Topology

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Contents

Here I have tried to put together a set of notes that I used for a course on Topology given for the M. Stat. second year students here at the Indian Statistical Institute during July–November, 2003. After covering standard material on general topology in the first half, I spent some time on quotient spaces, manifolds and finished part of the proof of the classification theorem for compact surfaces. I have spent very little time on metric spaces because the students already had taken a half course on metric spaces in their first year.

This would be followed up by a course on Algebraic topology in the next semester in which I plan to introduce homotopy, homology and complete the proof of the classification theorem for surfaces.

I drew plenty of help from the notes by Neil Strickland available on his webpage (`http://www.shef.ac.uk/~pm1nps/`), and from a book by Oleg Viro et al, which also is available on the net (`http://www.math.uu.se/~oleg/`).

Course outline. Topological space: definition,subspace topology, metric topology, examples, bases and subbases, product topology, quotient topology;

How to distinguish between two topological spaces – continuous map, open map, homeomorphism.

Separation properties: Hausdorff, normal, regular;

Compactness: sequential compactness, Heine-Borel property, Bolzano-Weierstrass property, continuous bijections from compact space to Hausdorff space are homeomorphisms, nets and filterbases, Tychonoff's theorem. Various compactifications. Locally compact spaces.

Connectedness: connectedness and path connectedness, components, number of connected components as an invariant;

Metric spaces: completeness, extension of continuous maps to completeions, separability, countability properties of the topology. Baire category theorem and applications. Metrizability of product spaces.

Construction of continuous functions: Urysohn lemma, Tietze's extension theorem, partition of unity and their usefulness (bundles/vector bundles), paracompactness.

Classification of compact connected surfaces.

Part I

1 Preliminaries

1.1 Set theoretic preliminaries

Axiom of choice.

First form *Let $\mathscr{C}$ be a collection of disjoint nonempty sets. Then there exists a set C such that $C \cap A$ is singleton for all $A \in \mathscr{C}$.*

Second form *Let $\mathscr{C}$ be a collection of nonempty sets (not necessarily disjoint). Then there exists a function $\gamma : \mathscr{C} \to \cup_{A \in \mathscr{C}} A$ such that $\gamma(A) \in A$ for all $A \in \mathscr{C}$.*

Definition 1.1 A relation '$\leq$' on a set A is called a **pre-order** if

1. $a \leq a \ \forall \ a \in A$,
2. $a \leq b$ and $b \leq c \Rightarrow a \leq c, \ \forall \ a, b, c \in A$.

It is called a **partial order** if in addition, one has

3. $a \leq b$ and $b \leq a \Rightarrow a = b$.

A pre-ordered set is called a **chain** if any two elements can be compared.

A partially ordered set that is also a chain is called a **totally ordered** set.

A relation '$<$' on a set A is called an **order relation** if it satisfies

1. $a \neq b \Rightarrow a < b$ or $b < a$ (but not both),
2. $a < b \Rightarrow a \neq b$,
3. $a < b$ and $b < c \Rightarrow a < c$.

A set equipped with an order relation is said to be **simply ordered**.

A set is said to be **well-ordered** if every nonempty set has a first (smallest) element.

Zermelo's theorem *Every nonempty set can be well-ordered.*

Zorn's lemma *Let X be preordered. If each chain in X has an upper bound, then X has at least one maximal element.*

Theorem 1.2 *Axiom of choice, Zermelo's theorem and Zorn's lemma are equivalent.*

Some applications.

1. existence of Hamel basis in vector spaces,
2. any ideal in a commutative ring with identity is contained in a maximal ideal,
3. nonemptyness of infinite cartesian products.

LECTURE 1

2 Topological spaces

The goal in most areas in mathematics, crudely speaking, is to classify various 'spaces'. In most situations, these spaces would be sets of points, and to distinguish between two sets, most natural thing to look at is the 'number of elements' or the cardinality. But this invariant is too crude and primitive and does not work on most occasions. What does one do then? Fortunately, all the spaces that occur usually come with some extra structures. One can then look at these structures or properties and try and distinguish between two given spaces. This structure could be algebraic — for example the space could be a group, or a ring, or a field. Or the space could be equipped with some order relation. In what follows, we will study the notion of topology, which in most situations spaces come naturally equipped with. And see how one can go about classifying spaces on the basis of their topological properties.

2.1 Definition and examples

Definition 2.1 $(X, \mathscr{T})$ is a **topological space** if $\mathscr{T} \subseteq \mathscr{P}(X)$ and obeys

1. ϕ, $X \in \mathscr{T}$,
2. $\mathscr{T}$ is closed under arbitrary unions and finite intersections.

Let $\mathscr{T}_1$, $\mathscr{T}_2$ be two topologies on X. If $\mathscr{T}_1 \subseteq \mathscr{T}_2$, then one uses one of the following statements:

1. $\mathscr{T}_1$ is coarser than $\mathscr{T}_2$,
2. $\mathscr{T}_1$ is weaker than $\mathscr{T}_2$,
3. $\mathscr{T}_2$ is finer than $\mathscr{T}_1$,
4. $\mathscr{T}_2$ is stronger than $\mathscr{T}_1$.

Before going into examples, let us introduce two other notions that will help us understand better some of the examples. A **metric space** is a set X along with a function $d : X \times X \to \mathbb{R}_+$ obeying the following three properties: (1) $d(x, y) = d(y, x)$, (2) $d(x, y) = 0 \Rightarrow x = y$, (3)

$d(x,z) \leq d(x,y) + d(y,z)$. An ϵ-ball around a point $x \in X$ is the set $\{y \in X : d(x,y) < \epsilon\}$ and is denoted by $N_\epsilon(x)$. A point $x \in E \subseteq X$ is an **interior point** of E if $N_\epsilon(x) \subseteq E$ for some $\epsilon > 0$. A set $U \subseteq X$ is **open** if every point of U is an interior point. Check that this defines a topology on X.

Suppose $(X, \mathscr{T}_X)$ is a topological space, and let $Y \subseteq X$. Define $\mathscr{T}_Y$ to be the following subcollection of $\mathscr{P}(Y) : \{U \cap Y : U \in \mathscr{T}_X\}$. This defines a topology on Y. This is called the **subspace topology** on Y inherited from X.

Examples.

1. X: any set, $\mathscr{T} = \{\phi, X\}$.

2. X: set, $\mathscr{T} = \mathscr{P}(X)$.

3. $X = \mathbb{R}$, with the usual metric $d(x,y) = |x - y|$.

4. Same as above, with $X = \mathbb{R}^n$, or $X = \mathbb{C}^n$.

5. S^n.

6. $\mathbb{D}^n$.

7. $SL(n, \mathbb{R})$.

8. p-adic metric on $\mathbb{Z}$, $\mathbb{Q}$.

Exercise 2.2 $\mathscr{T}_\alpha$: family of topologies on a set X. Then $\cap_\alpha \mathscr{T}_\alpha$ is a topology on X.

Give an example where $\cup_\alpha \mathscr{T}_\alpha$ is not a topology.

Let $\mathscr{C} \subseteq \mathscr{P}(X)$. Define $\mathscr{T}_\mathscr{C}$ to be the intersection of all topologies on X that contain X. Show that this is the weakest topology containing $\mathscr{C}$.

2.2 Bases

$\mathscr{B}$ is a **basis for a topology** on X if

1. $x \in X \Rightarrow \exists B \in \mathscr{B}$ with $x \in B$, (i. e. $\cup_{B \in \mathscr{B}} B = X$)

2. if $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, then there is a $B \in \mathscr{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

Let $\mathscr{B}$ be a basis for a topology. Define $\mathscr{T} \subseteq \mathscr{P}(X)$ as follows: $U \in \mathscr{T}$ if for all $x \in U$, there is a $B \in \mathscr{B}$ with $x \in B \subseteq U$. Check that

1. $\mathscr{T}$ is a topology,

2. $\mathscr{T} = \mathscr{T}_\mathscr{B}$,

3. $\mathscr{T} = \{$ arbitrary unions of finite intersections of elements from $\mathscr{B}\}$,

4. $\mathscr{T} = \{$ arbitrary unions of elements from $\mathscr{B}\}$.

Exercise 2.3 1. $\mathscr{T}$: topology on X. $\mathscr{B} \subseteq \mathscr{T}$. Then $\mathscr{B}$ is a basis for $\mathscr{T}$ if and only if for all $U \in \mathscr{T}$ and for all $x \in U$, there is a $B \in \mathscr{B}$ for which $x \in B \subseteq U$.

2. Show that $\{N_\epsilon(x) : \epsilon > 0, x \in \mathbb{C}^n\}$ is a basis for the standard topology on $\mathbb{C}^n$.

Exercise 2.4 1. $\mathscr{C} \subseteq \mathscr{P}(X)$, $\cup_{A\in\mathscr{C}} A = X$. Define $\mathscr{B} = \{\text{all finite intersections of elements of } \mathscr{C}\}$. Then $\mathscr{B}$ is a basis for a topology on X.
2. $\mathscr{T}_{\mathscr{B}}$ is the smallest topology containing $\mathscr{C}$.

2.3 More examples

1. Lower limit topology on $\mathbb{R}$.
2. Order topology on a simply ordered set.
3. Product topology on $X \times Y$.
4. Product topology on infinite cartesian products.
5. Quotient topology.

Exercise 2.5 Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be bases for topologies $\mathscr{T}_1$ and $\mathscr{T}_2$. Then $\mathscr{T}_1$ is finer than $\mathscr{T}_2$ if and only if for each $x \in X$ and $x \in B_2 \in \mathscr{B}_2$, there is a $B_1 \in \mathscr{B}_1$ such that $x \in B_1 \subseteq B_2$.

Using this, show that the lower limit topology on $\mathbb{R}$ is strictly finer than the order (or standard) topology.

2.4 Closed sets, limit points etc

Definition 2.6 Let $E \subseteq X$. The **closure** $\bar{E}$ of E is defined to be the intersection of all closed sets containing E. A set E is said to be **dense** (in X) if $\bar{E} = X$. The **boundary** ∂E of E is defined to be the set $\bar{E} \cap \bar{E^c}$.

A point $x \in X$ is said to be a **limit point** of E if every neighbourhood of x contains a point of E other than x, i.e. if for every neighbourhood U of x, the set $U \cap E\backslash\{x\}$ is nonempty. One denotes the set of limit points of a set E by E'.

A point $x \in E$ is called an **interior point** of E if there is an open neighbourhood U of x such that $U \subseteq E$. The interior $\operatorname{int} E$ of E is the set of all its interior points.

Exercise 2.7 Show that

1. $\bar{E} = E \cup E'$; E is closed if and only if $E' \subseteq E$.
2. $E \subseteq F \Rightarrow \bar{E} \subseteq \bar{F}$,
3. $E'' \subseteq E \cup E'$, $\bar{\bar{E}} = \bar{E}$,
4. $(E \cup F)' = E' \cup F'$, but in general, $\cup_i E_i' \subsetneqq (\cup_i E_i)'$,
5. $\overline{(E \cup F)} = \overline{E} \cup \overline{F}$, but in general, $\cup_i \overline{E_i} \subsetneqq \overline{(\cup_i E_i)}$.
6. $\operatorname{int} E$ is the union of all open sets contained in E,
7. $\partial E = \overline{E} \backslash \operatorname{int} E$.

Lecture 2

2.5 Specifying topology in terms of closure, limit points, boundary and interior

Theorem 2.8 *Let X be a set, $c : \mathscr{P}(X) \to \mathscr{P}(X)$ be a map satisfying the following:*

1. $c(\phi) = \phi$,
2. $E \subseteq c(E)$ *for all* $E \subseteq x$,
3. $c(c(E)) = c(E)$ *for all* $E \subseteq X$,
4. $c(E \cup F) = c(E) \cup c(F)$ *for all* $E, F \subseteq X$.

Then the collection $\mathscr{T} = \{c(E)^c : E \subseteq X\}$ is a topology on X and $c(E) = \bar{E}$ with respect to this topology.

Theorem 2.9 *Let X be a set, $\ell : \mathscr{P}(X) \to \mathscr{P}(X)$ be a map satisfying the following:*

1. $\ell(\phi) = \phi$,
2. $\ell(\ell(E)) \subseteq E \cup \ell(E)$ *for all* $E \subseteq X$,
3. $\ell(E \cup F) = \ell(E) \cup \ell(F)$ *for all* $E, F \subseteq X$,
4. *for each* $x \in X$, $x \notin \ell(\{x\})$.

Then the collection $\mathscr{T} = \{(E \cup \ell(E))^c : E \subseteq X\}$ is a topology on X and $\ell(E) = E'$ with respect to this topology.

Theorem 2.10 *Let X be a set, $i : \mathscr{P}(X) \to \mathscr{P}(X)$ be a map satisfying the following:*

1. $i(X) = X$,
2. $i(E) \subseteq E$,
3. $i(i(E)) = i(E)$,
4. $i(E \cap F) = i(E) \cap i(F)$ *for all* $E, F \subseteq X$.

Then the collection $\mathscr{T} = \{i(E) : E \subseteq X\}$ is a topology on X and $i(E) = \mathrm{int}\, E$ with respect to this topology.

Theorem 2.11 *Let X be a set, $b : \mathscr{P}(X) \to \mathscr{P}(X)$ be a map satisfying the following:*

1. $b(\phi) = \phi$,
2. $b(E) = b(E^c)$ *for all* $E \subseteq x$,
3. $b(b(E)) \subseteq b(E)$ *for all* $E \subseteq X$,
4. $E \cap F \cap b(E \cap F) = E \cap F \cap (b(E) \cap b(F))$ *for all* $E, F \subseteq X$.

Then the collection $\mathscr{T} = \{(E \cup b(E))^c : E \subseteq X\}$ is a topology on X and $b(E) = b(E)$ with respect to this topology.

2.6 Continuous maps

Definition 2.12 A map $f : X \to Y$ is said to be **continuous** if $f^{-1}(E)$ is open in X for all open sets E in Y. the map f is said to be **open** if $f(U)$ is open in Y for all open sets U in X, and is said to be **closed** if $f(F)$ is closed in Y for all closed sets F in X. The map f is a **homeomorphism** if it is a bijection, continuous and open.

Theorem 2.13 *Let $f : X \to Y$ be continuous.*

1. *if $g : Y \to Z$ is continuous then $g \circ f$ is continuous.*
2. *if $E \subseteq X$ has the subspace topology, then $f|_E : E \to Y$ is continuous,*
3. *$f : X \to f(X)$ is continuous,*
4. *if f is an open bijection, and $E \subseteq X$, then $f|_E : E \to f(E)$ is a homeomorphism.*

Proof: □

Theorem 2.14 *Let $f : X \to Y$ be a map. The following conditions are equivalent:*

1. *f is continuous,*
2. *$f^{-1}(F)$ is closed for all closed sets F in Y,*
3. *$f^{-1}(V)$ is open for all V belonging to a generating set for the topology of Y,*
4. *for each $x \in X$, and every neighbourhood $V(f(x))$ of $f(x)$, there exists a neighbourhood $U(x)$ of x such that $f(U(x)) \subseteq V(f(x))$,*
5. *$f(\overline{E}) \subseteq \overline{f(E)}$ for all $E \subseteq X$,*

6. $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ *for all* $B \subseteq Y$.

Proof: (2)$\Rightarrow$ (6):
$f^{-1}(\bar{B})$ is closed and $f^{-1}(B) \subseteq f^{-1}(\bar{B})$. Hence $\overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B})$.

