing through a point p determines an element v_γ in the dual of $C^\infty(p)$ by the following prescription:

$$v_\gamma f = \frac{d}{dt} f \circ \gamma|_{t=0}, \quad f \in C^\infty(p).$$

This element v_γ, which will usually be denoted by $\gamma'(0)$, satisfies the following:

$$v_\gamma(fg) = f(p)v_\gamma(g) + v_\gamma(f)g(p),$$

that is, v_γ is a tangent vector at the point p. We will see that any tangent vector arises this way.

Coordinate functions.

Let p be a point in a smooth manifold M, and let (ϕ, U) be a chart around p. Then the function x_i given by

$$x_i(p) = i^{\text{th}} \text{ coordinate of } \phi(p)$$

is called the i^{th} coordinate function. Note that a coordinate function makes sense only with respect to a given chart.

Suppose that $p \in M$ with (ϕ, U) being a coordinate chart around it. Denote by e_i the vector in $\mathbb{R}^n$ with 1 at the i^{th} place and 0 elsewhere. Let $\epsilon > 0$ be a positive real such that the ϵ-neighbourhood of $\phi(p)$ is contained in $\phi(U)$. Now define a curve on M as follows:

$$\gamma(t) = \phi^{-1}(\phi(p) + te_i), \quad \epsilon \in (-\epsilon, \epsilon).$$

As we have observed earlier, this will give rise to a tangent vector $\gamma'(0)$ at the point p. This vector will be denoted by $\frac{\partial}{\partial x_i}$ or ∂_i or ∂_{x_i}. It is easy to see now that

$$\begin{aligned}
\partial_{x_i} x_j &= \left.\frac{d}{dt}\right|_{t=0} x_j\left(\phi^{-1}(\phi(p) + te_i)\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} (j^{\text{th}} \text{ coordinate of } \phi(p) + te_i) \\
&= \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}
\end{aligned} \tag{1.1.1}$$

Lemma 1.1.2 *The set $\{\partial_{x_1}, \ldots, \partial_{x_n}\}$ is a basis for T_pM, the space of tangent vectors at the point p.*

Proof: Using equation (1.1.1), one can easily see that the given set is linearly independent. It is therefore enough to show that they span the whole of T_pM. Let (ϕ, U) be a coordinate chart around p. Take any $f \in C^\infty(p)$. Composing f with ϕ and using the Taylor's expansion, we can write f in the following form:

$$f(q) = f(p) + \sum (x_i(q) - x_i(p))g_i(q), \quad q \in U,$$

where g_i's are some functions in $C^\infty(U)$. Applying ∂_i to both sides, we get $\partial_i|_p(f) = g_i(p)$. Now, for any vector $v_p \in T_pM$, we have

$$\begin{aligned}
v_p(f) &= \sum v_p(x_i)g_i(p) \\
&= \sum v_p(x_i)\partial_i|_p(f) \\
&= \left(\sum v_p(x_i)\partial_i\right)(f).
\end{aligned}$$

Thus $v_p = \sum v_p(x_i)\partial_i$. □

Exercise. Let $v \in T_pM$. Then show that there is a smooth curve γ passing through p such that $\gamma'(0) = v$.

(Take $\gamma(t) = \phi^{-1}\left(\phi(p) + t\sum v_p(x_i)e_i\right)$.)

Definition 1.1.3 Let M and N be two smooth manifolds of dimension m and n respectively and let $F : M \to N$ be a smooth map. Then the *derivative* of F at the point p, to be denoted by $DF(p)$, is a linear map from T_pM to $T_{F(p)}N$ given by the following prescription:
Let $v \in T_pM$, $g \in C^\infty(F(p))$. Take a smooth curve γ passing through p with $\gamma'(0) = v$. Then

$$(DF(p)v)(g) := (F \circ \gamma)'(0)g = (g \circ F \circ \gamma)'(0)$$

In the special case when $N = \mathbb{R}$, $DF(p)$ maps T_pM to $\mathbb{R}$, i.e. $DF(p) \in (T_pM)^*$. Thus DF can be thaught of as a map from M to $\cup_p(T_pM)^*$. In this case, DF is called a smooth differential 1-form, and one usually writes dF for DF.

Exercise. Let $x_1, \ldots, x_n$ denote the coordinate functions. Then show that $\{dx_1(p), \ldots, dx_n(p)\}$ is the dual basis of $(T_pM)^*$ corresponding to the basis $\{\partial_{x_1}, \ldots, \partial_{x_n}\}$ of T_pM.

Further examples of differentiable manifolds.

1. **Tangent bundle.** Let M be a smooth manifold of dimension m with coordinate charts $\{(\phi_\alpha, U_\alpha)\}_\alpha$. Denote by TM the space $\{(p, v_p) : p \in M, v_p \in T_pM\}$. Define a family of coordinate charts $\{(\psi_\alpha, V_\alpha)\}_\alpha$ on TM as follows: Denote by π the projection on M, i.e. $\pi(p, v_p) = p$. Let $V_\alpha = \pi^{-1}(U_\alpha)$, and $\psi_\alpha(p, v_p) = (\phi(p), v_p)$.

2. **Grassmann manifolds.** Let $< .,. >$ be a positive definite inner product on $\mathbb{R}^n$. For every k-dimensional subspace E of $\mathbb{R}^n$, let $\rho_E : \mathbb{R}^n \to E$ and $\rho_E^\perp : \mathbb{R}^n \to E^\perp$ be the orthogonal projection. Consider the set $\mathcal{G}_\mathbb{R}(n;k)$ of all k-dimensional subspaces of $\mathbb{R}^n$. For $E \in \mathcal{G}_\mathbb{R}(n;k)$, set $U_E = \{F \in \mathcal{G}_\mathbb{R}(n;k) : F \cap E^\perp = 0\}$ and define $\phi_E : U_E \to L(E, E^\perp)$ by

$$\phi_E(F) = P_{E^\perp} \circ (P_{FE})^{-1}, \quad (P_{FE} = P_E|_F : F \to E).$$

   Check that $\mathcal{G}_\mathbb{R}(n;k)$ is a smooth manifold of dimension $k(n-k)$ with atlas $\{(U_E, \phi_E)\}$.

   $\mathcal{G}_\mathbb{R}(n+1;1)$ is called the real projective plane, and is denoted usually by $\mathbb{R}P^n$.