(6)$\Rightarrow$ (2):
Let $F \subseteq X$ be closed. Then $\overline{f^{-1}(F)} \subseteq f^{-1}(F) \subseteq \overline{f^{-1}(F)}$. Hence $\overline{f^{-1}(F)} = f^{-1}(F)$ so that $f^{-1}(F)$ is closed.

(1) false $\Rightarrow$ (5) false:
Take $E = f^{-1}(U^c)$.

(2)$\Rightarrow$ (5):
$E \subseteq f^{-1}(f(E)) \subseteq f^{-1}(\overline{f(E)})$, which is closed. Hence $\overline{E} \subseteq f^{-1}(\overline{f(E)})$. Therefore $f(\overline{E}) \subseteq f(f^{-1}(\overline{f(E)})) \subseteq \overline{f(E)}$. □

Theorem 2.15 *Let* $f : X \to Y$ *be a map. The following conditions are equivalent:*

1. f *is open,*
2. $f(\text{int}\, E) \subseteq \text{int}\, f(E)$ *for all* $E \subseteq X$,
3. $f(E)$ *is open for all* E *belonging to a basis for the topology of* X.

Proof: □

Exercise 2.16 Can you give an example of a function f for which $f(E)$ is open for all E belonging to a subbasis but f is not open?

Theorem 2.17 *A bijective map* $f : X \to Y$ *is a homeomorphism if and only if* $f(\overline{E}) = \overline{f(E)}$ *for all* $E \subseteq X$.

Exercise 2.18 Let C be the Cantor set. Show that

1. C is closed,
2. any point of C is its limit point,
3. C does not have any interior point,
4. C is homeomorphic to the product space $\{0, 2\}^{\mathbb{N}}$,
5. there is a continuous surjection from C to $[0, 1]^k$ for any $k \in \mathbb{N}$.

2.7 Convergence

Let X be a topological space. A map $f : \mathbb{N} \to X$ is called a **sequence** in X. A sequence $\{x_n\}$ is said to have a **limit** x (or converges to a point x) if for any neighbourhod U of x, there is an $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. A point x is said to be a **limit point** of $\{x_n\}$ if for any neighbourhod U of x and any $N \in \mathbb{N}$, there is an $n > N$ such that $x_n \in U$.

The following example illustrates that in general, $\{x_n\}$ has a limit point x does not imply that there is a subsequence that converges to x. Let $X = \mathbb{N} \times \mathbb{N} \cup \{p\}$. Define $U \subseteq X$ to be an open neighbourhood of p if there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $\{n\} \times \mathbb{N} \backslash U$ is finite. For a point $(m, n) \in \mathbb{N} \times \mathbb{N}$, call any subset of $\mathbb{N} \times \mathbb{N}$ containing (m, n) an open neighbourhood of (m, n). These will generate a topology on X. Now take a sequence $\{x_n\}$ in $\mathbb{N} \times \mathbb{N}$ such that each element of $\mathbb{N} \times \mathbb{N}$ appears exactly once in the sequence. The p is a limit point of $\{x_n\}$ but there is ano subsequence that converges to p.

Theorem 2.19 *Let $\{x_n\}$ be a sequence in a topological space X. Then x_n converges to x if and only if every subsequence of $\{x_n\}$ has a further subsequence that converges to x.*

Exercise 2.20 Show that almost sure convergence does not in general come from a topology.

A preordered set Λ is called a **directed set** if for any two elements a and b, there is an element c such that $a \leq c$, $b \leq c$. A **net** in X is a map from a directed set Λ to X. A net $\{x_\alpha\}$ is said to have a **limit** x (or converges to a point x) if for any neighbourhod U of x, there is a $\beta \in \Lambda$ such that $x_\alpha \in U$ for all $\alpha \geq \beta$. A point x is said to be a **limit point** of $\{x_\alpha\}$ if for any neighbourhod U of x and any $\beta \in \mathbb{N}$, there is an $\alpha \geq \beta$ such that $x_\alpha \in U$.

A **filterbase** in X is a family $\mathscr{F}$ of subsets of X such that each $E \in \mathscr{F}$ is nonempty, and for any two elements $E, F \in \mathscr{F}$, there is an element $G \in \mathscr{F}$ such that $G \subseteq E \cap F$. A filterbase $\mathscr{F}$ is said to have a **limit** x (or converges to a point x) if for any neighbourhod U of x, there is a $E \in \mathscr{F}$ such that $E \subseteq U$. A point x is said to be a **limit point** of $\mathscr{F}$ if for any neighbourhod U of x and any $E \in \mathscr{F}$, the set $E \cap U$ is nonempty.

There is a natural association between the notions of a net and a filterbase. Starting from a net, if one looks at the colection of tails, one gets a filterbase. Conversely, suppose we have a filterbase $\mathscr{F}$. Define $\Lambda = \{(x, E) : x \in X, E \in \mathscr{F}, x \in E\}$. Declare $(x, E) \leq (y, F)$ if $F \subseteq E$. This makes Λ a directed set. The map $(x, E) \mapsto x$ is a net in X.

2.8 Uses/applications of topology

In Analysis:
construction of continuous(/other) functions using notion of convergence on C[0,1] (or other suitable function spaces),
compare relative sizes of two sets to prove existence of objects with some specified property (Baire category).
As we go along, we will learn about more applications.

In Geometry:
In order to know about the kind of problems one tries to answer in Topology/Geometry, it would be a good idea at this point to have a look at what is perhaps the oldest 'unsolved' problem in Topology:
Poincaré Conjecture.
If M is a compact simply connected 3-manifold having the same homology groups as those of a 3-sphere, then M is homeomorphic to the 3-sphere.

If one replaces the number 3 in the above statement by 'n', one gets what is known as the generalized Poincaré conjecture.

Attempts to prove the Poincaré conjecture has led to a deeper understanding of the topology of manifolds. One of the first incorrect proofs was due to Poincaré himself, stated four years prior to formulation of his conjecture, and to which Poincaré subsequently found a counterexample. In 1934, Whitehead proposed another incorrect proof, then discovered a counterexample (the Whitehead link) to his own theorem. The $n = 1$ case of the generalized conjecture is trivial, the $n = 2$ case is classical and was known to 19th century mathematicians; $n = 4$ was proved by Freedman (1982), for which he was awarded the 1986 Fields medal, $n = 5$ is due to Zeeman (1961); $n = 6$ was established by Stallings(1962), and $n \geq 7$ was shown by Smale in 1961. Smale subsequently extended his proof to include all $n \geq 5$. The $n = 3$ case, the original conjecture, has now (2003) been claimed to have been proved by Russian Mathematician G. Perelman using techniques (Ricci flow) from geometry.

Our goal in this course would be to fully understand the statement of this conjecture.

Lecture 3

2.9 Few loose ends from the previous lectures

Nets and filterbases.

Proposition 2.21 *Let $f : X \to Y$ be a map. The f is continuous if and only if for every $x \in X$, $f(\mathscr{U}(x))$ converges to x.*

Proposition 2.22 *Let $f : X \to Y$ be a map. The f is continuous if and only if for any $x \in X$ and for any filterbase $\mathscr{U}$ converging to $x \in X$, $f(\mathscr{U})$ converges to x.*

Quotient topology. As we study various properties, we will see how product spaces, subspaces and quotient spaces behave with respect to these properties. We will see that quotient spaces can be quite wildly behaved at times, to the extent that at times one may need to go out of the usual topology and geometry and use Hilbert space operators (and operator algebras) to be able to deal with them.

Product and subspace topology revisited.

Proposition 2.23 *Product topology is the weakest topology on the Cartesian product that makes all the projection maps continuous.*

Proposition 2.24 *Let $X = \prod_\alpha X_\alpha$ and Let $b = (b_\alpha) \in X$. Let D denote the set of all points in X which differ from b only at finitely many coordinates. Then D is dense in X.*

Proof: By definition, finite dimensional open cylinder sets form a basis for the product topology. And any such set will contain a point of D. Hence the result follows. $\square$

Proposition 2.25 *Subspace topology is the weakest topology that makes the inclusion map continuous.*

Cantor set.

3 Connectedness

3.1 Definition and properties

Definition 3.1 A space X is **connected** if it is not the union of two disjoint open (proper) subsets.

A subset Y of X is connected if it is connected as a subspace of X.

Proposition 3.2 *Let $f : X \to Y$ be continuous and X be connected. Then $f(X)$ is also connected.*

Proof: Suppose U and V are open sets in Y such that $f(X) \cap U$ and $f(X) \cap V$ are disjoint and $f(X) = (f(X) \cap U) \cup (f(X) \cap V)$. Now $X = f^{-1}(f(X)) = f^{-1}(f(X) \cap U) \cup f^{-1}(f(X) \cap V)$. $f^{-1}(f(X) \cap U)$ and $f^{-1}(f(X) \cap V)$ are disjoint, and as $f^{-1}(f(X) \cap U) = f^{-1}(U)$, $f^{-1}(f(X) \cap V) = \cup f^{-1}(V)$, both are open. Since X is connected, this implies at least one of $f^{-1}(U)$ and $f^{-1}(V)$ has to be empty. □

Exercise 3.3 [Connected subsets of $\mathbb{R}$] Let E be a connected subset of $\mathbb{R}$. Let $a, b \in E$ and $a < c < b$. Show that $c \in E$. Use this to prove that the only connected subsets of $\mathbb{R}$ are the intervals.

Proposition 3.4 *The following are equivalent:*

1. X is connected,

2. only subsets of X that are both open and closed are ϕ and X,

3. there does not exist any surjective continuous map from X to $\{1, 2\}$.

Proof: Equivalence of the first two parts are obvious.

Assume that X is connected, and let $f : X \to \{1, 2\}$ be continuous. $f^{-1}\{1\}$ and $f^{-1}\{2\}$ are disjoint open sets in X, and their union is X. Since X is connected, one of these have to be empty.

Next, assume X is not connected, so that there are two disjoint nonempty open sets U and V such that $X = U \cup V$. Define $f : X \to \{1, 2\}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in U, \\ 2 & \text{if } x \in V \end{cases}.$$

This is a continuous surjection. □

Using this result, one can now give a shorter proof of the earlier proposition. If $f(X)$ is not connected, there would exist a continuous surjection g from $f(X)$ to $\{1, 2\}$. But then $g \circ f$ is a continuous surjection from X to $\{1, 2\}$.

Lemma 3.5 *X_α: family of subsets of X. If each X_α is connected and $\cap X_\alpha \neq \phi$, then $\cup X_\alpha$ is connected.*

Lemma 3.6 *$A \subseteq X$. If A is connected and $A \subseteq B \subseteq \bar{A}$, then B is connected.*

Proof: Take a continuous map g from B to $\{1, 2\}$. Now look at $g|_A$. $g(B) = g(\bar{A}) \subseteq \overline{g(A)}$ (here $\overline{A}$ denotes closure of A in B). Since A is connected, $g(A)$ is singleton. Hence so also is $g(B)$.□

The above lemma tells us that by adding limit points, one can not destroy the connectedness of a set.

Theorem 3.7 $X = \prod X_\alpha$ *is connected if and only if each* X_α *is connected.*

Proof: If $\prod X_\alpha$ is connected, then each X_α is connected, being images under the projection maps.

Let Λ be the index set. Let us first look at the case $|\Lambda| = 2$. Fix a point (x_1, x_2). Now show that for each $y \in X_2$, the space $\{x_1\} \times X_2 \cup X_1 \times \{y\}$ is connected and contains the point (x_1, x_2). Hence their union, which is $X_1 \times X_2$, is also connected. By repeated application, the result now follow for the case $|\Lambda| < \infty$.

Assume next that $|\Lambda| = \infty$. Choose and fix a point $b = (b_\alpha) \in X$. Let $F = \{\alpha_1, \ldots, \alpha_n\}$ be a finite subset of Λ, Denote by X_F the set $\{(x_\alpha) \in X : x_\alpha = b_\alpha$ for all $\alpha \notin F\}$. Show that this is homeomorphic to $X_{\alpha_1} \times \cdots \times X_{\alpha_n}$ and hence is connected. The union of the sets X_F is the set D in proposition 2.24, and is therefore dense in X. Hence by lemma 3.6, X is connected.□

Lecture 4

3.2 Components.

A **component** is a maximal connected subset. We will next give a more direct description of components in X.

Define an equivalence relation $\sim$ on X as follows: $x \sim y$ if there exists a connected subset E of X containing both x and y. Let $C_1(x)$ denote the equivalence class of x, i.e. $C_1(x) = \{y \in X : x \sim y\}$. Define another subset $C_2(x)$ by

$$C_2(x) = \bigcup_{\substack{E \text{ connected} \\ x \in E}} E.$$

By lemma 3.5, $C_2(x)$ is connected.

Lemma 3.8 $C_1(x) = C_2(x)$.

Proof: The inclusion $C_1(x) \subseteq C_2(x)$ follows directly from the definition of $C_1(x)$. For the reverse inclusion, take a $z \in C_2(x)$. Then $z \in E$ for some connected subset E of X that contains x. Therefore $z \sim x$, so that $z \in C_1(x)$. $\square$

Corollary 3.9 *$C_1(x)$ is connected.*

By definition, there does not exist any connected set that contains x and a point outside $C_1(x)$. Thus $C_1(x)$ is a **maximal connected subset** of X. We will omit the subscript and just write $C(x)$.

Lemma 3.10 *A component in X is always closed.*

Proof: A component is of the form $C(x)$. By lemma 3.6, $\overline{C(x)}$ is also connected. By maximality of $C(x)$, one must have $C(x) = \overline{C(x)}$. $\square$

Here are two examples where a component may fail to be open.

Example 3.11 Take $X = \mathbb{Q}$. Then for any $r \in X$, $C(r) = \{r\}$.

Example 3.12 Take an infinite Cartesian product $X = \prod_\alpha X_\alpha$. Take $x = (x_\alpha) \in X$. Then $C(x) = \prod_\alpha C(x_\alpha)$. (for any $y \in C(x)$, $y_\alpha = \pi_\alpha(y) \subseteq \pi_\alpha(C(x))$. $\pi_\alpha(C(x))$ is a connected subset of X_α containing x_α. Hence $\pi_\alpha(C(x)) \subseteq C(x_\alpha)$. Thus $y_\alpha \in C(x_\alpha)$. The other inclusion follows from theorem 3.7). It follows from this that if infinitely many of the X_α's are not connected, then $C(x)$ is not open.

A space X is said to be **totally disconnected** if $C(x) = \{x\}$ for all $x \in X$.

Theorem 3.13 *Let X and Y be two topological spaces that are homeomorphic. then they have the same number of components.*

3.3 Path connectedness

A **path** in a topological space X is a continuous image of a connected subset of $\mathbb{R}$.

A space X is said to be **path connected** if for any two points $x, y \in X$, there is a special kind of connected subset, namely, a path in X containing the two points x and y.

A **path component** of X is a maximal path connected subset of X.

Examples.

1. $X = \{(x, \sin\log x) : 0 < x < \infty\}$: path connected.

2. $Y = \{0\} \times [0, 1] \cup X$: connected but not path connected.

3. $Z = \{0\} \times \{y \in \mathbb{R} \backslash \mathbb{Q} : y \in [0, 1]\} \cup X$: connected with uncountably many path components.

Exercise 3.14 Let X be a topological space. Suppose for any two points x and y, there is a path γ passing through one such that the other point is a limit point of $\{\gamma(t) : t\}$. Then Show that X is connected.