3. **Level sets.** Let $f : U \subseteq \mathbb{R}^{n+k} \to \mathbb{R}^n$ be a smooth map. Let $q \in \mathbb{R}^n \cap f(U)$ and $S = f^{-1}\{q\}$. Assume that for every $p \in S$, the map $DF(p) : \mathbb{R}^{n+k} \to \mathbb{R}^n$ has rank n. Then S can be given a smooth atlas making it into a smooth

manifold of dimension k. S is called a level set. We write $f = (f_1, \ldots, f_n)$. Let $p \in S$. Without any loss in generality, we can assume that the submatrix $((\frac{\partial f_i}{\partial x_j}(p)))$ is invertible. Consider now the map $\phi : U \to \mathbb{R}^{n+k}$ given by $\phi(\boldsymbol{x}) = (f(\boldsymbol{x}), x_{n+1}, \ldots, x_{n+k})$. Then we have

$$D\phi(p) = \begin{pmatrix} \left(\left(\frac{\partial f_i}{\partial x_j}(p)\right)\right) & * \\ 0 & I_{k\times k} \end{pmatrix},$$

a nonsingular lower triangular matrix. Hence by the Inverse Function Theorem, there is a neighbourhood V of p in U such that ϕ is a diffeomorphism from V onto $\phi(V)$. Let us now introduce a new set of coordinates on V:

$$x_i'(\boldsymbol{x}) = \begin{cases} f_i(\boldsymbol{x}) & \text{if } i \le n, \\ x_i & \text{if } i > n. \end{cases}$$

Then with respect to these coordinates, we see that $S \cap V = \{\boldsymbol{x} \in V : x_i'(\boldsymbol{x}) = q_i \text{ for } i \le n\}$. On $S \cap V$, we introduce a new coordinate system by $y_i(\boldsymbol{x}) = x_{i+n}'(\boldsymbol{x})$, $1 \le i \le k$. Thus the relative neighbourhoods $S \cap V$ of S locally look like $\mathbb{R}^k$.

Exercise. S is a smooth manifold of dimension k.

Examples of level sets:

1. S^n.
2. $O(n) = \{A \in M_n(\mathbb{R}) : A^tA = AA^t = I\}$.
3. $SO(n) = \{A \in O(n) : \det A = 1\}$.

Partition of unity.

A partition of unity on a smooth manifold M is a collection $\{\phi_\alpha\}_{\alpha\in\Lambda}$ of nonzero smooth functions $\phi_\alpha : M \to \mathbb{R}$ such that

(1) each $\operatorname{supp}\phi_\alpha$ is compact and is contained in some coordinate neighbourhood,

(2) the family $\{\operatorname{supp}\phi_\alpha\}$ is locally finite,

(3) $\phi_\alpha(p) \ge 0$, $\sum \phi_\alpha(p) = 1$ for all $p \in M$.

Lemma 1.1.4 *A smooth manifold admits a partition of unity if and only if each of its components is a Hausdorff manifold with a countable basis for its topology.*

Lemma 1.1.5 *For a smooth manifold admitting a partition of unity, if U is an open set and C is a closed set contained in U, then there exists a smooth function $f : M \to \mathbb{R}$ such that $f = 1$ on C, and $f = 0$ on U^c.*

1.2 Vector fields

A vector field is an association $p \mapsto X_p \in T_pM$. It is called *smooth* if for any $f \in C^\infty(M)$, the function $Xf : p \mapsto X_pf$ is smooth. Thus any smooth vector field induces a map on the algebra $C^\infty(M)$. It is easy to see that $X(fg) = (Xf)g + f(Xg)$ for all $f, g \in C^\infty(M)$, i.e. $\nabla_X : f \mapsto Xf$ is a derivation on $C^\infty(M)$. Conversely, one can check that any derivation on $C^\infty(M)$ gives rise to a smooth vector field. One could also view a vector field X as a map ϕ from a manifold M to its tangent bundle TM by defining $\phi(p) = (p, X_p)$. Then ϕ satisfies $\pi \circ \phi = \text{id}$.

Exercise. Show that any smooth map $\phi : M \to TM$ satisfying $\pi \circ \phi = \text{id}$ must be of the form $\phi(p) = (p, X_p)$, where X is a smooth vector field on M.

Let X and Y be two smooth vector fields on M. Define $X+Y$ by the prescription $(X+Y)f = Xf + Yf$. Then $X+Y$ is also a smooth vector field. One could also define the product of X and Y as the following map on $C^\infty(M)$: $(XY)f = X(Yf)$. Check that XY, though linear, is not a derivation. Thus the space $\mathcal{D}^1$ (or $\mathcal{D}^1(M)$) of all smooth vector fields on M is a vector space but not an algebra with the multiplication described above. However, it is an algebra with the product of X and Y defined by $[X, Y](f) = X(Yf) - Y(Xf)$. It is in fact a Lie algebra, since for any three vector fields X, Y and Z, one has

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Exercise. Show that $\mathcal{D}^1$ is a $C^\infty(M)$-module.

At each point p, a vector field X gives a tangent vector X_p, and we know that there is a curve γ passing through p such that $\gamma'(0) = X_p = X_{\gamma(0)}$. What we do not know however is that whether $\gamma'(t) = X_{\gamma(t)}$ for the neighbouring values of t. If there is a curve γ with this property, i.e. $\gamma'(t) = X_{\gamma(t)}$ for all t in some neighbourhood of 0, then it is called an *integral curve* of X at p. It can be shown that integral curves always exist.

Proposition 1.2.1 *Let $X \in \mathcal{D}^1$, $p \in M$. There exists an interval $(-\epsilon, \epsilon)$ around* 0 *in $\mathbb{R}$ and a unique integral curve $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$.*

Proof: Choose a coordinate neighbourhood (ϕ, U) around p, and let $\gamma : (-\epsilon, \epsilon) \to M$ be a smooth curve on this neighbourhood. Write $\gamma_i(t)$ for $x_i(\gamma(t))$, and X^i for $X(x_i) \circ \phi^{-1}$. Then γ satisfies the required conditions if and only if the system of differential equations

$$\frac{d}{dt}\gamma_i(t) = X^i(\gamma_1, \ldots, \gamma_n), \quad 1 \leq i \leq n,$$

admits a solution, which it does from the theory of differential equations. $\square$

Exterior differentiation

Write $\mathcal{D}_1(M)$ for the space of $C^\infty(M)$-linear maps from $\mathcal{D}^1(M)$ to $C^\infty(M)$. For $f \in C^\infty(M)$, $df(p)$ maps T_pM to $\mathbb{R}$. For a smooth vector field X, one can think of $df(X)$ to be the function $p \mapsto X_pf$. Thus df maps $\mathcal{D}^1(M)$ to $C^\infty(M)$. It is easy to see that for $g \in C^\infty(M)$, $df(gX)(p) = (gX)_pf = g(p)X_pf = (g\,df(X))(p)$. Thus df is $C^\infty(M)$-linear i.e. is an element of $\mathcal{D}_1(M)$. Elements of $\mathcal{D}_1(M)$ are called $(0,1)$-tensors. More generally, a $C^\infty(M)$-linear map ϕ from

$$\underbrace{\mathcal{D}_1(M) \times \cdots \times \mathcal{D}_1(M)}_{r} \times \underbrace{\mathcal{D}^1(M) \times \cdots \times \mathcal{D}^1(M)}_{s}$$

to $C^\infty(M)$ is called an (r,s)-tensor, contravariant of rank r and covariant of rank s. The space of all such ϕ's is denoted by $\mathcal{D}^r_s(M)$ or just by $\mathcal{D}^r_s$. Each element $\psi \in \mathcal{D}^r_s(M)$ gives rise to a multilinear map from

$$\underbrace{(T_pM)^* \times \cdots \times (T_pM)^*}_{r} \times \underbrace{T_pM \times \cdots \times T_pM}_{s}$$

to $\mathbb{R}$ by the prescription

$$\psi_p(df_1(p), \ldots, df_r(p), X^1_p, \ldots, X^s_p) := \psi(df_1, \ldots, df_r, X^1, \ldots, X^s)(p),$$

that is, each ψ gives rise to an element ψ_p of

$$\underbrace{T_pM \otimes \cdots \otimes T_pM}_{r} \otimes \underbrace{(T_pM)^* \otimes \cdots \otimes (T_pM)^*}_{s},$$

which is usually denoted by $\mathcal{D}^r_s(p)$.

Notations. $\mathcal{D}^r := \mathcal{D}^r_0$, $\mathcal{D}_s := \mathcal{D}^0_s$, $\mathcal{D}^* = \oplus_{r\geq 0}\mathcal{D}^r$, $\mathcal{D}_* = \oplus_{s\geq 0}\mathcal{D}_s$, $\mathcal{D} = \oplus_{r,s\geq 0}\mathcal{D}^r_s$.

Exterior differential forms.

Let $\mathcal{A}_s$ be the space of all alternating $C^\infty(M)$-multilinear maps from $\underbrace{\mathcal{D}^1 \times \ldots \times \mathcal{D}^1}_{s}$ to $C^\infty(M)$. Write $\mathcal{A}_0 = C^\infty(M)$ and $\mathcal{A} = \oplus_{s\geq 0}\mathcal{A}_s$. Elements of $\mathcal{A}$ are called *exterior differential forms* on M. Elements of $\mathcal{A}_s$ are *exterior differential s-forms*. For a manifold M of dimension m, one has $\mathcal{A}_{m+1} = \mathcal{A}_{m+2} = \ldots = \{0\}$. Notice that $\mathcal{A}_s \subseteq \mathcal{D}_s$ for all s ($\mathcal{A}_1 = \mathcal{D}_1$).

One can define an action of the n-th permutation group $\mathcal{S}_n$ on $\mathcal{D}_n$ as follows:

$$\sigma \cdot d(X_1, \ldots, X_n) = d(X_{\sigma^{-1}(1)}, \ldots, X_{\sigma^{-1}(n)}), \quad d \in \mathcal{D}_n, \sigma \in \mathcal{S}_n.$$

Observe that $d \in \mathcal{A}_n$ if and only if $\sigma \cdot d = \epsilon(\sigma) d$. Define $A_n : \mathcal{D}_n \to \mathcal{D}_n$ by the prescription

$$A_n(d_n) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \epsilon(\sigma) \sigma \cdot d_n.$$

Also, define $A : \mathcal{D}_* \to \mathcal{D}_*$ by $A = \oplus_{n \geq 0} A_n$ $(A_0 := I)$. From the observation, it follows that

$$A_n(\mathcal{D}_n) = \mathcal{A}_n, \quad A(\mathcal{D}_*) = \mathcal{A}, \quad A^2 = A,$$

i.e. A is a projection of $\mathcal{D}_*$ onto $\mathcal{A}$. Now for ω_1 and ω_2 in $\mathcal{A}$, define their *exterior product* by

$$\omega_1 \wedge \omega_2 = A(\omega_1 \otimes \omega_2).$$

This makes $\mathcal{A}$ an associative algebra, isomorphic to $\mathcal{D}_* / \ker A$. This is called the *Grassman algbra* of the manifold M.

Theorem 1.2.2 *There exists a unique $\mathbb{R}$-linear map $d : \mathcal{A} \to \mathcal{A}$ such that*

1. *for $f \in \mathcal{A}_0 = C^\infty(M)$, df is the 1-form given by $df(X) = Xf$,*
2. *$d\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ for all $n \geq 0$,*
3. *$d \circ d = 0$,*
4. *$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge d\omega_2$, where $\omega_1 \in \mathcal{A}_r$, $\omega_2 \in \mathcal{A}$.*

Proof: For $\omega \in \mathcal{A}_p$, $X_i \in \mathcal{D}^1$, define $d\omega$ as follows:

$$\begin{aligned}(p+1) d\omega(X_1, \ldots, X_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} X_i \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}) \\ &\quad + \sum_{i<j} \omega([X_i, X_j], \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}).\end{aligned}$$

□

We will next introduce three very important concepts, namely, *connection*, *covariant derivative* and *parallel transport.*

Connection.

A vector field X, we have seen, can be identified with the derivation ∇_X on $C^\infty(M)$ given by $\nabla_X(f) = Xf$. What does $\nabla_X f$ at a point p tell us? $\nabla_X f(p) = X_p(f)$, which is the rate of change of the (scalar-valued) function f along a curve γ passing through p where $\gamma'(0) = X_p$. We will next be interested in measuring the change of a (tangent-) vector valued smooth map, i.e. a smooth vector field Y along an integral curve of X at p. We are immediately in trouble if we try something like $\lim_{h\to 0} \frac{Y_{\gamma(t+h)} - Y_{\gamma(t)}}{h}$, because $Y_{\gamma(t+h)}$ and $Y_{\gamma(t)}$ are not even in the same space. But for the time being, let us forget about this problem and suppose we have at each point p a quantity $(\nabla_X Y)(p)$ that measures the change of Y at p along an integral curve of X. What will it be? It will be a tangent vector at p. Thus for each p, $(\nabla_X Y)(p) \in T_pM$, and Y being smooth, we would expect it to be smooth, which means, $\nabla_X Y$ will be a smooth vector field again. Now, what are the properties that this $\nabla_X Y$ must possess? It should surely be linear:

$$\nabla_X(Z+W) = \nabla_X Z + \nabla_X W. \tag{1.2.2}$$

∇_X measures rate of change, so it is some kind of a derivative. So

$$\begin{aligned}\nabla_X(fZ) &= (\nabla_X f)Z + f(\nabla_X Z)\\ &= (Xf)Z + f(\nabla_X Z).\end{aligned} \tag{1.2.3}$$