Give an example of a space that is connected but does not have this property.

Theorem 3.15 *A set is path connected if and only if it is connected and each point has a path connected neighbourhood.*

Proof: Assume X is connected and each point has a path connected neighbourhood. Take a (nonempty) path component C in X. Each point $x \in C$ has a path connected neighbourhood U. By maximality of C, one must have $U \subseteq C$. Thus C is open. Now C^c is an union of path components, each of which is open. So C^c is open. Since X is connected, C^c must be empty.

The other direction is trivial. □

3.4 Applications

1. $\mathbb{R}^n$ is connected.

2. Let E be any countable subset of $\mathbb{R}^n$. If $n > 1$, then $\mathbb{R}^n \backslash E$ is connected.

3. Any open subset of $\mathbb{R}^n$ is connected if and only if it is path connected.
4. S^n connected for all $n \in \mathbb{N}$.
5. $GL(n, \mathbb{R})$ is not connected.
6. connectedness (or lack of it) of $SL(n, \mathbb{R})$.

Problems.

1. Is $\mathbb{R}^n$ homeomorphic to $\mathbb{R}$ $(n > 1)$?
2. Is $\mathbb{D}^n \backslash \partial \mathbb{D}^n$ homeomorphic to $\mathbb{R}^n$?
3. Is S^1 homeomorphic to $\mathbb{R}$ (or $[0, 1]$)?
4. Is S^2 homeomorphic to $\mathbb{D}^2$?
5. Is $\mathbb{D}^2 \backslash \{(0, 0)\}$ homeomorphic to $\mathbb{D}^2$?

Answer to the last two questions above will require 'higher dimensional' generalizations of connectedness, to which we will come back later.

Lecture 5

4 Baire category theorem and applications

Definition 4.1 A set E is said to be **nowhere dense** if its closure has no interior point. A set is said to be of the **first category** if it is a countable union of nowhere dense sets. A set that is not of the first category is said to be of the **second category**.

Exercise 4.2 Show that E is nowhere dense if and only if $(\bar{E})^c$ is dense.

Example 4.3 1. Any finite subset or $\mathbb{Z}$ or $\mathbb{N}$ in $\mathbb{R}$.
2. The Cantor set in $[0, 1]$.

Exercise 4.4 Let $X = C([0, 1])$, and $d(f, g) = \sup_x |f(x) - g(x)|$. Let $0 < \alpha < 1$ and $0 < \beta$. Define

$$E = \{f \in X : \exists t \in [0, \alpha] \text{ such that } |f(x) - f(t)| \leq \beta(x - t)\ \forall x \in [t, 1]\}.$$

Show that E is closed and nowhere dense.

Exercise 4.5 Show that

1. if E is a set of the first category, then so is every subset of E.
2. if each E_n is a set of the first category, then $\cup_{n\in\mathbb{N}} E_n$ is also a set of the first category.

Exercise 4.6 A topological space is called a **Baire space** if countable intersections of open dense sets are dense.

Prove that

1. any open subset of a Baire space is a Baire space,
2. if every point has a neighbourhood that is a Baire space, then the space is a Baire space,
3. the set of irrationals is a Baire space in the subspace topology inherited from $\mathbb{R}$.

Corollary 4.7 (Baire category theorem) *Let X be a Baire space and E be a subset of X of the first category. Then E^c is dense in X.*

Proof: Let $E = \cup_n E_n$ where each E_n is nowhere dense. Then $E^c = \cap E_n^c$. Now each $\bar{E_n}^c$ is open dense. By the theorem above, $\cap \bar{E_n}^c$ is dense. But $\cap \bar{E_n}^c \subseteq \cap E_n^c = E^c$. Therefore E^c is dense. □

Theorem 4.8 *Let (X, d) be a complete metric space. Then X is a Baire space*

Remark 4.9 Both the conditions, i.e. countable nature of the collection $\{U_k\}$ and that each U_k is open is crucial, as the following two examples illustrate. Take $X = \mathbb{R}$ with usual metric, and take $U_x = \mathbb{R}\backslash\{x\}$ for $x \in \mathbb{R}$. Here $\cap_x U_x = \phi$. For the same space X, take the colection to be $\{U_1, U_2\}$, where $U_1 = \mathbb{Q}$, $U_2 = \mathbb{R}\backslash\mathbb{Q}$.

Proof: Start with an $x \in X$ and a neighbourhood U of x. We will show that $U \cap (\cap_k U_k)$ is nonempty.

Since U_1 is open dense, $U \cap U_1$ is a nonempty open set. Choose a ball B_1 of radius $\epsilon_1 < 1$ such that $\bar{B_1} \subseteq U \cap U_1$. Since U_2 is open dense, $B_1 \cap U_2$ is a nonempty open set. Choose a ball B_2 of radius $\epsilon_2 < 1/2$ such that $\bar{B_2} \subseteq B_1 \cap U_2 \subseteq U \cap (U_1 \cap U_2)$. Repeat this process. Now choose a sequence x_n such that each x_n comes from B_n. Then $d(x_n, x_m) < \epsilon_N$ for $m, n \geq N$. As $\lim_N \epsilon_N = 0$, the sequence x_n is Cauchy, hence converges to some $x \in X$.

Now $x_n \in B_N$ for all $n \geq N$. Hence $x \in \bar{B_N} \subseteq U \cap (\cap_{k=1}^N U_k)$. This happens for all $N \in \mathbb{N}$. Hence $x \in U \cap (\cap_k U_k)$. □

Exercise 4.10 1. Suppose $f \in C([0,1])$ has a finite right derivative at a point $t \in [0,1)$. Show that for some $n \in \mathbb{N}$,

$$|f(x) - f(t)| \leq n(x - t) \quad \text{for all } x \in [t, 1].$$

2. Let E be the set of nowhere differentiable functions in $C([0,1])$ and let F_n be the set defined in exercise 4.4 for $\alpha = 1 - \frac{1}{n}$ and $\beta = n$. Show that $E^c \subseteq \cup_n F_n$.
3. Show that E is dense.

Theorem 4.11 (Uniform boundedness principle) *Let (X, d) be a complete metric space, and $\mathscr{F}$ be a family of real-valued continuous functions on X. Assume that for each $x \in X$, there is a positive real M_x such that*

$$|f(x)| < M_x \quad \forall f \in \mathscr{F}.$$

Then there is an open set U in X and a constant $M > 0$ such that

$$\sup_{\substack{x \in U \\ f \in \mathscr{F}}} |f(x)| \leq M.$$

Proof: Write

$$E_{m,f} = \{x \in X : |f(x)| \leq m\}, \quad E_m = \cap_{f \in \mathscr{F}} E_{m,f}.$$

The given condition says that $X = \cup_m E_m$. Hence for some M, the set E_M is not nowhere dense. Since E_M is closed, it follows that there is an open set $U \subseteq E_M$. □

Corollary 4.12 (UBP in functional analysis) *Let X be a Banach space,and let $\mathscr{F}$ be a family of continuous linear maps from X to another normed linear space Y. Suppose for each $x \in X$, there exists an $M_x \in \mathbb{R}_+$ such that*

$$\sup_{f \in \mathscr{F}} ||f(x)|| < M_x.$$

Then

$$\sup_{f \in \mathscr{F}} ||f|| < \infty.$$

5 Separation properties

5.1 Separation axioms

A topological space is T_1 if given any two points, at least one of them has a neighbourhood not containing the other. A space is called **Hausdorff** if given any two points x and y, there are disjoint open sets U and V such that $x \in U$, $y \in V$. A Hausdorff space is called **regular** if given a point x and a closed set F not containing x, there are disjoint open sets U and V such that $x \in U$ and $F \subseteq V$. A Hausdorff space is called **normal** if given any two disjoint closed sets E and F, there are disjoint open sets U and V such that $E \subseteq U$ and $F \subseteq V$. A space is said to be **completely regular** if given a point x and a closed set F not containing x, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \in F$.

Examples.

5.2 Properties

Proposition 5.1 *Let X be a Hausdorff space. Then*

1. *any finite subset is closed,*
2. *x is a limit point of a set E if and only if every neighbourhod of x contains infinitely many points from E.*

Proposition 5.2 *X is Hausdorff if and only if the diagonal $\{(x, x) : x \in X\}$ is closed in $X \times X$.*

Exercise 5.3 Show that
1. X is Hausdorff if and only if for each $x \in X$, one has $\cap\{\overline{U} : U \text{ open}, x \in U\} = \{x\}$.
2. X is Hausdorff if and only if given any $x \in X$, for each $y \neq x$, there is a neighbourhood U of x such that $y \notin \overline{U}$.
3. If $(X, \mathscr{T})$ is Hausdorff, then any stronger topology on X is also Hausdorff.

Theorem 5.4
1. *Any subspace of a Hausdorff space is Hausdorff,*
2. *the product space $\prod X_\alpha$ is Hausdorff if and only if each X_α is Hausdorff,*
3. *if X is Hausdorff and $f : X \to Y$ is a closed bijection, then Y is also Hausdorff.*

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Proposition 5.5 *Let X and Y be topological spaces, Y be Hausdorff and let $f, g : X \to Y$ be continuous. If D is a dense subset of X and $f|_D = g|_D$, then $f = g$.*

Proof: Observe that the set $F = \{x \in X : f(x) = g(x)\}$ is closed, since it is the inverse image of the diagonal $\{(x,x) : x \in X\}$ under the map $x \mapsto (f(x), g(x))$ which is continuous. The given condition says that $D \subseteq F$. Since D is dense in X, we have $F = X$. $\square$

Regular spaces.

Example 5.6 Take $X = \mathbb{R}$ and let $\mathscr{T}$ be the topology generated by $\{(a,b) : a < b\} \cup \{\mathbb{Q}\}$. This topology is stronger than the usual topology on $\mathbb{R}$. The set of irrational is closed in this topology, but one can not separate this set and the point 1. In other words, this topology is not regular, even though the usual topology on $\mathbb{R}$ is.

Lemma 5.7 *X is regular if and only if for every $x \in X$ and each open neighbourhood U of x, there is an open set V such that $x \in V \subseteq \overline{V} \subseteq U$.*

Theorem 5.8 *1. Any subspace of a regular space is regular,*

2. the product space $\prod X_\alpha$ is regular if and only if each X_α is regular.

Normal spaces.

Lemma 5.9 *X is normal if and only if for every pair (F, U) with F closed, U open and $F \subseteq U$, there is an open set V such that $F \subseteq V \subseteq \overline{V} \subseteq U$.*

Exercise 5.10 Show that

1. X is normal if and only if for any two disjoint closed sets E and F, there exist open sets U and V such that $\overline{U}$ and $\overline{V}$ are disjoint and $E \subseteq U$, $F \subseteq V$.
2. X is normal if and only if for any two disjoint closed sets E and F, there exists an open set U such that $E \subseteq U$ and $\overline{U} \cap F = \phi$.

Theorem 5.11 *1. Any closed subspace of a normal space is normal,*

2. If the product space $\prod X_\alpha$ is normal then each X_α is normal.

3. *if X is normal and $f : X \to Y$ is a continuous closed surjection, then Y is also normal.*

Exercise 5.12 Let us denote by $\mathbb{R}_\ell$ the set $\mathbb{R}$ equipped with the lower limit topology.

1. Let E be a closed set in $\mathbb{R}_\ell$, and let $p \in E^c$. Let $q = \inf\{x \in E : x > p\}$. Show that $q \in E$.
2. Show that $\mathbb{R}_\ell$ is normal.
3. $D = \mathbb{Q} \times \mathbb{Q}$ is dense in $\mathbb{R}_\ell \times \mathbb{R}_\ell$.
4. $\{(x, -x) : x \in \mathbb{R}\backslash\mathbb{Q}\}$ is a closed discrete subset of $\mathbb{R}_\ell \times \mathbb{R}_\ell$.
   (one calls a subspace **discrete** if it is discrete in the subspace topology.)

Exercise 5.13 Let X be a normal topological space.

1. Let D be a dense subset and S be a closed discrete subspace of X. Show that there is an injective map from $\mathscr{P}(S)$ to $\mathscr{P}(D)$.
2. Show that X can not simultaneously have a countable dense subset and an uncountable closed discrete subspace.

5.3 Continuous functions

Exercise 5.14 Let (X, d) be a metric space. Let $x \in X$ and $E, F \subseteq X$. Define

$$d(x, E) = d(E, x) = \inf\{d(x, y) : y \in E\}, \quad d(E, F) = d(F, E) = \inf\{d(y, z) : y \in E, z \in F\}.$$

Show that

1. $d(x, E) = 0$ iff $x \in \overline{E}$.
2. Give an example where $d(E, F) = 0$ but $\overline{E}$ and $\overline{F}$ are disjoint.
3. The map $x \mapsto d(x, E)$ is continuous.
4. Given a closed set E and a point $x \notin E$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in E$.
5. Given two disjoint closed sets E and F, there exists a continuous function $f : X \to [0, 1]$ such that $f(y) = 0$ for all $y \in E$ and $f(z) = 1$ for all $z \in F$.

Corollary 5.15 *Let (X, d) be a metric space. Then $C(X)$ separates points.*

Lemma 5.16 *Let X be normal. Let D be the set of diadic rationals in $[0, 1]$. Let $\{U_r\}_{r\in D}$ be a family of open sets in X such that $\overline{U_r} \subseteq U_s$ whenever $r < s$. Define a function f on X by*

$$f(x) = \inf\{r : x \in U_r\}.$$

Then f is continuous and takes values in $[0, 1]$.

There is nothing special with the set D here. Any dense subset of $[0, 1]$ will do.
Proof: Take $x \in X$ and $\epsilon > 0$. First, observe that $f(y) \le t$ for all $y \in U_t$. Now choose $s \in D$ such that $f(x) < s < f(x) + \epsilon$ and $x \in U_s$. Let $r \in D$ be such that $f(x) - \epsilon < r < f(x)$. Then $f(y) < r$ implies $y \in U_r$. Therefore $y \in U_r^c$ implies $f(y) \ge r$. Now take $V = U_s \backslash \overline{U_r}$. Check that $x \in V$ and for all $y \in V$, $f(y)$ lies between r and s. □

Theorem 5.17 (Urysohn lemma) *Let X be Hausdorff. Then X is normal if and only if given any two disjoint closed sets E and F in X, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in E$ and $f(x) = 1$ for all $x \in F$.*

Corollary 5.18 *Normal spaces are completely regular.*

Corollary 5.19 *Let X be normal. Then $C(X)$ separates points of X.*

Corollary 5.20 *Let X be normal and let E be a closed subset of X. Let $f : E \to [-r, r]$ be a continuous function. Then there exists a continuous function $g : X \to [-\frac{r}{3}, \frac{r}{3}]$ such that $|f(x) - g(x)| \leq \frac{2r}{3}$ for all $x \in E$.*

Lecture 7

Theorem 5.21 (Tietze's extension theorem) *Let X be Hausdorff. Then X is normal if and only if given any closed set E in X and a continuous function $f : E \to [0,1]$, there exists a continuous function $g : X \to [0,1]$ whose restriction to E is f.*

Proof: Using the previous corollary, construct successively continuous functions $f_0, f_1, \ldots$ such that

$$|f(x) - (f_0(x) + \ldots + f_n(x))| \le \left(\frac{2}{3}\right)^{n+1} \qquad \forall x \in E,$$

$$|f_n(x)| \le \frac{1}{3}\left(\frac{2}{3}\right)^n.$$

$g := \sum_n f_n$ will be the required function. □

Remark 5.22 One can replace the interval $[0,1]$ in the above theorem by $\mathbb{R}$.