$(\nabla_X \cdot)(p)$ is roughly speaking a generalized tangent vector, so $\nabla_X(\cdot) + \nabla_Y(\cdot) = \nabla_{X+Y}(\cdot)$, that is,

$$\nabla_X(Z) + \nabla_Y(Z) = \nabla_{X+Y}(Z). \tag{1.2.4}$$

Finally, notice that if γ is a curve through p such that $\gamma'(0) = X_p$, then $\widetilde{\gamma}(t) := \gamma(tf(p))$ is a curve through p such that $\widetilde{\gamma}'(0) = f(p)\gamma'(0) = f(p)X_p$. Therefore the rate of change along $\widetilde{\gamma}$ is $f(p)$ times the rate of change along γ, i.e.

$$\nabla_{fX}(Y) = f\nabla_X Y. \tag{1.2.5}$$

Definition 1.2.3 A map $\nabla_X : \mathcal{D}^1 \to \mathcal{D}^1$ satisfying equations (1.2.2)–(1.2.5) above is called a *connection*.

Certain things are immediate from the above definition. For example

1. If X or Y vanish on an open set U around p, then $(\nabla_X Y)(p) = 0$.

2. If $X_p = 0$, then $(\nabla_X Y)(p) = 0$.

3. If Y vanishes on any curve γ through p with $\gamma'(0) = X_p$, then $(\nabla_X Y)(p) = 0$.

Covariant derivative.

Let γ be a smooth curve and Y be a smoothly varying vector field along γ (i.e. $Y(t) \in T_{\gamma(t)}M$ and $t \mapsto Y(t)f$ is smooth for all smooth functions f on M). The covariant derivative of Y along the curve γ is a measure of the change in Y along γ. If the manifold M is already equipped with a connection ∇, then one can define a covariant derivative in the manner described below.

Lemma 1.2.4 *Let γ and Y be as above. Then there is a smooth vector field X such that $X_{\gamma(t)} = \gamma'(t)$ for all t, and a smooth vector field $\widetilde{Y}$ such that $\widetilde{Y}_{\gamma(t)} = Y(t)$ for all t.*

We will need the following auxiliary lemma for the proof.

Lemma 1.2.5 *Let $G : \mathbb{R} \to \mathbb{R}$ be differentiable, and let J be an interval with nonempty interior. Let $\gamma : J \to M$ be a smooth one-one curve such that $\gamma(J)$ is contained in one single coordinate neighbourhood. Then there is a $g \in C^\infty(M)$ such that $g(\gamma(t)) = G(t)$.*

Proof of the earlier lemma: Assume $\gamma(I)$ is contained in a single coordinate neighbourhood U. Now $\gamma'(t) = \sum \gamma'(t)(x_i)\partial_i = \sum (x_i \circ \gamma)'(t)\partial_i$. By the above lemma, we can choose $g_i \in C^\infty(M)$ such that $g_i(\gamma(t)) = (x_i \circ \gamma)'(t)$. Write $X_p = \sum g_i(p)\partial_i$. Then $X_{\gamma(t)} = \gamma'(t)$. Using overlapping neighbourhoods and partitions of unity, extend X to the whole of M.

Take $G(t) = Y(t)x_k$ and use the above lemma to get the second part. □

Having got hold of X and $\widetilde{Y}$, define the covariant derivative of Y along γ to be $(\nabla_X \widetilde{Y})(\gamma(t))$, and denote it by $\frac{DY}{dt}$.

Exercise. Show that: (1) the above definition is independent of the choice of X and $\widetilde{Y}$, (2) $\frac{D}{dt}(Y + Z) = \frac{DY}{dt} + \frac{DZ}{dt}$, (3) $\frac{D}{dt}(fY) = (\gamma'(t)f)Y + f\frac{DY}{dt}$.

Given the notion of a covariant derivative, one can obtain a connection by the following prescription. Define $(\nabla_X Y)(p)$ to be the covariant derivative of $Y_{\gamma(t)}$ along γ where γ is a curve through p with $\gamma'(0) = X_p$.

Exercise. Show that the ∇ defined above is independent of the choice of γ.

Definition 1.2.6 A vector field $Y(t)$ along a curve γ is said to be parallel along it if $\frac{DY}{dt} = 0$ for all t.

Christoffel symbols. Consider a coordinate neighbourhood U of p. $(\partial_1, \ldots, \partial_n)$ form a basis for T_qM for all $q \in U$. Therefore $\nabla_{\partial_i}\partial_j$ has an expansion of the form $\nabla_{\partial_i}\partial_j = \sum_k \Gamma_{ij}^k \partial_k$ where Γ_{ij}^k are smooth functions on U. Γ_{ij}^k's are called the Christoffel symbols.

Proposition 1.2.7 *Let ϕ_i, $1 \le i \le n$, be differentiable functions from $[a,b]$ to $\mathbb{R}$ such that*

$$\frac{d\phi_k}{dt} + \sum_{i,j} \gamma'(t)(x_i)\Gamma_{ij}^k(\gamma(t))\phi_j = 0, \quad 1 \le k \le n. \tag{1.2.6}$$

Then $Y : t \mapsto \sum \phi_i(t)\partial_i|_{\gamma(t)}$ is a smoothly varying vector field parallel along γ.

Proof: Choose and fix a point $\gamma(t)$ and a coordinate neighbourhood U around $\gamma(t)$. Let X^i and Y^j be smooth functions on U such that $X^i(\gamma(s)) = \gamma'(s)(x_i)$ and $Y^j(\gamma(s)) = \phi_j(s)$. Now set $X = \sum X^i\partial_i$, $\widetilde{Y} = \sum Y^i\partial_i$. Then X and $\widetilde{Y}$ are smooth vector fields such that $X_{\gamma(s)} = \gamma'(s)$ and $\widetilde{Y}_{\gamma(s)} = Y(s)$ for all $\gamma(s) \in U$. Now

$$\begin{aligned} \nabla_X \widetilde{Y} &= \sum X^i \nabla_{\partial_i}(\sum Y^j\partial_j) \\ &= \sum X^iY^j\nabla_{\partial_i}\partial_j + \sum X^i\partial_i(Y^j)\partial_j \\ &= \sum X^iY^j\Gamma_{ij}^k\partial_k + \sum X^i\partial_i(Y^j)\partial_j. \end{aligned}$$

Hence

$$\begin{aligned} (\nabla_X\widetilde{Y})_{\gamma(s)} &= \sum_k \Big(\sum_{i,j} \gamma'(s)(x_i)\phi_j(s)\Gamma_{ij}^k(\gamma(s)) + \gamma'(s)(Y^k)\Big)\partial_k \\ &= \sum_k \Big(\sum_{i,j} \gamma'(s)(x_i)\phi_j(s)\Gamma_{ij}^k(\gamma(s)) + (Y^k \circ \gamma)'\Big)\partial_k \\ &= \sum_k \Big(\sum_{i,j} \gamma'(s)(x_i)\phi_j(s)\Gamma_{ij}^k(\gamma(s)) + \phi_k'\Big)\partial_k. \end{aligned}$$

Thus (1.2.6) is precisely the condition that $(\nabla_{\gamma'(t)}Y(t))_{\gamma(t)} = 0$. □

Proposition 1.2.8 *Let M be a smooth manifold equipped with a connection (or, equivalently, a covariant derivative). Let $\gamma : J \to M$ be a smooth curve. Then for any $s, t \in J$, there exists an isomorphism τ_{st} (which depends on the curve γ) from $T_{\gamma(s)}M$ to $T_{\gamma(t)}M$ such that for any fixed s and a tangent vector $v \in T_{\gamma(s)}M$, $Y(t) := \tau_{st}v$ is the unique smoothly varying vector field parallel along γ.*

Proof: Choose any vector $y = \sum y_k \partial_k \in T_{\gamma(s)}M$. Solve the system of differential equations (1.2.6) with initial conditions $\phi_k(s) = y_k$, $1 \le k \le n$, to get ϕ_k^y and the vector field $Y^y(t) = \sum \phi_k^y(t)\partial_k$. Define

$$\tau_{st} : y \mapsto Y^y(t).$$

□

The family $\tau = \{\tau_{st}\}$ is called the *parallel transport* along γ.

Let X be a smooth vector field, γ a smooth curve passing through p such that $\gamma'(0) = X_p$, τ the parallel transport along γ and $Y(t)$ a smooth vector field along γ. Then the covariant derivative and the connections on M are related to τ via the following relations:

$$\begin{aligned} \frac{DY}{dt} &= \lim_{h\to 0} \frac{\tau_{t\,t+h}^{-1}(Y(t+h)) - Y(t)}{h}, \\ (\nabla_X Y)(p) &= \lim_{h\to 0} \frac{\tau_{0\,h}^{-1}(Y(h)) - Y(0)}{h}. \end{aligned}$$

We have discussed the notions of connection, covariant derivative and parallel transport, and have seen that they are equivalent concepts. But we have so far been quiet about the question of their existence. We will see in the next section that any smooth manifold admits a connection (and hence covariant derivative and parallel transport along curves).

1.3 Riemannian manifolds

We next introduce the notion of a Riemannian metric on a smooth manifold.

Definition 1.3.1 A *Riemannian metric* on a smooth manifold M is a family of symmetric positive bilinear forms $\langle\cdot,\cdot\rangle_p$ on T_pM that varies smoothly with p (meaning, the function $p \mapsto \langle \partial_i|_p, \partial_j|_p\rangle_p$ is smooth in a neighbourhood of p). A smooth manifold equipped with a Riemannian metric is called a *Riemannian manifold.*

Exercise. Show that in a Riemannian manifold, the function $p \mapsto \langle X_p, Y_p\rangle_p$ is smooth for all smooth vector fields X and Y.

When there is no ambiguity, we will omit the subscript 'p'.

Definition 1.3.2 Let M and N be two Riemannian manifolds. A map $f : M \to N$ is called an *isometry* if f is a diffeomorphism and $df(p) : (T_pM, \langle\cdot,\cdot\rangle) \to (T_{f(p)}N, \langle\cdot,\cdot\rangle)$ is an isometry.

Examples.

1. $M = \mathbb{R}^n$ with $\langle \partial_i, \partial_j \rangle = \delta_{ij}$,

2. $M = S^n$ with $\langle \cdot, \cdot \rangle$ being the inner product inherited from $\mathbb{R}^{n+1}$.

3. $M = H^2$ with the metric given by

$$\begin{aligned} \langle \partial_x, \partial_x \rangle_{(x,y)} &= \langle \partial_y, \partial_y \rangle_{(x,y)} &= y^{-2}, \\ \langle \partial_x, \partial_y \rangle_{(x,y)} &= \langle \partial_y, \partial_x \rangle_{(x,y)} &= 0. \end{aligned}$$

4. $M = D$ with the metric given by

$$\begin{aligned} \langle \partial_x, \partial_x \rangle_{(x,y)} &= \langle \partial_y, \partial_y \rangle_{(x,y)} &= \frac{1}{(1-(x^2+y^2))^2}, \\ \langle \partial_x, \partial_y \rangle_{(x,y)} &= \langle \partial_y, \partial_x \rangle_{(x,y)} &= 0. \end{aligned}$$

The last two examples illustrate that the same smooth manifold can have different Riemannian structures, and hence can be different as Riemannian manifolds.

Lemma 1.3.3 *A smooth manifold can always be given a Riemannian structure.*

Proof: Let $\{\xi_\alpha\}_{\alpha \in \Lambda}$ be a partition of unity in M. Consider a system of coordinate charts (U_α, ϕ_α) such that $\operatorname{supp} \xi_a lpha \subseteq U_\alpha$. Take any $p \in M$. Then $p \in U_\alpha$ for some α. Let $x_1^\alpha, \ldots, x_n^\alpha$ be the coordinate functions with respect to (U_α, ϕ_α). Define for v_p and w_p in T_pM, $\langle v_p, w_p \rangle^{(\alpha)} := \sum v_p(x_i^\alpha) w_p(x_i^\alpha)$, and

$$\langle v_p, w_p \rangle := \sum \xi_\alpha(p) \langle v_p, w_p \rangle^{(\alpha)}.$$

This defines a Riemannian metric on M. (**Check!**). □

Riemannian connection.

We have observed that the notion of a connection gives rise to the notion of parallel vector fields along curves. Now suppose we have a curve γ on a Riemannian manifold M, and two vector fields X and Y parallel along γ. Then it is natural to expect the angle between them to be preserved by the parallel translation, i.e.

$$\langle X, Y \rangle\rangle_{\gamma(t)} = \text{constant}. \tag{1.3.7}$$

This is the compatibility condition. In other words, we say that a connection on a Riemannian manifold is a Riemannian connection if for any smooth curve γ and any two parallel vector fields X and y along γ, the quantity $\langle X, Y \rangle\rangle_{\gamma(t)}$ is independent of t.