Proof: Here we start with a continuous function $f : E \to \mathbb{R}$. First look at the function $\phi = \arctan f : E \to (-\frac{\pi}{2}, \frac{\pi}{2})$. Extend it using Tietze's extension theorem to a continuous function $\psi : X \to [-\frac{\pi}{2}, \frac{\pi}{2}]$. Now E and $F := \psi^{-1}\{\pm 1\}$ are two disjoint closed sets in X. Using Urysohn lemma, get a continuous function θ such that $\theta|_F = 0$ and $\theta|_E = 1$. The the function $\psi\theta$ is continuous, is an extension of ϕ and takes values in the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. The function $g := \tan(\psi\theta)$ would be our required function. □

5.4 Countability properties

A Hausdorff space is called **second countable** if it has a countable basis.

Proposition 5.23 1. *second countability is invariant under continuous open surjections.*

2. *subspaces of a second countable space are second countable.*

3. *$\prod_{\alpha\in\Lambda} X_\alpha$ is second countable iff Λ is countable and each X_α is second countable.*

Proof: Proof of the first two parts are trivial. For part 3, first assume that the product space $\prod_{\alpha\in\Lambda} X_\alpha$ is second countable. By part 2, each X_α is second countable. Let $\mathscr{B}$ be a countable basis for the product space. Construct a map $\psi : \Lambda \to \mathscr{B}$ as follows: For each α, let $\psi(\alpha)$

be an element in $\mathscr{B}$ such that $\pi_\alpha(\psi(\alpha))$ is different from ϕ and X_α. By definition of product topology, pre-image of every point $V \in \mathscr{B}$ is finite. But this forces Λ to be countable.

Assume next that Λ is countable and each X_α is second countable. It is then easy to produce a countable basis for the product topology. □

A family of open sets $\{U_\alpha\}_\alpha$ is said to be an **open cover** for a set E if $E \subseteq \cup U_\alpha$. If one just says 'an open cover', one means an open cover for the whole space. If $\{U_\alpha\}_\alpha$ is an open cover for E and $\{V_\beta\}$ is a subcollection that is an open cover for E, then one calls $\{V_\beta\}$ an **open subcover** (of $\{U_\alpha\}$).

Proposition 5.24 *If X is second countable, then every open cover has a countable subcover.*

Proof: Let $\{U_i\}_{i\in\mathbb{N}}$ be a countable basis and let $\{V_\alpha\}$ be an open cover. Look at the set $S = \{j \in \mathbb{N} : U_j \subseteq V_\alpha \text{ for some } \alpha\}$. Since each V_α is a union of U_i's, $\{U_i\}_{i\in S}$ is again an open cover. Now choose α_i such that $U_i \subseteq V_{\alpha_i}$ for all $i \in S$. $\{V_{\alpha_i}\}_{i\in S}$ would be a countable cover. □

Exercise 5.25

1. Let X be regular and let E and F be two disjoint closed sets in X. Suppose $\{U_n\}$ and $\{V_n\}$ are two countable families of open sets such that $E \subseteq \cup U_n$, $F \subseteq \cup V_n$ and $\overline{U_n} \cap F = \phi$, $\overline{V_n} \cap E = \phi$ for all n.

   Show that there exist disjoint open sets U and V such that $E \subseteq U$ and $F \subseteq V$.
2. Use part 1 to show that a regular space with a countable basis is normal.

A Hausdorff space is called **separable** if it has a countable dense subset.

Proposition 5.26

1. *Continuous image of a separable space is separable.*
2. *Open subspaces of separable spaces are separable.*
3. *$\prod_{\alpha\in\Lambda} X_\alpha$ is separable iff each X_α is separable and $|\Lambda| \leq |\mathbb{R}|$.*

Proof: 1. If D is a countable dense subset of the domain space, $f(D)$ would be a countable subset of the range space.

2. If D is dense and U is open, then $U \cap D$ is dense in U.

3. Assume that the product space is separable. Hence each X_α is Hausdorff and has a countable dense subset, i.e. each X_α is separable. Let D be a countable dense subset of the product space. For each α, let (U_α, V_α) be a pair of nonempty disjoint open sets in X_α. Now construct a function $\psi : \Lambda \to \mathscr{P}(D)$ by $\psi(\alpha) = D \cap U_\alpha$. For $\alpha \neq \beta$, the intersection $D \cap \langle U_\alpha\rangle \cap \langle V_\alpha\rangle$ is empty whereas $D \cap \langle U_\beta\rangle \cap \langle V_\alpha\rangle$ is not. Therefore ψ is injective. Hence cardinality of Λ can not exceed that of $\mathscr{P}(D)$, which is same as that of $\mathbb{R}$.

Next, let us assume that each X_α is separable and $|\Lambda| \leq |\mathbb{R}|$. We can assume without loss of generality that $\Lambda \subseteq [0, 1]$ (just as a set). Let $\{d_\alpha(n) : n \in \mathbb{N}\}$ be a countable dense subset for X_α. Let $J_1, J_2, \ldots, J_k$ be disjoint intervals in $[0, 1]$ with rational endpoints, and let $n_1, \ldots, n_k$ be natural numbers. Define $p(J_1, \ldots, J_k; n_1, \ldots, n_k) = (x_\alpha)$ where

$$x_\alpha = \begin{cases} d_\alpha(n_i) & \text{if } \alpha \in J_i, \\ d_\alpha(1) & \text{if } \alpha \notin \cup J_i. \end{cases}$$

Then the set

$$D = \{p(J_1, \ldots, J_k; n_1, \ldots, n_k) : k \in \mathbb{N},\ J_i\text{'s disjoint intervals with rational endpoints, } n_i \in \mathbb{N}\}$$

is dense in the product space. □

Exercise 5.27 Give an example of a separable space and a subspace that is not separable.

Proposition 5.28 *Let (X, d) be a separable metric space. Then any subspace is also separable.*

Proof: Let Y be a subspace of X and let $\{x_1, x_2 \ldots\}$ be a countable dense subset of X. Choose and fix an element $z \in Y$. Define a countable subset $D = \{y_{nk}\}$ of Y as follows:

$$y_{nk} = \begin{cases} z & \text{if } N_{1/k}(x_n) \cap Y = \phi, \\ \text{any member of } N_{1/k}(x_n) \cap Y & \text{otherwise.} \end{cases}$$

We will show that D is dense in Y.

Take $y \in Y$ and $k \in \mathbb{N}$. We will show that there is an n such that $y_{nk} \in N_{2/k}(y)$. Since $\{x_1, x_2 \ldots\}$ is dense in X and $y \in X$, there is an n such that $x_n \in N_{1/k}(y)$. But this means that $N_{1/k}(x_n) \cap Y$ is nonempty. Therefore $y_{nk} \in N_{1/k}(x_n) \cap Y$. Since $y_{nk} \in N_{1/k}(x_n)$ and $x_n \in N_{1/k}(y)$, we have $y_{nk} \in N_{2/k}(y)$. □

Lecture 8

6 Covering properties

6.1 Compactness

A topological space is said to be **compact** if every open cover has a finite subcover.

Proposition 6.1 *Let X be a topological space and let E be a subset. E is compact in the subspace topology if and only if for every open cover for E in X, there is a finite subcover.*

Proof: Assume E is compact in the subspace topology. Take an open cover $\{U_\alpha\}$ of E in X. The sets $U_\alpha \cap E$ are all open in the subspace topology and they cover E. Therefore there is a finite subcollection $\{U_{\alpha_n} \cap E\}$ that covers E. But then $\{U_{\alpha_n}\}$ is an open cover for E in X.

Converse is similar. $\square$

Proposition 6.2 *Any closed subset of a compact space is compact.*

Proof: Let X be compact and let E be a closed subset of X. Take an open cover $\{U_\alpha\}$ of E. Then $\{U_\alpha\} \cup \{E^c\}$ is an open cover for X. Since X is compact, this has a finite subcover, say $\{V_\beta\}$. If E^c is not a member of this, then this is a subcover of the original cover for E that we started with. If E^c is a member in this collection, then removing E^c, we get a subcover of the open cover for E. $\square$

Proposition 6.3 *Any compact subset of a Hausdorff space is closed.*

Proof: Let X be Hausdorff and let K be a compact subset of X. Take $y \in K^c$. For each $x \in K$, there are disjoint open sets U_x and V_x such that $x \in U_x$, $y \in V_x$. Now, the collection $\{U_x\}_{x \in K}$ is an open cover for K. So it has a finite subcover, say, $\{U_{x_1}, \ldots, U_{x_n}\}$. Write $V = \cap_{i=1}^n V_{x_i}$. Then V is an open neighbourhood of y and $V \subseteq K^c$. Thus y is an interior point of K^c. $\square$

Corollary 6.4 *A subset of a compact Hausdorff space is compact if and only if it is closed.*

Proposition 6.5 *In a Hausdorff space, compact sets can be separated from points by disjoint open sets.*

Proof: Let K be compact and let $y \in K^c$. For each $x \in K$, get disjoint open sets U_x and V_y such that $x \in U_x$ and $y \in V_x$. Now get hold of a finite subcover $\{U_{x_1}, \ldots, U_{x_n}\}$ of the cover $\{U_x\}_{x \in K}$ for K. Now write $U = \cup_{i=1}^n U_{x_i}$ and $V = \cap_{i=1}^n V_{x_i}$. The sets U and V are open, disjoint and separate K and y. $\square$

Proposition 6.6 *Two disjoint compact sets in a Hausdorff space can be separated by disjoint open sets.*

Proof: Let E and F be two disjoint compact sets. For each $x \in E$, get a pair of disjoint open sets U_x and V_x such that $x \in U_x$ and $F \subseteq V_x$. Now look at the open cover $\{U_x\}$ for E and get a finite subcover $\{U_{x_i}\}_{1 \leq i \leq n}$. Finally, set $U = \cup U_{x_i}$ and $V = \cap V_{x_i}$. $\square$

Corollary 6.7 *Compact Hausdorff spaces are normal.*

Proposition 6.8 *Continuous image of compact sets are compact.*

Proof: Let K be a compact subset of X and $f : X \to Y$ be continuous. Take an open cover $\{U_\alpha\}$ for $f(K)$ in Y. Then $\{f^{-1}(U_\alpha)\}$ is an open cover for K in X. Since K is compact, this admits a finite subcover, say, $\{f^{-1}(U_{\alpha_i})\}$. Then $\{U_{\alpha_i}\}$ is a finite subcover of $\{U_\alpha\}$ for $f(K)$. $\square$

Proposition 6.9 *Let X be compact, Y be Hausdorff and let $f : X \to Y$ be a continuous bijection. Then f is a homeomorphism.*

Proof: This is a simple corollary of the earlier results. One needs to show that f is open. Take an open set U in X. Then U^c is a closed subset of X, hence compact. Therefore $f(U^c)$ is compact. Since Y is Hausdorff, $f(U^c)$ is closed. But $f(U^c) = f(U)^c$. Hence $f(U)$ is open. $\square$

Let $\mathscr{F}$ and $\mathscr{G}$ be two filterbases. One says $\mathscr{F}$ is **subordinated to** $\mathscr{G}$ if for any $E \in \mathscr{G}$, there is an $F \in \mathscr{F}$ such that $F \subseteq E$. A filterbase $\mathscr{F}$ is called a **maximal filterbase** if whenever $\mathscr{G}$ is subordinated to $\mathscr{F}$, $\mathscr{F}$ is also subordinated to $\mathscr{G}$.

Lemma 6.10 *A filterbase $\mathscr{F}$ in X is maximal iff for any subset A of X, there is an $F \in \mathscr{F}$ such that $F \subseteq A$ or $F \subseteq A^c$.*

Proof: Suppose there is an $A \subseteq X$ such that $A \cap F \neq \phi \neq A^c \cap F$ for all $F \in \mathscr{F}$. Define $\mathscr{G} = \{A \cap F : F \in \mathscr{F}\}$. Then $\mathscr{G}$ is a filterbase properly subordinated to $\mathscr{F}$, i.e. $\mathscr{F}$ is not maximal.

Conversely, assume now that for all $A \subseteq X$, there is an $F \in \mathscr{F}$ such that $F \subseteq A$ or $F \subseteq A^c$. Let $\mathscr{G}$ be a filterbase subordinated to $\mathscr{F}$. Take $G \in \mathscr{G}$. Suppose, if possible that there is an $F \in \mathscr{F}$ such that $F \subseteq G^c$. Since $\mathscr{G}$ is subordinated to $\mathscr{F}$, there is an $E \in \mathscr{G}$ such that $E \subseteq F$. Thus $E \subseteq G^c$, which means $E \cap G = \phi$, which can not happen because they both come from $\mathscr{G}$. Thus $F \cap G \neq \phi$ for all $F \in \mathscr{F}$. By the property above, this implies that there is an $F \in \mathscr{F}$ such that $F \subseteq G$, so that $\mathscr{F}$ is subordinated to $\mathscr{G}$. Thus $\mathscr{F}$ is maximal. $\square$

Corollary 6.11 *Let $\mathscr{F}$ be a maximal filterbase in X and let $f : X \to Y$ be a map. Then $f(\mathscr{F})$ is a maximal filterbase in Y.*

Exercise 6.12 A maximal filterbase has a limit point iff it converges to that point.

Exercise 6.13 Show that a filterbase $\mathscr{F}$ has a limit point x iff it has a maximal subordinated filterbase $\mathscr{M}$ that converges to x.

A collection of subsets $\{F_\alpha\}_\alpha$ is said to have the **finite intersection property** if every finite subcollection has nonempty intersection.

Theorem 6.14 *Let X be a topological space. The following conditions are equivalent:*

1. *X is compact,*
2. *any family of closed sets having finite intersection property has nonempty intersection,*
3. *each filterbase in X has a limit point,*
4. *each maximal filterbase in X converges.*

Proof: (2)$\Rightarrow$ (3): Let $\mathscr{F}$ be a filterbase. Look at the collection $\{\overline{F} : F \in \mathscr{F}\}$. Since $\mathscr{F}$ is a filterbase, this collection has finite intersection property. Hence $\cap_{F\in\mathscr{F}}\overline{F}$ is nonempty. Take an x in this intersection. Any open neighbourhod of x then intersects $\overline{F}$, and hence intersects F. $\square$

Exercise 6.15 A filterbase $\mathscr{F}$ in $\prod X_\alpha$ converges to a point (x_α) iff $\pi_\alpha(\mathscr{F})$ converges to x_α for each α.

Theorem 6.16 (Tychonoff's theorem) *$\prod X_\alpha$ is compact if and only if X_α is compact.*

Proof: If the product is compact, then each X_α, being a continuous image, is compact.

For the converse, take a maximal filterbase in the product space and use the lemma above. $\square$

Lecture 9

Proposition 6.17 *Let X be a compact space and let $f : X \to \mathbb{R}$ be continuous. Then both* $\sup f$ *and* $\inf f$ *are finite and these are attained.*

Proof: Since $f(X)$ is compact, both $\sup f$ and $\inf f$ are finite. For $y \in X$, denote by F_y the set $\{x \in X : f(x) \leq y\}$. Each F_y is closed. Write $b = \inf f$. For $y > b$, F_y is nonempty. Since $F_y \subseteq F_z$ for $y \leq z$, the family $\{F_y\}_{y>b}$ has finite intersection property. Therefore $\cap_{y>b} F_y$ is nonempty. Take any x in this intersection. It is clear that $f(x) = b$.

Similarly one can show that the supremum is also attained. □

Theorem 6.18 *Let (X, d) be a compact metric space and let $\mathscr{U}$ be an open cover for X. Then there exists a $\lambda = \lambda_{\mathscr{U}} > 0$ such that any open ball $N_\lambda(x)$ is contained in some $U \in \mathscr{U}$.*

Proof: Since $\mathscr{U}$ is an open cover, for each $x \in X$, there is a positive real $r(x)$ such that $N_{r(x)}(x) \subseteq U$ for some $U \in \mathscr{U}$. Now look at the cover $\{N_{\frac{r(x)}{2}}(x) : x \in X\}$ and get a finite subcover $\{N_{\frac{r(x_i)}{2}}(x_i) : 1 \leq i \leq n\}$. Let $\lambda = \min\{\frac{r(x_i)}{2} : 1 \leq i \leq n\}$. Now take an open ball $N_\lambda(x)$. Since $\{N_{\frac{r(x_i)}{2}}(x_i) : 1 \leq i \leq n\}$ is a cover, $x \in N_{\frac{r(x_i)}{2}}(x_i)$ for some i. But then $N_\lambda(x) \subseteq N_{r(x_i)}(x_i)$. But $N_{r(x_i)}(x_i) \subseteq U$ for some $U \in \mathscr{U}$. □

Such a number λ is called a **Lebesgue number** for the cover $\mathscr{U}$.

Theorem 6.19 *Let (X, d) be a metric space. The following conditions are all equivalent.*

1. *X is compact,*
2. *X is sequentially compact,*
3. *X is complete and totally bounded.*

Proof:

(1) ⇒ (2):

Take a sequence $\{x_n\}$ in X. Let F_n be the closure of $\{y \in X : y = x_k \text{ for some } k \geq n\}$. The colection $\{F_n\}$ has finite intersection property. So there is an $x \in \cap_n F_n$. It is now easy to extract a subsequence of $\{x_n\}$ that converges to x.

(2) ⇒ (3):

Completeness is obvious.

If X is not totally bounded, there exists some $\epsilon > 0$ such that finite number of ϵ-balls can not cover X. Now choose $x_1, x_2, \ldots$ successively such that $x_n \notin \cup_{i=1}^{n-1} N_\epsilon(x_i)$. Then $\{x_n\}$ can not have any convergent subsequence.

(3) $\Rightarrow$ (2):

Take a sequence $\{x_n\}$ in X. Since X is completely bounded, finite number of 1-balls cover X. At least one of these balls will contains infinitely many terms of the sequence. In other words, one can choose a subsequence that lies entirely in one single 1-ball. Repeating the same argument, get a subsequence of this subsequence that lies in one single $\frac{1}{2}$-ball. Repeat the argument. Suppose $\{y_{nk}\}_k$ is the sequence obtained at the n^{th} stage. Let $z_n = y_{nn}$. Then $\{z_n\}$ is a subsequence of the original sequence, and by the way the y_{ij}'s were chosen, it follows that $\{z_n\}$ is Cauchy. Now by completeness of X, $\{z_n\}$ converges.

(2) $\Rightarrow$ (1):

Let $\{U_\alpha\}$ be an open cover of X such that no U_α is X. Define

$$\phi(x) = \sup\{\epsilon > 0 : N_\epsilon(x) \subseteq U_\alpha \text{ for some } \alpha\}.$$

Then $\phi(x) > 0$ for all $x \in X$. Suppose $\phi(x) = \infty$ for some x. Since X is bounded, this will imply that $U_\alpha = X$ for some α. Hence we must have $0 < \phi(x) < \infty$ for all $x \in X$.

We will next show that $\phi(x) \leq \phi(y) + d(x, y)$. If not, there exist $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that $\phi(x) - \epsilon_1 > \phi(y) + \epsilon_2 + d(x, y)$. By definition of ϕ, there is an $r > \phi(x) - \epsilon_1$ such that $N_r(x) \subseteq U_\alpha$ for some α. But then for any $z \in N_{\phi(y)+\epsilon_2}(y)$, one has $d(x, z) \leq d(x, y) + \phi(y) + \epsilon_2 < r$, which means $N_{\phi(y)+\epsilon_2}(y) \subseteq N_r(x) \subseteq U_\alpha$. This can not happen by definition of $\phi(y)$. Therefore $\phi(x) \leq \phi(y) + d(x, y)$. Interchanging x and y, we get $\phi(y) \leq \phi(x) + d(x, y)$. Together, these give us $|\phi(x) - \phi(y)| \leq d(x, y)$.

Our next claim is that $\inf \phi > 0$. If $\inf \phi = 0$, then there is a sequence $\{x_n\}$ such that $\phi(x_n)$ converges to 0. By sequential compactness, choose a convergent subsequence $\{x_{n_k}\}$ converging to, say, x. Since $|\phi(x_{n_k}) - \phi(x)| \leq d(x_{n_k}, x)$, it follows that $\phi(x) = 0$, which is impossible. Thus $\inf \phi > 0$.

Now choose an r such that $0 < r < \inf \phi$. Then for any $x \in X$, there is an α such that $N_r(x) \subseteq U_\alpha$. Since X is completely bounded, there are finitely many points $x_1, \ldots, x_k$ such that $X \subseteq \cup_i N_r(x_i)$. Let α_i be such that $N_r(x_i) \subseteq U_{\alpha_i}$. Then $\{U_{\alpha_i}\}$ is a finite subcover of $\{U_\alpha\}$. □

Exercise 6.20 Give an example of a set X, and two metrics d_1 and d_2 on X inducing the same topology such that (X, d_1) is complete but (X, d_2) is not.

Give an example of a set X, and two metrics d_1 and d_2 on X inducing the same topology such that (X, d_1) is totally bounded but (X, d_2) is not.

A set E in a metric space is said to be **bounded** if $\sup\{d(x, y) : x, y \in E\}$ is finite.

Proposition 6.21 *Let (X, d) be a metric space and let E be a compact subset of X. Then X is closed and bounded.*

Proof: Since X is hausdorff, we already know that E is closed. Now choose and fix a point $y \in E$ and look at the function $f(x) := d(x, y)$. This is a continuous function from X to $\mathbb{R}$. Hence it attains its supremum over the compact set E. Call it M. Then for any two points $x, x' \in E$, one has $d(x, x') \leq d(x, y) + d(x', y) \leq 2M$. Thus E is bounded. $\square$

Proposition 6.22 *Closed bounded intervals in $\mathbb{R}$ are compact.*

Proof: Use theorem 6.19. $\square$

Proposition 6.23 *A subset E of $\mathbb{R}^n$ is compact iff it is closed and bounded.*

Proof: It is enough to show that any closed and bounded subset E of $\mathbb{R}^n$ is compact. Suppose $E \subseteq \mathbb{R}^n$ is closed and bounded. Boundedness would imply that E is contained in a set of the form $\prod_i [a_i, b_i]$, where $a_i, b_i \in \mathbb{R}$ for all i. But each $[a_i, b_i]$ is compact. Hence their product $\prod_i [a_i, b_i]$ is also compact. Now E is a closed subset of a compact space. Therefore E is compact. $\square$

Proposition 6.24 *Let (X, d) be a metric space. Let $E \subseteq X$ be closed and $K \subseteq X$ be compact. If $E \cap K = \phi$, then $d(E, K) > 0$.*

Proof: Look at the function $x \mapsto d(E, x)$. This is continuous, so attains its minimum over K at some point, say x in K. But $d(E, x)$ can not be zero, because $d(E, x) = 0$ implies $x \in \overline{E} = E$. $\square$

6.2 Locally compact spaces

A subset E of a topological space is called **relatively compact** if its closure is compact. A space X is said to be **locally compact** if every point has a relatively compact open neighbourhood.

Lemma 6.25 *Let X be a topological space and let $A \subseteq E \subseteq X$. Then $\overline{A}^E = E \cap \overline{A}$.*

Proof: $E \cap \overline{A}$ is closed in E and containes A. Therefore $\overline{A}^E \subseteq E \cap \overline{A}$. For the reverse inclusion, take an $x \in E \cap A'$ and a E-neighbourhood $E \cap V$ containing x. Since V is a neighbourhood of x and $x \in A'$, V contains a point, say y, of A other than x. But $A \subseteq E$. Therefore $y \in E \cap V$. Therefore $x \in \overline{A}^E$. $\square$

Proposition 6.26 *Let X be a Hausdorff space. Then the following conditions are equivalent:*

1. *X is locally compact,*
2. *for each $x \in X$, and for each open neighbourhood U of x, there exists a relatively compact open set V such that $x \in V \subseteq \overline{V} \subseteq U$,*

3. *for each compact K in X and open set U with $K \subseteq U$, there exists a relatively compact open set V such that $K \subseteq V \subseteq \overline{V} \subseteq U$,*

4. *X has a basis consisting of relatively compact open sets.*

Proof: (1)⇒(2):

Take $x \in X$ and an open neighbourhood U of x. Since X is locally compact, there is a relatively compact open neighbourhood W. Since $\overline{W}$ is compact and Hausdorff, it is regular. Hence there is an open set G in $\overline{W}$ containing x whose closure (in $\overline{W}$) is contained in $\overline{W} \cap U$. Now, G must be of the form $\overline{W} \cap A$ where A is open in X. Take $V = W \cap A$. We will show that $\overline{V}$ is compact and $\overline{V} \subseteq U$. First, observe that $V \subseteq \overline{W}$, so that $\overline{V}^{\overline{W}} = \overline{W} \cap \overline{V}$. Since $V \subseteq W$, it follows that $\overline{V} \subseteq \overline{W}$, and hence $\overline{V}^{\overline{W}} = \overline{V}$. Therefore $\overline{V}$ is compact. Finally, $V \subseteq G$. Hence $\overline{V} = \overline{V}^{\overline{W}} \subseteq \overline{G}^{\overline{W}} \subseteq U$.

(2)⇒(3):

For each $x \in K$, get a relatively compact open set V_x with $x \in V_x \subseteq \overline{V_x} \subseteq U$. The collection $\{V_x : x \in K\}$ covers K. Get a finite subcover $\{V_{x_i} : 1 \leq i \leq n\}$. Take $V = \cup_i V_{x_i}$.

(3)⇒(4):

Let $\mathscr{B}$ be the collection of all relatively compact open sets in X. It is clear from (3) that $\mathscr{B}$ is a basis for the topology of X.

(4)⇒(1):

This part follows directly from the definition. □

Lecture 10

Corollary 6.27 *Let X be a locally compact Hausdorff space and let E be an open subset. Then E is also locally compact.*

Proof: This follows from part (2) of the previous proposition. $\square$

Lemma 6.28 *Let X be a locally compact Hausdorff space and let E be a closed subset. Then E is also locally compact.*

Proof: Take $x \in E$. Since X is locally compact, there is an open neighbourhood V in X with compact closure. Write $U = V \cap E$. We will show that closure of U in E is compact. First, observe that $\overline{U}^E \subseteq \overline{U}^X$. Now $\overline{U}^E$, being the intersection of a closed set in X and E, is closed in X. Therefore it is now enough to show that $\overline{U}^X$ is compact. But this follows from the fact that $\overline{U}^X \subseteq \overline{V}^X \cap E$ and the right hand side here is a closed subset of the compact set $\overline{V}^X$. $\square$

Proposition 6.29 1. *Let X be locally compact, Y be Hausdorff and $f : X \to Y$ be a continuous open map. Then $f(X)$ is locally compact.*

2. *Product of locally compact spaces is locally compact iff all but finitely many of them are compact.*

Proof: 1. Take $y = f(x) \in f(X)$. Let U be a relatively compact open neighbourhood of x in X. Write $V = f(U)$. From the given condition, V is open in $f(X)$. Since f is continuous, $f(\overline{U})$ is compact. Since $f(X)$ is Hausdorff, $f(\overline{U})$ is closed. Hence $\overline{V} \subseteq f(\overline{U})$. But this means $\overline{V}$ is compact and local compactness of $f(X)$ follows.

2. Assume each X_α is locally compact and F is a finite subset of the index set Λ such that X_α is compact for all $\alpha \notin F$. Take $x = (x_\alpha) \in \prod X_\alpha$. For each $\alpha \in F$, let V_α be a relatively compact open neighbourhood of x_α. For $\alpha \notin F$, set $V_\alpha = X_\alpha$. Now let $V = \prod V_\alpha$. Then V is a relatively compact open neighbourhood of $x = (x_\alpha)$.

Conversely, now assume that the product space is locally compact. From part (1), it follows that each X_α is locally compact. Since any open set in the product space looks like $\prod V_\alpha$ where the $V_\alpha = X_\alpha$ for all but finitely many α's, and there exists relatively compact open sets in the product space, by Tychonoff's theorem, the conclusion follows. $\square$

Proposition 6.30 *A locally compact Hausdorff space is completely regular.*

Proof: Let F be closed and $x \in F^c$. By the earlier proposition, there are relatively compact open sets U and V such that $x \in U \subseteq \overline{U} \subseteq V \subseteq \overline{V} \subseteq F^c$. Now $\overline{V}$ is compact Hausdorff, hence is normal. So there is a continuous function $f : \overline{V} \to [0,1]$ such that $f(x) = 1$ and $f|_{\overline{V}\backslash U} = 0$. Define $\widetilde{f} : X \to [0,1]$ as follows:

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \in \overline{V}, \\ 0 & \text{if } x \notin \overline{V}. \end{cases}$$

Check that $\widetilde{f}|_{\overline{V}} = f$ and $\widetilde{f}|_{U^c} = 0$. Since both $\widetilde{f}|_{\overline{V}}$ and $\widetilde{f}|_{U^c}$ are continuous and they coincide on $\overline{V} \cap U^c = \overline{V}\backslash U$, it follows that $\widetilde{f}$ is continuous. □

6.3 Compactification

Let X be a topological space. A pair $(\hat{X}, i)$, where $\hat{X}$ is a compact Hausdorff space and $i : X \to \hat{X}$ is a homeomorphism between X and a subspace of $\hat{X}$, is called a **compactification** of X if the image $i(X)$ is dense in $\hat{X}$. Often one identifies X with $i(X)$ and just says $\hat{X}$ is a compactification of X.

The concept of compactification is important, because one normally has nice and strong results on compact spaces. If it is possible to enlarge a space so as to make it compact, at the same time keeping track of the extra point(s) added so that one can later undo the changes, one can often prove results about the original space. So it is important to know which spaces can be compactified, and how do the various compactifications look like.

Examples.

1. S^1 as a compactification of $\mathbb{R}$,
2. $[0,1]$ as a compactification of $\mathbb{R}$,
3. Riemann sphere as compactification of $\mathbb{C}$,
4. $\{(0,x) : x \in [0,1]\} \cup \{(x, \sin\log x) : x \in [0,1]\}$ as a compactification of $\mathbb{R}$. More generally, let f and g be two continuous functions from $\mathbb{R}$ to $[0,1]$, such that f is one-one.. Define $i : \mathbb{R} \to [0,1]^2$ by $i(x) = (f(x), g(x))$. $(\overline{i(\mathbb{R})}, i)$ is a compactification of $\mathbb{R}$.