Proposition 1.3.4 *A connection ∇ is Riemannian if and only if for any smooth curve γ and vector fields X and Y along γ, we have*

$$\gamma'(t)\langle X, Y\rangle = \langle \nabla_{\gamma'(t)}X, Y\rangle + \langle X, \nabla_{\gamma'(t)}Y\rangle. \tag{1.3.8}$$

Proposition 1.3.5 *A connection ∇ is Riemannian if and only if for any three vector fields X, Y and Z, one has*

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle. \tag{1.3.9}$$

Proof of proposition 1.3.4: Choose a point t_0 in the domain of γ and an orthonormal basis $\{v_1, \ldots, v_n\}$ of $T_{\gamma(t_0)}M$. Write $V_i(t) = \tau_{t_0 t}v_i$. Then by (1.3.7), $\{V_1(t), \ldots, v_n(t)\}$ is an orthonormal basis for $T_{\gamma(t)}M$ for all t. Write $X_{\gamma(t)} = \sum X^i(t)V_i(t)$ and $Y_{\gamma(t)} = \sum Y^i(t)V_i(t)$. Then $\langle X, Y\rangle_{\gamma(t)} = \sum X^i(t)Y^i(t)$, $(\nabla_{\gamma'}X)_{\gamma(t)} = \sum(\frac{dX^i}{dt})V_i(t)$, and $(\nabla_{\gamma'}Y)_{\gamma(t)} = \sum(\frac{dY^i}{dt})V_i(t)$, so that (1.3.8) is satisfied. The other implication is straightforward. $\square$

A connection ∇ is said to be symmetric if

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{1.3.10}$$

Theorem 1.3.6 *Any Riemannian manifold admits a unique symmetric connection compatible with its metric.*

Proof: Using (1.3.9) and (1.3.10) repeatedly, one gets the following:

$$\begin{aligned} \langle \nabla_X Y, Z\rangle &= \frac{1}{2}\Big(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &\qquad + \langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle - \langle [Y, Z], X\rangle\Big). \end{aligned} \tag{1.3.11}$$

Take $X = \partial_i$, $Y = \partial_j$ and $Z = \partial_r$ to get

$$\sum_k \Gamma^k_{ij} g_{kr} = \frac{1}{2}(\partial_i(g_{jr}) + \partial_j(g_{ir}) - \partial_r(g_{ij})).$$

Denoting by $((g^{ij}))$ the inverse of the matrix $((g_{ij}))$, we obtain

$$\Gamma^k_{ij} = \frac{1}{2}(\partial_i(g_{jr}) + \partial_j(g_{ir}) - \partial_r(g_{ij}))g^{rk}. \tag{1.3.12}$$

$\square$

Remark 1.3.7 *Equation (1.3.11) defines a connection uniquely on any pseudo-riemannian manifold.*

1.4 Computations

Consider example 4 in section 1.3. Let us compute the Christoffel symbols for This manifold. We know that

$$\begin{aligned} g_{ij}(x_1,x_2) &= (1-x_1^2-x_2^2)^{-2}\delta_{ij}, \\ g^{ij}(x_1,x_2) &= (1-x_1^2-x_2^2)^{2}\delta_{ij}. \end{aligned}$$

Therefore

$$\frac{\partial g_{ii}}{\partial x_j} = 4x_j(1-x_1^2-x_2^2)^{-3}.$$

Using equation (1.3.12), we obtain

$$\begin{aligned} \Gamma_{ij}^k &= \frac{1}{2}(1-x_1^2-x_2^2)^2\left(\frac{\partial g_{ki}}{\partial x_j}+\frac{\partial g_{kj}}{\partial x_i}-\frac{\partial g_{ij}}{\partial x_k}\right) \\ &= \begin{cases} \frac{2x_1}{1-x_1^2-x_2^2} & \text{if } i=1, j=1, k=1, \\ -\frac{2x_2}{1-x_1^2-x_2^2} & \text{if } i=1, j=1, k=2, \\ \frac{2x_2}{1-x_1^2-x_2^2} & \text{if } i=1, j=2, k=1, \\ \frac{2x_1}{1-x_1^2-x_2^2} & \text{if } i=1, j=2, k=2, \\ \frac{2x_2}{1-x_1^2-x_2^2} & \text{if } i=2, j=1, k=1, \\ \frac{2x_1}{1-x_1^2-x_2^2} & \text{if } i=2, j=1, k=2, \\ -\frac{2x_1}{1-x_1^2-x_2^2} & \text{if } i=2, j=2, k=1, \\ \frac{2x_2}{1-x_1^2-x_2^2} & \text{if } i=2, j=2, k=2. \end{cases} \end{aligned} \tag{1.4.13}$$

Let us next consider the following two curves on D:

$$\begin{aligned} \gamma(t) &= \left(\frac{1}{\sqrt{2}}\cos t, -\frac{1}{\sqrt{2}}\sin t\right), \quad 0 \le t \le \pi, \\ \sigma(t) &= \left(\frac{1}{\sqrt{2}}-t, 0\right), \quad 0 \le t \le \sqrt{2}. \end{aligned}$$

We will compute the parallel transport from $T_{(\frac{1}{\sqrt{2}},0)}D$ to $T_{(-\frac{1}{\sqrt{2}},0)}D$ along the curves

γ and σ. Let us first focus our attention on γ. Using equation (1.4.13), we get

$$\Gamma_{ij}^k(\gamma(t)) = \begin{cases} 2\sqrt{2}\cos t & \text{if } i=1, j=1, k=1, \\ 2\sqrt{2}\sin t & \text{if } i=1, j=1, k=2, \\ -2\sqrt{2}\sin t & \text{if } i=1, j=2, k=1, \\ 2\sqrt{2}\cos t & \text{if } i=1, j=2, k=2, \\ -2\sqrt{2}\sin t & \text{if } i=2, j=1, k=1, \\ 2\sqrt{2}\cos t & \text{if } i=2, j=1, k=2, \\ -2\sqrt{2}\cos t & \text{if } i=2, j=2, k=1, \\ -2\sqrt{2}\sin t & \text{if } i=2, j=2, k=2. \end{cases}$$

Since $\gamma'(t)x_1 = -\frac{1}{\sqrt{2}}\sin t$ and $\gamma'(t)x_2 = -\frac{1}{\sqrt{2}}\cos t$, the system of differential equations (1.2.6) now reads

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}' = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Using proposition 1.2.8, we obtain

$$\tau_{0t}^{\gamma}(y) = \frac{1}{1+4t^2} \begin{pmatrix} 1 & 2t \\ -2t & 1 \end{pmatrix} y.$$

In particular,

$$\tau_{0\pi}^{\gamma}(y) = \frac{1}{1+4\pi^2} \begin{pmatrix} 1 & 2\pi \\ -2\pi & 1 \end{pmatrix} y.$$