Question. Suppose $(\hat{X}, i)$ is a compactification of X. What are all the ($\mathbb{R}$ or $\mathbb{C}$-valued) continuous functions on X that extend to $\hat{X}$? Since $\hat{X}$ is compact, one necessary condition will be the boundedness of the function. Now suppose $f : X \to \mathbb{R}$ is a bounded continuous function. Does there always exist a compactification $\hat{X}$ such that f extends to $\hat{X}$?

Theorem 6.31 (Alexandroff) *Let X be a locally compact Hausdorff space. Then*

1. *there exists a compactification $\hat{X}$ such that $\hat{X}\backslash\ X$ is singleton,*

2. *if Y and Z are two such compactifications, then there is a homeomorphism between the two whose restriction to (the image of) X is the identity map.*

Proof: 1. Let $\hat{X}$ be the disjoint union of X and a singleton $\{\infty\}$. Define a collection $\mathscr{B}$ of subsets of $\hat{X}$ as follows: $E \in \mathscr{B}$ if either E is an open subset of X or if it is the complement (in $\hat{X}$) of a compact set in X. Equip $\hat{X}$ with the topology generated by $\mathscr{B}$. Observe that this topology is Hausdorff and X is open in $\hat{X}$. Take an open cover $\{U_\alpha\}$ for $\hat{X}$. then $\infty \in U_\beta$ for some β. This U_β is of the form K^c, where K is some compact set in X. Now $\{U_\alpha \cap X\}$ is an open cover for K (in X). Since K is compact, There is a finite subcollection $\{U_{\alpha_1} \cap X, \ldots, U_{\alpha_n} \cap X\}$ that cover K. Then $\{U_\beta, U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ is a finite cover for $\hat{X}$. Finally, X is dense in $\hat{X}$ because every neighbourhood of ∞ contains points from X.

2. Let (Y, i) and (Z, j) be two one-point compactifications of X. Define $f : Y \to Z$ by the prescription: $f(\infty) = \infty$ and $f(i(x)) = j(x)$ for $x \in X$. It is now easy to see that this is a homeomorphism. □

Corollary 6.32 *A Hausdorff space is locally compact iff it is an open subset of a compact Hausdorff space.*

Exercise 6.33 Let $\hat{X}$ be the one-point compactification of X. Find all continuous functions on X that extend to continuous functions on $\hat{X}$.

Proposition 6.34 *Let X be locally compact Hausdorff, $K \subseteq X$ be compact, $U \subseteq X$ open and $K \subseteq U$. Then there exists a continuous function $f : X \to [0,1]$ such that $f|_K = 1$ and $f|_{U^c} = 0$.*

Proof: Let $\hat{X}$ be the one-point compactification of X. Then K is closed in $\hat{X}$ and U is open in $\hat{X}$. Since $\hat{X}$ is compact Hausdorff, it is normal. Hence There is a continuous function $g : \hat{X} \to [0,1]$ such that $g|_K = 1$ and $g|_{\hat{X} \setminus U} = 0$. Take f to be the restriction of g to X. □

One can give a proof of the Riesz-representation theorem (for constructing positive measures out of positive functionals on $C_c(X)$), which is useful, for example, in constructing the Haar measure on a locally compact group.

Lecture 11

7 Function spaces

Let X and Y be two topological spaces. We will denote by Y^X the space of all functions from X to Y. Let $A \subseteq X$ and $B \subseteq Y$. Let $N(A, B)$ denote the subset $\{f \in Y^X : f(A) \subseteq B\}$ of Y^X.

Exercise 7.1 Let A, $A_i \subseteq X$ and $B, B_i \subseteq Y$. Prove that

1. $\cap N(A_i, B) = N(\cup A_i, B)$,
2. $\cap N(A, B_i) = N(A, \cap B_i)$,
3. $\cap N(A_i, B_i) \subseteq N(\cup A_i, \cup B_i)$.

7.1 Topologies on Y^X

Let $\mathscr{P}$ be the collection $\{N(A, B) : A \text{ finite}, B \text{ open}\}$ and let $\mathscr{T}_P$ be the topology generated by $\mathscr{P}$.

Lemma 7.2 *A sequence $\{f_n\}$ converges to f in this topology iff for each $x \in X$, the sequence $\{f_n(x)\}$ converges to $f(x)$ in Y.*

Proof: Assume f_n converges to f in the topology $\mathscr{T}_P$. Take $x \in X$ and let U be an open set in Y around $f(x)$. Now look at the subbasic neighbourhood $N(\{x\}, U)$. Some stage onwards, all the f_n's lie in this neighbourhood. But this means $f_n(x) \in U$ for some $n \geq N$.

Converse is similar. $\square$

Next, let $\mathscr{C}$ be the collection $\{N(A, B) : A \text{ compact}, B \text{ open}\}$ and let $\mathscr{T}_C$ be the topology generated by $\mathscr{C}$. This topology is called the compact-open topology on Y^X. Observe that $\mathscr{T}_P$ is weaker than $\mathscr{T}_C$.

Lemma 7.3 *For $y \in Y$, let f_y denote the constant function $x \mapsto y$. Define $\phi : Y \to Y^X$ by $\phi(y) = f_y$. Then Y is homeomorphic to $\phi(Y)$.*

Proof: Let U be open in Y and $y \in U$. Let K be any compact set in X. Then one has $\phi(y) \in \phi(Y) \cap N(K, U) \subseteq \phi(U)$. Thus $\phi(U)$ is open in $\phi(Y)$. Take a subbasic open set $\phi(Y) \cap N(K, U)$ in $\phi(Y)$. It is easy to check that $\phi^{-1}(\phi(Y) \cap N(K, U)) = U$. Hence ϕ is continuous. Thus ϕ gives a homeomorphism between and $\phi(Y)$. $\square$

The above lemma says that Y can be thaught of as a subspace of Y^X (with compact-open topology or the topology of pointwise convergence). We will now be interested in the subspace $C(X,Y)$ of continuous maps from X to Y.

Lemma 7.4 *The compact-open topology on $C(X,Y)$ is Hausdorff (respectively, regular) iff Y is Hausdorff (respectively, regular).*

Proof: Assume that Y is Hausdorff. Take $f \neq g \in C(X,Y)$. Choose $x \in X$ such that $f(x) \neq g(x)$. Let U and V be two disjoint neighbourhoods of $f(x)$ and $g(x)$ respectively. Then $N(\{x\},U)$ and $N(\{x\},V)$ are two disjoint neighbourhoods of f and g.

Assume next that Y is regular. Take $f \in C(X,Y)$ and U open in $C(X,Y), f \in U$. Let $\cap N(K_i,U_i)$ be a (basic) neighbourhood of f contained in U. For each U_i, get hold of an open set V_i such that $f(K_i) \subseteq V_i \subseteq \overline{V_i} \subseteq U_i$. Then $\cap N(K_i,V_i)$ is an open neighbourhood of f whose closure is contained in U (**why?**).

The converse parts follow from the previous lemma and the fact that subspaces of Hausdorff spaces are Hausdorff and subspaces of regular spaces are regular. $\square$

Exercise 7.5 Show that if X is discrete, then $Y^X \cong \prod_{x\in X} Y_x$, where each $Y_x = Y$.

Exercise 7.6 Check whether the map $(f,x) \mapsto f(x)$ from $C(X,Y) \times X \to Y$ is continuous.

7.2 Arzela-Ascoli theorem

We will now assume that Y is a metric space. In this section, we will characterize relatively compact sets in $C(X,Y)$.

One says that a subset $E \subseteq C(X,Y)$ is **equicontinuous** if for any $x \in X$ and $\epsilon > 0$, there is an open neighbourhood U of x such that $f(U) \subseteq N_\epsilon(f(x))$ for all $f \in E$. It is easy to see that any finite subset of $C(X,Y)$ is equicontinuous, and if E and F both are equicontinuous, then $E \cup F$ is also equicontinuous.

For product spaces, we have seen that if we start with a filterbase $\mathscr{F}$ on the product space for which each $\pi_\alpha(\mathscr{F})$ converges, then the filterbase $\mathscr{F}$ itself converges. A similar statement would fail in general for the compact-open topology on $C(X,Y)$. However, under some additional restrictions on the filterbase, it would continue to hold as the next lemma says.

Lemma 7.7 *Let E be an equicontinuous subset of $C(X,Y)$. Denote by e_x the map $f \mapsto f(x)$. Let $\mathscr{F}$ be a filterbase in E such that $e_x(\mathscr{F})$ converges to $\ell_x \in Y$ for each $x \in X$. Then $\phi : x \mapsto \ell_x$ is continuous and $\mathscr{F}$ converges to ϕ.*

Before beginning the proof, here is a small exercise from real analysis.

Exercise 7.8 Suppose $\{f_n\}$ be a sequence of functions from $\mathbb{R}$ to $\mathbb{R}$ that converges to f. Assume that the family $\{f_n\}$ is equicontinuous. Show that f is continuous.

Proof: The proof consists of two parts: first one shows that ϕ is continuous, and then that $\mathscr{F}$ converges to ϕ. Proof of the first part is essentially the exercise above in the set up of filterbases.

Choose a neighbourhood $N_\epsilon(\ell_x)$ around ℓ_x. Since $e_x(\mathscr{F})$ converges to ℓ_x, there is an $F \in \mathscr{F}$ such that $e_x(F) \subseteq N_\epsilon(\ell_x)$. Also, since the family E is equicontinuous, there is a neighbourhood V around x such that $f(V) \subseteq N_\epsilon(f(x))$ for all $f \in E$. Now take an $y \in V$. We will show that $d(\ell_y, \ell_x) < 3\epsilon$. Since $e_y(\mathscr{F})$ converges to ℓ_y, there is a $G \in \mathscr{F}$ such that $e_y(G) \subseteq N_\epsilon(\ell_y)$. Let $H \subseteq F \cap G$, $H \in \mathscr{F}$. Then we have, for $f \in H$, $f(x) \in N_\epsilon(\ell_x)$, $f(y) \subseteq N_\epsilon(f(x))$ and $f(y) \subseteq N_\epsilon(\ell_y)$. These three together imply that $d(\ell_x, \ell_y) < 3\epsilon$, i.e. the function ϕ is continuous.

Next, take a neighbourhood $N(K, U)$ around ϕ. We will prove that there exists an $F \in \mathscr{F}$ such that $F \subseteq N(K, U)$. This would imply that if $\cap_i N(K_i, U_i)$ is a neighbourhood of ϕ, then there are $F_i \in \mathscr{F}$ such that $F_i \subseteq N(K_i, U_i)$. Which, in turn, will ensure that there is an $F \in \mathscr{F}$ such that $F \subseteq \cap_i N(K_i, U_i)$.

Since $\phi \in N(K, U)$, one has $\phi(K) \subseteq U$. For each $x \in K$, there would exist an $\epsilon_x > 0$ such that $N_{2\epsilon_x}(\ell_x) \subseteq U$, a neighbourhood V_x of x such that $f(V_x) \subseteq N_{\epsilon_x}(f(x))$ for all $f \in E$ and an $F_x \in \mathscr{F}$ such that $e_x(F_x) \subseteq N_{\epsilon_x}(\ell_x)$. Now the family $\{V_x\}$ covers K. Let $\{V_{x_1}, \ldots, V_{x_n}\}$ be a finite subcover. Let F be an element of $\mathscr{F}$ contained in the intersection $\cap_i F_{x_i}$. Take an $f \in F$. Take an $y \in K$. Then $y \in V_{x_i}$ for some i. Since $f \in F_{x_i}$, we have $f(x_i) \in N_{\epsilon_{x_i}}(\ell_{x_i})$. But $f(y) \in f(V_{x_i}) \subseteq N_{\epsilon_{x_i}}(f(x_i))$. Hence $f(y) \in N_{2\epsilon_{x_i}}(\ell_{x_i}) \subseteq U$. This shows that $F \subseteq N(K, U)$. □

Lemma 7.9 *If $E \subseteq C(X, Y)$ is equicontinuos, then so is $\overline{E}$.*

Proof: Suppose $g \in \overline{E}$. Then there is a filterbase $\mathscr{F}$ in E that converges to g. First part of the proof of the lemma above tell us that given a point $x \in X$ and an $\epsilon > 0$, there is an open neighbourhood V that depends only on the family E, such that $f(V) \subseteq N_\epsilon(f(x))$ for all $f \in E$ and $g(V) \subseteq N_{3\epsilon}(g(x))$. This gives equicontinuity of the set $\overline{E}$. □

Theorem 7.10 (Arzela-Ascoli) *Let Y be a metric space. Suppose $E \subseteq C(X, Y)$ is equicontinuous and for each $x \in X$, the set $e_x(E) = \{f(x) : f \in E\}$ is relatively compact in Y. Then E is relatively compact.*

Proof: Take a maximal filterbase $\mathscr{F}$ in $\overline{E}$. Since e_x is continuous, we have $e_x(\overline{E}) \subseteq \overline{e_x(E)}$. So $e_x(\mathscr{F})$ is a maximal filterbase in $\overline{e_x(E)}$. Since this is compact, $e_x(\mathscr{F})$ converges to some point ℓ_x in $\overline{e_x(E)} \subseteq Y$. By lemma 7.7, it follows that $\mathscr{F}$ converges to the function $\phi : x \mapsto \ell_x$, which belongs to $\overline{E}$. □

Here are two applications of the above theorem:

1. to prove weak convergence of a sequence of probability measures on $C[0,1]$ from the convergence of the finite dimensional distributions and tightness (see Billingsley: Weak convergence of probability measures),

2. consider the following system of first order differential equations:

$$\begin{aligned}
\frac{dx_1}{dt} &= f_1(x_1, \dots, x_n; t), \\
\cdots & \quad \cdots \\
\frac{dx_n}{dt} &= f_n(x_1, \dots, x_n; t).
\end{aligned}$$

   One requires the Arzela-Ascoli theorem for proving the existence of solutions of the above system assuming continuity of the functions f_i. (see Birkhoff & Rota: Ordinary differential equations).

LECTURE 12

7.3 Stone-Weierstrass theorem

In this section, we will impose further restrictions on X and Y: we will assume that X is a compact Hausdorff space and $Y = \mathbb{C}$. Denote $C(X, Y)$ by just $C(X)$. Most of the results will continue to hold for the case $Y = \mathbb{R}$. Others hold with very minor modifications.

Let d_0 be a metric on $\mathbb{C}$ equivalent to the standard Euclidean metric. Define a metric d on $C(X)$ by $d(f, g) = \sup_x d_0(f(x), g(x))$.

Lemma 7.11 *Topology coming from the metric d on $C(X)$ is the same as the compact-open topology.*

First, note that due to the extra structure available in $\mathbb{C}$, one can make $C(X)$ into a complex algebra by defining product and sum to be pointwise product and sum respectively. Also, the function $\mathbb{I} : x \mapsto 1$ acts as a multiplicative unity in $C(X)$. Thus $C(X)$ is a unital algebra. The positivity structure (complex conjugation) in $\mathbb{C}$ gives rise to an involution in $C(X)$: $f^*(x) := \overline{f(x)}$. Since X is compact, $\|f\| := \sup |f(x)|$ gives a norm on $C(X)$ that turns it into a Banach space with the following additional properties: it is a unital algebra under pointwise product, $\|fg\| \leq \|f\|\|g\|$, $\|f^* f\| = \|f\|^2$. Such an algebra is called a C^*-algebra. Since multiplication is commutative, it is a commutative C^*-algebra with identity.

Proposition 7.12 *The map $(f, g) \mapsto f + g$ and $f \mapsto -f$ are continuous.*

Exercise 7.13 Show that the sequence $1 - \sum_{j=1}^{n} \binom{\frac{1}{2}}{j}(1 - x)^j$ converges uniformly over the interval $[0, 1]$ to the function $\sqrt{x}$.