Doing the same calculations for the curve σ, we get

$$\Gamma_{ij}^k(\sigma(t)) = \begin{cases} \frac{2\sqrt{2}-4t}{1-2t^2+2\sqrt{2}t} & \text{if } i=1, j=1, k=1, \\ 0 & \text{if } i=1, j=1, k=2, \\ 0 & \text{if } i=1, j=2, k=1, \\ \frac{2\sqrt{2}-4t}{1-2t^2+2\sqrt{2}t} & \text{if } i=1, j=2, k=2, \\ 0 & \text{if } i=2, j=1, k=1, \\ \frac{2\sqrt{2}-4t}{1-2t^2+2\sqrt{2}t} & \text{if } i=2, j=1, k=2, \\ -\frac{2\sqrt{2}-4t}{1-2t^2+2\sqrt{2}t} & \text{if } i=2, j=2, k=1, \\ 0 & \text{if } i=2, j=2, k=2. \end{cases}$$

This, together with $\sigma'(t)x_1 = -1$, $\sigma'(t)x_2 = 0$ gives the following system of differential equations:

$$\begin{aligned} \phi_1' &= -\frac{2\sqrt{2}-4t}{1-2t^2+2\sqrt{2}t}\phi_1, \\ \phi_2' &= -\frac{2\sqrt{2}-4t}{1-2t^2+2\sqrt{2}t}\phi_2. \end{aligned}$$

Solving these and using proposition 1.2.8 again, we get

$$\tau^{\sigma}_{0t}(y) = \frac{1}{2(1-(\frac{1}{\sqrt{2}}-t)^2)}y,$$

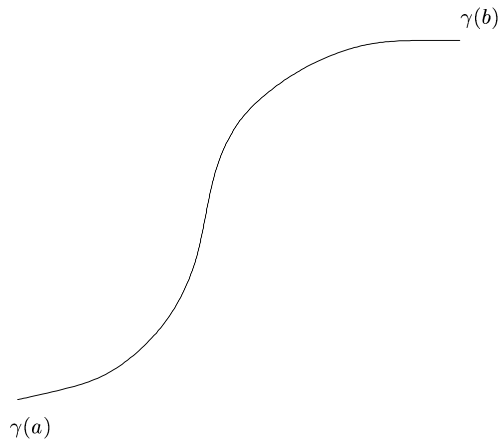
and in particular,

$$\tau^{\sigma}_{0\sqrt{2}}(y) = y.$$

Thus the parallel transports from $T_{(\frac{1}{\sqrt{2}},0)}D$ to $T_{(-\frac{1}{\sqrt{2}},0)}D$ along the curves γ and σ are clearly different.

1.5 Geodesics

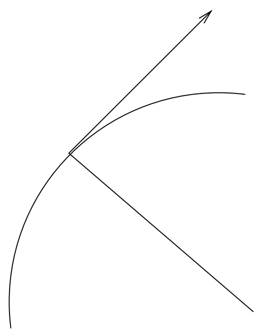
For a curve γ on M, $\gamma'(t)$ gives the rate of change along γ at the point $\gamma(t)$. If we think of $\gamma(t)$ as the position of a particle at time t, then $\gamma'(t)$ gives its *velocity* at t. The magnitude is given by $\|\gamma'(t)\|$, which is its speed at t. Therefore the distance covered by the particle from time a to time b is given by $\int_a^b \|\gamma'(t)\|dt$. This is the *length* of the curve γ from a to b.



Once we have the notion of the length of a curve, we can introduce the following metric on each connected component of a Riemannian manifold:

$$d(p,q) = \inf\{\text{length of } \gamma : \gamma \text{ is a piecewise smooth curve from } p \text{ to } q\}.$$

For a curve γ, γ' gives its velocity, and $\nabla_{\gamma'}\gamma'$ gives its acceleration. The terminology will be clearer if one keeps in mind the following analogy.



Suppose M is a manifold where a particle is moving around smoothly. If we plot its position against time, we get a smooth curve γ. Now, $\gamma'(t)$ is the rate at which its position changes with time, i.e. its velocity ('direction' being taken into account in an appropriate manner). How this velocity changes is obviously its acceleration, which is given by $\nabla_{\gamma'}\gamma'$.

A curve with zero acceleration is called a *geodesic*.

One immediate outcome of the definition is that, for a geodesic γ, the quantity $\|\gamma'(t)\|$ is constant, and so the arclength from t_1 to t_2 is given by $\int_{t_1}^{t_2} \|\gamma'(t)\| dt = \text{const} \cdot (t_2 - t_1)$.

Let $p \in M$, U be a coordinate neighbourhood around p and γ be a curve passing through p. Let us now rewrite the condition $\nabla_{\gamma'}\gamma' = 0$ in local terms. We have $\gamma'(t) = \sum \gamma'(t)(x_i)\partial_i$; hence

$$\begin{aligned} \nabla_{\gamma'}\gamma' &= \sum_{i,j} \gamma'(t)(x_i)\gamma"(t)(x_j)\nabla_{\partial_i}\partial_j + \sum_j \gamma'(t)(\gamma'(t)x_j)\partial_j \\ &= \sum_{i,j,k} \frac{d}{dt}(x_i \circ \gamma)\frac{d}{dt}(x_j \circ \gamma)\Gamma_{ij}^k \partial_k + \sum \frac{d^2}{dt^2}(x_j \circ \gamma)\partial_j, \end{aligned}$$

so that $\nabla_{\gamma'}\gamma' = 0$ means

$$\sum \frac{d^2}{dt^2}(x_k \circ \gamma) + \sum_{i,j} \frac{d}{dt}(x_i \circ \gamma)\frac{d}{dt}(x_j \circ \gamma)\Gamma_{ij}^k \circ \gamma = 0.$$

Now this system of second order differential equations with the initial conditions

$$x_k \circ \gamma(0) = x_k(p), \quad 1 \le k \le n$$

admits a solution. This means each point p of M has a geodesic passing through it in some neighbourhood.