Lemma 7.14 *Let A be a closed subalgebra of $C(X, \mathbb{R})$. If $f \in A$, then $|f| \in A$ (If $A \subseteq C(X, \mathbb{C})$, then one needs A to be $*$-closed).*

Proof: Since X is compact, $f(X)$ is contained in a bounded interval. By multiplying by an appropriate scalar, we can assume that $-1 \leq f(x) \leq 1$ for all $x \in X$. Since $f \in A$, f^2 is also in A. So it is enough to show that if $f \in A$ and $0 \leq f(x) \leq 1$ for all $x \in X$, then $\sqrt{f} \in A$. Now by the exercise above, there is a sequence of polynomials p_n such that $p_n(x)$ converges to $\sqrt{x}$ uniformly. This means $p_n(f)$ converges to $\sqrt{f}$. Since $p_n(f) \in A$ for all n, one has $\sqrt{f} \in A$. $\square$

Corollary 7.15 *Let A be a closed subalgebra of $C(X, \mathbb{R})$. If $f \in A$, then $f^{\pm} \in A$.*

Corollary 7.16 *Let A be a closed subalgebra of $C(X, \mathbb{R})$. If $f, g \in A$, then* $\min\{f, g\}$ *and* $\max\{f, g\}$ *are in A.*

Lemma 7.17 *Let A be a unital subalgebra of $C(X)$ that separates points of X. Let x and y be two distinct points of X and let $a, b \in \mathbb{R}$. Then there exists an $f \in A$ such that $f(x) = a$ and $f(y) = b$.*

Theorem 7.18 (Stone-Weierstrass) *Let A be a closed unital subalgebra of $C(X, \mathbb{R})$ that separates points of X. Then $A = C(X, \mathbb{R})$.*

Proof: Take an $f \in C(X)$ and an open neighbourhood $W = \cap_i N(K_i, U_i)$ of f. We will show that there is a $g \in A \cap W$. Write $K = \cup K_i$. The K is compact. Let $\epsilon_i = d(f(K_i), U_i^c)$. Since $f(K_i)$ is compact, U_i^c is closed and these are disjoint, it follows (**how?**) that $\epsilon_i > 0$. Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$. Then $\epsilon > 0$ and if for some g, one has $|f(x) - g(x)| < \epsilon$ for all $x \in K$, then $g \in W$.

Take an $x \in K$. For each $y \in K$, get a function $f_y \in A$ such that $f_y(x) = f(x)$ and $f_y(y) < f(y) + \epsilon$. Let V_y be a neighbourhood of y such that $f_y(z) < f(z) + \epsilon$ for all $z \in V_y$. Get a finite subcollection $\{V_{y_1}, \ldots, V_{y_n}\}$ that covers K. Now take $g_x := \min\{f_{y_1}, \ldots, f_{y_n}\}$. Then one has

$$g_x(x) = f(x), \qquad \text{and } g_x(z) < f(z) + \epsilon \;\; \forall z \in K.$$

Now let W_x be an open neighbourhood of x such that $g_x(y) > f(y) - \epsilon$ for all $y \in W_x$. Get a finite subcollection $\{W_{x_1}, \ldots, W_{x_m}\}$ that covers K. Finally, let $g = \max\{g_{x_1}, \ldots, g_{x_m}\}$. Now check that $f(z) - \epsilon < g(z) < f(z) + \epsilon$ for all $z \in K$. Also from the lemmas above, it follows that $g \in A$. □

Corollary 7.19 *Let A be a unital involutive subalgebra of $C(X, \mathbb{C})$. Then A is dense in $C(X, \mathbb{C})$.*

Proof: $A \cap C(X, \mathbb{R})$ is a unital subalgebra of $C(X, \mathbb{R})$ that separates points of X. Hence $A \cap C(X, \mathbb{R})$ is dense in $C(X, \mathbb{R})$. Now the result follows from the fact that A is the complexification of $A \cap C(X, \mathbb{R})$ and $C(X, \mathbb{C})$ is the complexification of $C(X, \mathbb{R})$. □

Corollary 7.20 *If X and Y are two compact Hausdorff spaces and $f : X \times Y \to \mathbb{R}$ is a continuous function, then for every $\epsilon > 0$ there exist $n > 0$ and continuous functions $f_1, f_2, \ldots, f_n$ in $C(X, \mathbb{R})$ and continuous functions $g_1, g_2, \ldots, g_n$ in $C(Y, \mathbb{R})$ such that $\|f - \sum f_i g_i\| < \epsilon$.*

Proof: Exercise. □

8 Topological groups

A group G equipped with a topology $\mathscr{T}$ is called a **topological group** if the topology is T_1 and if the maps $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous.

Examples.

1. Any group with discrete topology,
2. $(\mathbb{R}, +)$ or $(\mathbb{C}, +)$ with their usual topologies,
3. $(M_n(\mathbb{R}), +)$, $(M_n(\mathbb{C}), +)$.
4. $(GL(n, \mathbb{R}), \cdot)$, $(GL(n, \mathbb{C}), \cdot)$.
5. $U(n)$, $SU(n)$, $O(n)$, $SO(n)$.
6. $SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$.

Notation. Let E and F be subsets of a group G. If we use multiplicative notation, then EF will mean the subset $\{xy : x \in E, y \in F\}$ and E^{-1} will mean the subset $\{x^{-1} : x \in E\}$. In additive notation, same sets will be denoted by $E + F$ and $-E$ respectively. By xF, we will mean the set $\{x\}F$. Similarly Fx will stand for $F\{x\}$.

8.1 Elementary properties

1. Let $x, y \in G$. For any open neighbourhood U of xy, there is an open set V around x and an open set W around y such that $VW \subseteq U$.
2. For any neighbourhood U of x^{-1}, where $x \in G$, there is a neighbourhood V of x such that $V^{-1} \subseteq U$.
3. Let $x, y \in G$. For any neighbourhood U of xy^{-1}, there are neighbourhoods V of y and W of z such that $VW^{-1} \subseteq U$.
4. Choose and fix any $x \in G$. The maps $y \mapsto xy$ and $y \mapsto yx$ are homeomorphisms.
5. The map $x \mapsto x^{-1}$ is a homeomorphism.
6. Let F be closed in G and $x \in G$. Then the sets xF, Fx and F^{-1} are closed.
7. Let U be open in G and $E \subseteq G$. Then the sets EU, UE and U^{-1} are open.
8. Any topological group is homogeneous.
   (A topological space X is called **homogeneous** if for any $x, y \in X$, there is a homeomorphism $\phi : X \to X$ that maps x to y).

9. Let V be a neighbourhood around 1 in G. Then there is a neighbourhood U of 1 such that $1 \in U = U^{-1} \subseteq V$.

10. Let K be a compact subset of G, U open and $K \subseteq U$. Then there is an open set V such that $VK \subseteq U$.

11. If K and L are compact, then KL is also compact.

12. Let $E \subseteq G$ and V be a neighbourhood of the identity. Then $\overline{E} \subseteq EV$.

13. Let U be a neighbourhood of the identity. Then there is another neighbourhood V such that $1 \in V \subseteq \overline{V} \subseteq U$

14. Let $E \subseteq G$. Then $\overline{E} = \cap_V EV = \cap_V VE$, the intersection being taken over all open neighbourhoods V of the identity.

    Proof: Take $x \in \overline{E}$. For any nbd V of 1, xV^{-1} is a nbd of x. Since $x \in \overline{E}$, the set $xV^{-1} \cap E$ is nonempty. Hence there exists $v \in V$ and $y \in E$ such that $xv^{-1} = y$. But this means that $x = yv \in EV$. Thus $\overline{E} \subseteq EV$ for any nbd V of 1.

    Next, assume $x \notin \overline{E}$. Then there is a nbd V of 1 such that $xV \subseteq \overline{E}^c$. We will show that $x \notin EV^{-1}$. If there is an $y \in E$ and $v \in V$ such that $x = yv^{-1}$, then $xv = y$, so that $y \in \overline{E}^c$. □

15. A topological group is Hausdorff.

    Proof: It is enough to show that 1 and and element $x \neq 1$ has disjoint nbds. First choose a nbd U of 1 such that $x \notin U$. Let V be a nbd of 1 such that $VV^{-1} \subseteq U$. Then V and xV are two required nbds. □

16. A topological group is regular.

    Proof: Let F be a closed set not containing 1. We will show that F and 1 has disjoint nbds. F^c is an open nbd of 1. Choose a nbd V of 1 such that $V^2 (:= V \cdot V) \subseteq F^c$. Since $\overline{V} = \cap_U VU \subseteq V^2 \subseteq F^c$. Therefore V and $\overline{V}^c$ are two required nbds. □

17. A locally compact topological group is normal.

    This will be proved later, after introducing the notion of paracompactness.

18. Let H be an open subgroup of G. Then H is also closed.

19. Let H be a closed subgroup of G. Then G/H is a Hausdorff space.

    Proof: It is easy to see that singletons are closed in G/H, so that it is T_1. Take $xH \neq yH$ in G/H. Then $x \notin yH$. By regularity of G, there is a nbd W of 1 such that Wx is disjoint from yH. Choose a nbd V of 1 such that $V^{-1}V \subseteq W$. Now if VxH and VyH are not disjoint, then there are elements $v, v' \in V$ and $h, h' \in H$ such that $vxh = v'yh'$. But then

$v'^{-1}vx = yh'h^{-1} \in yH$, whereas $v'^{-1}vx \in Wx$, which is disjoint from yH. Thus VxH and VyH must be disjoint and are two required nbds. $\square$

20. Let G be a topological group and let H be a closed normal subgroup. Then G/H is a topological group in the quotient topology.

21. Let H be a closed subgroup of G. If G is (locally) compact, then both H and G/H are (locally) compact.

    Proof: If G is compact, then H is compact because it is a closed subset of a compact space and G/H is compact because it is the image of G under a continuous map.

    Assume next that G is locally compact.

    Since G is locally compact, there is a relatively compact neighbourhood U of 1. $V = U \cap H$ is an open neighbourhood of 1 in H. The closure $\overline{V}^H$ is contained in $\overline{U} \cap H$ which is compact. Hence $\overline{V}^H$ itself is compact. Since H is homogeneous, it is locally compact.

    Let $x \in G/H$ and let U be an open neighbourhood of x in G/H. Since G/H has quotient topology, $V = \pi^{-1}(U)$ is open in G. Let $y \in G$ be such that $\pi(y) = x$. Since G is locally compact and V is an open neighbourhood of y in G, there is an open neighbourhood W of y in G such that $\overline{W} \subseteq V$ and $\overline{W}$ is compact. By definition of quotient topology, $\pi(W)$ is open in G/H, and since π is continuous, $\pi(\overline{W})$ is compact in G/H. But G/H is Hausdorff. So $\pi(\overline{W})$ is closed. It is now easy to see that $\pi(W)$ is a relatively compact open neighbourhood of x in G/H, and $\overline{\pi(W)} \subseteq U$. $\square$

Summary

1. Axiom of choice, Zorn's lemma and Zarmelo's theorem: remember the statements so that you can use them if required.

2. Number of connected components is a fundamental invariant for a topological space. It can be used to conclude, for example, that the following pair of spaces are not homeomorphic: $(GL(n, \mathbb{R}), \mathbb{R}^{n^2})$. In the second part of the course, we will see 'higher dimensional' analogues of this invariant.

3. Hausdorffness and regularity are preserved while passing on to subspaces and while forming product spaces.

4.

property	preserved under
Compactness	continuous maps
Hausdorffness	closed bijections
Normality	continuous closed surjections

5. Compact Hausdorff spaces are normal and locally compact Hausdorff spaces are completely regular.

6. Locally compact groups are normal. Any topological group is regular.

7. A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

8. Two theorems related to compactness: Tychonoff's theorem and Arzela-Ascoli theorem. Proofs of both uses the following criterion for compactness: *a space is compact iff any maximal filterbase in it converges.* (This is essentially same as the criterion involving finite intersection property of family of closed sets, but put in a more ready-to-use form).

9. Proof of Stone-Weierstrass theorem is based on the fact that closed subalgebras of $C(X, \mathbb{R})$ are closed under lattice (min, max) operations, the proof of which in turn depends on the fact that the square root function in $[0, 1]$ can be uniformly approximated by polynomials.

10. Applications of Baire category theorem: proving existence of continuous nowhere differentiable functions, and in proving uniform boundedness principle.

11. Urysohn's lemma gives a way to construct (nontrivial) continuuous functions on normal spaces. It will be used later to construct what are known as *partitions of unity*, which is an extremely powerful tool in topology.

12. In many books or papers, you will see the terms filter and ulrafilter. Here is how they are related to filterbases and maximal filterbases:
A **filter** $\mathscr{F}$ is a collection of subsets in a topological space such that $\phi \notin \mathscr{F}$ and if E and

F are in $\mathscr{F}$, then $E \cap F \in \mathscr{F}$. An **ultrafilter** is a filter $\mathscr{F}$ that is maximal, i.e. if $\mathscr{G}$ is another filter and $\mathscr{F} \subseteq \mathscr{G}$, then $\mathscr{F} = \mathscr{G}$.

Check that a filter is a filterbase and a ultrafilter is a maximal filterbase.

Conversely, suppose you have a filterbase $\mathscr{F}'$. Define

$$\mathscr{F} = \{E : \exists F \in \mathscr{F}' \text{ such that } F \subseteq E\}.$$

Then $\mathscr{F}$ is a filter. In case $\mathscr{F}'$ is a maximal filterbase, $\mathscr{F}$ is an ultrafilter.

All the properties of filters and ultrafilters carry over nicely to the corresponding properties of filterbases and maximal filterbases.

Lemma 6.10 for ultrafilters becomes:
$\mathscr{F}$ is an ultrafilter iff for any $E \subseteq X$, either $E \in \mathscr{F}$ or $E^c \in \mathscr{F}$.

Part II

Lecture 13

9 Partitions of unity

9.1 Piecing together continuous functions

A family $\mathscr{E} = \{E_\alpha\}_\alpha$ of subsets of a topological space X is said to be **point-finite** if every point $x \in X$ is contained in at most finitely many E_α's. The family $\mathscr{E}$ is called **nbd-finite** if for any $x \in X$, there is an open nbd U such that U intersects at most finitely many E_α's.

Proposition 9.1 *Let $\mathscr{E} = \{E_\alpha\}_\alpha$ be a nbd-finite family of subsets of X. Then $\{\overline{E_\alpha}\}$ is also nbd-finite and $\cup_\alpha \overline{E_\alpha}$ is closed.*

Proof: Take an $x \in X$. Since $\mathscr{E}$ is nbd-finite, there is a nbd U that intersects only finitely many E_α's. Let F be the set of α's for which U intersects E_α. Now observe that since U is open, for any set A, if $U \cap A$ is empty, then so is $U \cap \overline{A}$. Hence $U \cap \overline{E_\alpha} = \phi$ for all $\alpha \in F^c$.