We can actually prove something stronger. Given any nonzero vector $v_p \in T_pM$, we can find a geodesic γ passing through p for which $\gamma'(0) = v_p$. To see this, one has to rewrite the system of differential equations in the following way: write f_k for $x_k \circ \gamma$ and g_k for $\frac{d}{dt}(x_k \circ \gamma)$. Then we have

$$\begin{aligned} \frac{df_k}{dt} &= g_k, \\ \frac{dg_k}{dt} &= -\sum_{i,j}(\Gamma_{ij}^k \circ \gamma) g_i g_j, \quad 1 \le k \le n, \end{aligned} \tag{1.5.14}$$

with initial conditions

$$\begin{aligned} f_k(0) &= x_k(p), \\ g_k(0) &= v_p(x_k), \quad 1 \le k \le n. \end{aligned}$$

Choose the coordinate neighbourhood U in such a manner that

$$\phi(U) = \{(x_1, \ldots, x_n) : |x_i - x_i(p)| < c\},$$

where c is some positive number. For $0 < c_1 < c$ and $0 < K < \infty$, in the interval $|f_i| < c_1$, $|g_i| < K$, $1 \le i \le n$, the right hand side of the above equations satisfy a Lipschitz condition. From the theory of differential equations, one can then conclude that *the system (1.5.14) admits a unique solution on some interval* $(-\epsilon, \epsilon)$ *such that* $|f_i(t)| < c_1$, $|g_i(t)| < K$ *for all* i, $t \in (-\epsilon, \epsilon)$. It is clear then that there is a geodesic $\gamma : J \to M$ passing through p with $\gamma'(0) = v_p$. If $\gamma_1 : J_1 \to M$ and $\gamma_2 : J_2 \to M$ are two such geodesics, by the uniqueness of solutions of a system of differential equations, they must coincide on $J_1 \cap J_2$. It follows then that there there exists a unique maximal geodesic $\gamma : J \to M$ such that $\gamma(0) = p$, $\gamma'(0) = v_p$. (A *maximal geodesic* is one that is not a proper restriction of any other geodesic).

1.6 Exponential Map

Theorem 1.6.1 *Let M be a minifold with a connection and p be a point in M. Then there exists an open neighbourhood N_0 of 0 in T_pM and an open neighbourhood of p in M such that the map $v \mapsto \gamma_v(1)$ is a diffeomorphism of N_0 onto N_p.*

The map described in the above theorem is called the *exponential map*, and one writes $\mathrm{Exp}_p v$ for $\gamma_v(1)$.

An open neighbourhood N_0 of 0 in T_pM is said to be a *normal neighbourhood* if

1. Exp maps N_0 diffeomorphically onto some neighbourhood of p,
2. N_0 is star-shaped (i.e. $u \in N_0$ and $0 \le t \le 1$ implies $tu \in N_0$).

1.7 Curvature

Let M be a Riemannian manifold of dimension two, and p be a point in M. Denote by $N_r(0)$ the ball of radius r around 0 in T_pM, and let $B_r(p)$ be its image under the exponential map. The sectional curvature of M at p is defined to be the following limit:

$$\kappa = \lim_{r\to 0} 12 \frac{\operatorname{area} N_r(0) - \operatorname{area} B_r(p)}{\operatorname{area} N_r(0)}.$$

Let M be a manifold with affine connection ∇. For $X, Y \in \mathcal{D}^1$, write

$$\begin{aligned} T(X,Y) &= \nabla_X Y - \nabla_Y X - [X,Y], \\ R(X,Y) &= \nabla_X \nabla_Y - \nabla_Y \nabla_X \nabla_{[X,Y]}. \end{aligned}$$

The following map from $\mathcal{D}_1 \times \mathcal{D}^1 \times \mathcal{D}^1$ to $C^\infty(M)$ is customarily denoted by the same symbol T, and is called the *torsion tensor field*:

$$(\omega, X, Y) \mapsto \omega(T(X,Y)).$$

The *curvature tensor field* is the map

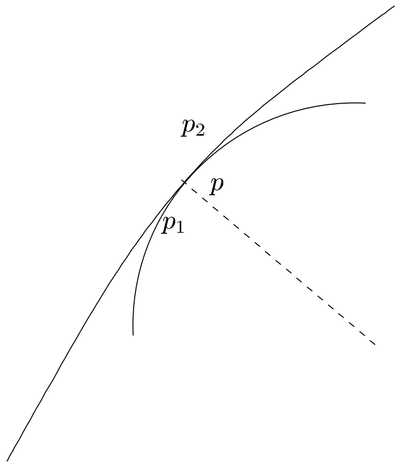
$$(\omega, X, Y, Z) \mapsto \omega(R(X,Y)Z)$$

from $\mathcal{D}_1 \times \mathcal{D}^1 \times \mathcal{D}^1 \times \mathcal{D}^1$ to $C^\infty(M)$, and is again denoted by the same symbol R.

Theorem 1.7.1 *Let M be a Riemannian manifold with curvature tensor field R. Let $p \in M$, S a two-dimensional subspace of T_pM and x, y be two independent vectors in S. Denote by $x \vee y$ denote the parallelogram enclosed by x and y. Then the sectional curvature of M at p along the section S is given by*

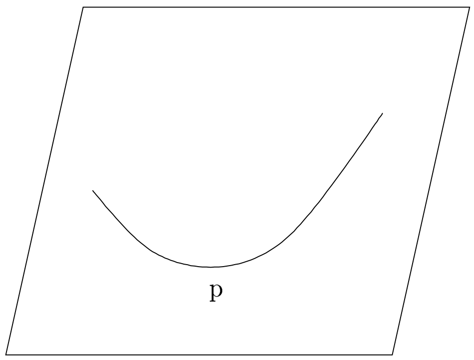
$$\kappa_S = -\frac{\langle R(x,y)x, y\rangle}{(\operatorname{area} x \vee y)^2}.$$

There is another more intuitive notion of curvature. Let us first consider the curvature of a curve in two dimensions. Take a curve γ passing through a point p. Now look at the circle that best fits γ at the point p (take two points p_1 and p_2 on the curve in a neighbourhood of p, and look at the circle passing through p, p_1 and p_2.



As p_1 and p_2 are brought arbitrarily close to p, this circle tends to a limiting circle). Its radius r gives a measure of curvature of γ at p. Of course, we want the curvature of a straight line to be zero; so we set $\kappa = 1/r$.

Now take any surface M embedded in $\mathbb{R}^3$ and a point $p \in M$. Taking any curve on M passing through p and measuring its curvature gives a measurement of curvature of M at the point p (along the direction γ). But obviously, we should not take into consideration all curves. We just look at curves that lie on a plane passing through the normal at p. Once we fix such a plane, the curve iis fixed too (except for reparametrization), and we have a value of curvature along that plane.



What is amazing is that for a point p, the minimum and the maximum values of this curvature are attained along two perpendicular planes. What is even more amazing is that no matter what surface you take, its curvature at a point p along a plane depends just on the angle it makes with the two perpendicular planes mentioned above. More specifically, if the plane makes an angle θ with the plane along which curvature is the minimum, then the curvatue along this plane is

$$\kappa_{\text{min}} \cos^2 \theta + \kappa_{\text{max}} \sin^2 \theta.$$

The sectional curvature of M at p turns out to be just the product $\kappa_{\text{min}} \cdot \kappa_{\text{max}}$.