Take an $x \notin \cup_\alpha \overline{E_\alpha}$. Since $\{\overline{E_\alpha}\}$ is nbd-finite, there is an open nbd U of x and $\alpha_1, \ldots, \alpha_n$ such that $U \cap \overline{E_\alpha} = \phi$ for all $\alpha \notin \{\alpha_1, \ldots, \alpha_n\}$. Now for each i, choose an open nbd U_i around x such that $U_i \cap \overline{E_{\alpha_i}} = \phi$. Let $V = U \cap U_1 \cap \ldots \cap U_n$. Then V is an open nbd of x that does not intersect $\cup_\alpha \overline{E_\alpha}$. □

Proposition 9.2 *Let $\mathscr{U} = \{U_\alpha\}_\alpha$ be an open cover for X. Then a subset E of X is open (resp. closed) if and only if each $E \cap U_\alpha$ is open (resp. closed) in U_α.*

Proof: Exercise. □

Corollary 9.3 *Let $\mathscr{U} = \{U_\alpha\}_\alpha$ be an open cover for X and let Y be another topological space. Let $f_\alpha : U_\alpha \to Y$ be a family of continuous functions such that for each pair (α, β), one has $f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$. Then there exists a unique continuous function $f : X \to Y$ such that $f|_{U_\alpha} = f_\alpha$ for all α.*

Proposition 9.4 *Let $\mathscr{F} = \{F_\alpha\}_\alpha$ be a nbd-finite family of closed sets that cover X. Then a subset E of X is open (resp. closed) if and only if each $E \cap F_\alpha$ is open (resp. closed) in F_α.*

Proof: Assume $E \cap F_\alpha$ is closed in F_α for each α. This means that each $E \cap F_\alpha$ is closed in X and since $\{F_\alpha\}$ is nbd-finite, so is the family $\{E \cap F_\alpha\}$. Hence $E = \cup(E \cap F_\alpha)$ is closed.

If $E \cap F_\alpha$ is open in F_α, then each $E^c \cap F_\alpha = F_\alpha \backslash E \cap F_\alpha$ is closed in F_α, so that by what we have already proved, E^c is closed. □

Corollary 9.5 *Let $\mathscr{F} = \{F_\alpha\}_\alpha$ be a nbd-finite closed cover for X and let Y be another topological space. Let $f_\alpha : F_\alpha \to Y$ be a family of continuous functions such that for each pair (α, β), one has $f_\alpha|_{F_\alpha \cap F_\beta} = f_\beta|_{F_\alpha \cap F_\beta}$. Then there exists a unique continuous function $f : X \to Y$ such that $f|_{F_\alpha} = f_\alpha$ for all α.*

9.2 Paracompactness

Let $\mathscr{U} = \{U_\alpha : \alpha \in \Lambda\}$ and $\mathscr{V} = \{V_\alpha : \alpha \in \gamma\}$ be two open covers for X. $\mathscr{V}$ is said to be a **refinement** of $\mathscr{U}$ if for any $V \in \mathscr{V}$, there is a $U \in \mathscr{U}$ such that $V \subseteq U$. $\mathscr{V}$ is said to be a **precise refinement** of $\mathscr{U}$ if $\Lambda = \Gamma$ and for each $\alpha \in \Lambda$, $V_\alpha \subseteq U_\alpha$.

Proposition 9.6 *Let $\mathscr{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover and let $\mathscr{V} = \{V_\gamma : \gamma \in \Gamma\}$ be a point-finite refinement. Then there $\mathscr{U}$ admits a precise point-finite refinement.*

Proof: For each $\gamma \in \Gamma$, let $J_\gamma = \{\lambda \in \Lambda : V_\gamma \subseteq U_\lambda\}$. Let $\phi : \Gamma \to \Lambda$ be a map such that $\phi(\gamma) \in J_\gamma$ (Axiom of choice). Now define, for each $\lambda \in \Lambda$, $W_\lambda = \cup_{\gamma:\phi(\gamma)=\lambda} V_\gamma$. Then $\{W_\lambda : \lambda \in \Lambda\}$ is a precise refinement of $\mathscr{U}$.

Check that this is point finite. □

A Hausdorff space X is said to be **paracompact** if every open cover admits an open nbd-finite refinement. In other words, given any open cover $\{U_\alpha\}$, there should exist a nbd-finite open cover $\{V_\beta\}$ such that for every V_β, there is a U_α such that $V_\beta \subseteq U_\alpha$. The following results will provide one with a large supply of paracompact spaces.

Proposition 9.7 *Every metric space is paracompact.*

The proof is a little lengthy and will be omitted.

Proposition 9.8 *Let X be a locally compact σ-compact space. Then X is paracompact.*

Proof: There are compact sets K'_n such that $X = \cup_n K'_n$. Without loss in generality, one may assume that $K'_n \subseteq K'_{n+1}$ for all n. Note that since X is locally compact, each point has a relatively compact open nbd. So if K is a compact subset, then one can get a finite cover $\{U_1, \ldots, U_n\}$ consisting of relatively compact open sets. Writing $U = \cup U_i$, one gets an open set U that contains K and has compact closure. Using this observation, let us show that *there is a sequence of compact sets K_n such that $K_n \subseteq int K_{n+1}$ for all n.* Let U_1 be an relatively compact open set containing K'_1. Write $K_1 = \overline{U_1}$. having chosen $K_1, \ldots, K_n$, take K_{n+1} to be the closure of a relatively compact open set that contains $K_n \cup K'_{n+1}$.

Next, define $L_1 = K_1$ and $L_n = K_n \backslash \operatorname{int} K_{n-1}$ for $n > 1$. Check that L_n does not intersect L_m for $|m-n| > 1$. Our aim next is to get compact sets F_n such that $L_n \subseteq \operatorname{int} F_n$ and F_n does not intersect F_m for too many m's. Let $G_n = \operatorname{int} K_{n+1} \backslash K_{n-2}$. Then G_n is open, $L_n \subseteq G_n$ and $G_n \cap G_m = \phi$ for $|n-m| > 2$. Now for each $x \in L_n$, get a relatively compact open nbd U_x with $x \in U_x \subseteq \overline{U_x} \subseteq G_n$. Let $\{U_{x_1}, \ldots, U_{x_k}\}$ be a finite subcover. Define $F_n = \cup_{i=1}^{k} \overline{U_{x_i}}$. Then each F_n is compact, $L_n \subseteq \operatorname{int} F_n \subseteq F_n \subseteq G_n$ and $F_n \cap F_m = \phi$ for $|n-m| > 2$.

Finally, now take an open cover $\mathscr{U}$ of X. Let $\{U_i^{(n)}\}_{1 \leq i \leq k_n}$ be a finite subcollection that covers L_n. Then $\{U_i^{(n)} \cap \operatorname{int} F_n : n \geq 1, 1 \leq i \leq k_n\}$ is a nbd-finite refinement of $\mathscr{U}$. $\square$

Proposition 9.9 *Every paracompact space is regular.*

Proof: Let X be paracompact. Take an $x \in X$ and a closed set F such that $x \notin F$. Since X is Hausdorff, for each $y \in F$, there is a disjoint pair of nbds U_y and V_y of x and y respectively. Now look at the open cover $\{V_y : y \in F\} \cup \{F^c\}$ and get a nbd-finite refinement, say, $\mathscr{U}$. Let $\mathscr{U}'$ be the subcollection of all those $U \in \mathscr{U}$ for which U intersects F. First observe that $\mathscr{U}'$ covers F. Also, for any $U \in \mathscr{U}'$, one has $U \subseteq V_y$ for some $y \in F$, so that $x \notin \overline{U}$. Write $W = \cup_{U \in \mathscr{U}'} U$ Now since $\mathscr{U}'$ is nbd-finite, one has $\overline{W} = \cup_{U \in \mathscr{U}'} \overline{U}$. And since $x \notin \overline{U}$ for all $U \in \mathscr{U}'$, we have $x \notin \overline{W}$. Thus W and $\overline{W}^c$ are two required nbds. $\square$

Proposition 9.10 *Every paracompact space is normal.*

Proof: Let X be paracompact and let E and F be two disjoint closed sets. The argument then is identical to the proof of the previous proposition, with x replaced by E and instead of Hausdorffness, one uses regularity. $\square$

Corollary 9.11 *Any locally compact group is normal.*

Proof: Take a relatively compact open nbd U of 1 such that $U = U^{-1}$. Let $H = \cup_{n \geq 1} U^n$. Then H is a subgroup that is locally compact, σ-compact. Hence by the results above, H is normal. Also notice that H is open in G. It is now easy to use this to prove that G is also normal. $\square$

LECTURE 14

9.3 Existence

Let X be a Hausdorff space. A family $\{\phi_\alpha\}$ of cotinuous functions from X to $[0,1]$ is called a **partition of unity** on X if the $\{\operatorname{supp}\phi_\alpha\}$ form a nbd-finite closed cover for X, and $\sum_\alpha \phi_\alpha(x) = 1$ for all $x \in X$.

Let $\mathscr{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover for X. A partition of unity $\{\phi_\alpha\}_{\alpha\in\Lambda}$ is said to be **subordinated to** $\mathscr{U}$ if $\operatorname{supp}\phi_\alpha \subseteq U_\alpha$ for each α.

Exercise 9.12

1. Let X be normal and let $\{U, V\}$ is an open cover for X. Show that there is an open set W such that $\overline{W} \subseteq U$ and (W, V) still covers X.
2. Let X be normal and let $\{U_1, \dots, U_n\}$ be an open cover for X. Show that there is an open cover $\{V_1, \dots, V_n\}$ such that $\overline{V_i} \subseteq U_i$ for all i.

Theorem 9.13 (Shrinking lemma) *Let X be normal and let $\{U_\alpha\}$ be a point-finite open cover. Then there exists a point-finite open cover $\{V_\alpha\}$ such that for each α, one has $\overline{V_\alpha} \subseteq U_\alpha$.*

Proof: Let us call the index set Λ. Let '$<$' be a well-order on Λ. We say a subset S of Λ has property P if there exist open sets V_α for $\alpha \in S$ such that $\overline{V_\alpha} \subseteq U_\alpha$ for all $\alpha \in S$ and the collection $\{V_\alpha\}_{\alpha\in S} \cup \{U_\alpha\}_{\alpha\in S^c}$ covers X. We will show that if for any $\alpha \in \Lambda$, $S_\alpha := \{\beta \in \Lambda : \beta < \alpha\}$ has property P, then $\{\beta \in \Lambda : \beta \le \alpha\}$ also has property P. By transfinite induction, the result will follow.

Since S_α has property P, the collection $\{V_\beta\}_{\beta\in S_\alpha} \cup \{U_\beta\}_{\beta\in S_\alpha^c}$ covers X. Hence $(\cup_{\beta<\alpha} V_\beta) \cup (\cup_{\beta>\alpha} U_\beta)^c \subseteq U_\alpha$. Since X is normal, there is an open set V_α such that $(\cup_{\beta<\alpha} V_\beta) \cup (\cup_{\beta>\alpha} U_\beta)^c \subseteq V_\alpha \subseteq \overline{V_\alpha} \subseteq U_\alpha$. Therefore $\{V_\beta\}_{\beta\le\alpha} \cup \{U_\beta\}_{\beta>\alpha}$ covers X. □

Theorem 9.14 *Let X be normal and let $\mathscr{U}$ be a point-finite open cover for X. Then there exists a partition of unity on X subordinated to $\mathscr{U}$.*

Proof: Write $\mathscr{U} = \{U_\alpha\}_{\alpha\in\Lambda}$. Get open covers $\{V_\alpha\}_{\alpha\in\Lambda}$ and $\{W_\alpha\}_{\alpha\in\Lambda}$ such that $\overline{W_\alpha} \subseteq V_\alpha \subseteq \overline{V_\alpha} \subseteq U_\alpha$ for each α. Use Urysohn lemma to get continuous functions $g_\alpha : X \to [0,1]$ such that $g_\alpha|_{W_\alpha} = 1$ and $g_\alpha|_{\overline{V_\alpha}^c} = 0$. Now define

$$\phi_\alpha(x) = \frac{g_\alpha(x)}{\sum_\beta g_\beta(x)}.$$

□

Theorem 9.15 *Let X be paracompact and let $\mathscr{U}$ be an open cover for X. Then there exists a partition of unity on X subordinated to $\mathscr{U}$.*

Proposition 9.16 *Let X be normal and let $\mathscr{U}$ be a finite open cover for X. Then there exists a partition of unity on X subordinated to $\mathscr{U}$.*

Lecture 15

For the rest of the course, our stress will be on getting familiar with lots of examples of topological spaces, most of them sitting inside some $\mathbb{R}^n$, so that when we study the problem of classification of these spaces, we can understand and appreciate better various results. In the process, we will also learn about various construction techniques that topologists use all the time to produce new, more complicated spaces out of tractable ones.

10 Quotient spaces

Let X be a topological space and let $\sim$ be an equivalence relation on X. Let $Y = X/\sim$ and let $\pi : X \to Y$ be the canonical projection. The **quotient topology** on Y is defined to be the collection of all those subsets U of Y for which $\pi^{-1}(U)$ is open in X. Thus quotient topology on Y is the finest topology for which π is continuous.

What one is doing here is the following. Suppose $\mathscr{P}$ is the partition of X coming from $\sim$. Now look at those elements of $\mathscr{T}_X$ that are unions of elements in $\mathscr{P}$. This subcollection is again a topology on X. Now carry this topology to $X/\sim$ via the projection map π.

Let X and Y be two topological spaces. A map $f : X \to Y$ is said to be an **identification map** if f is onto and Y has the quotient topology.

Main examples of quotient spaces that arise naturally are homogeneous spaces of topological groups, and orbit spaces for actions of (topological) groups on (topological) spaces.

Another class of examples that do not quite occur 'naturally', but are very crucial to a mathematician/topologist are spaces obtained from a given topological space by collapsing one (or more) subspace(s) to a point. In other words, suppose we have a topological space X, and let A be a subspace. Now define an equivalence relation $\sim$ on X by declaring $x \sim y$ if either $x, y \in A$ or $x = y$. The resulting quotient space $Y = X/\sim$ is called the space obtained from X **by collapsing** A **to a point**.

Here is a special case of the above construction. Let X be a topological space, F be a subset, and let $\alpha : F \to X$ be a continuous map. Now take the product space $X \times [0, 1]$ and define $A_x = \{(x, 0), (\alpha(x), 1)\}$ for $x \in F$. The space $X \times [0, 1]/\{A_x\}_{x\in F}$ will normally be denoted by $X \times [0, 1]/\alpha$.

Exercise 10.1 Show that

1. if X is regular and A is closed, then X/A is Hausdorff,
2. if X is normal and A is closed, then X/A is normal.

Yet another class of examples come from **gluing** of two (or more) topological spaces. Let X, Y be two topological spaces, $A \subseteq X$, and let $\phi : A \to Y$ be a continuous map. Define an equivalence relation $\sim$ on $X \sqcup Y$ by declaring $x \sim y$ if $x = y$ or $\phi(x) = y$ (or $\phi(y) = x$). Then the space $Y \cup_\phi X := X \sqcup Y/\sim$ is the space obtained by gluing (or attaching) X and Y using the attaching map ϕ.

Proposition 10.2 *Let Z be another topological space and let $f : Y \to Z$ be a map. Then f is continuous if and only if $f \circ \pi$ is continuous.*

Proof: Assume $f : Y \to Z$ is continuous. Then for any U open in Z, $f^{-1}(U)$ is open in Y. By definition, this means $\pi^{-1}(f^{-1}(U))$ is open in X. But $\pi^{-1}(f^{-1}(U)) = (f \circ \pi)^{-1}(U)$. Thus $f \circ \pi$ is continuous.

Conversely, assume that $f \circ \pi$ is continuous, so that for any open U in Z, $\pi^{-1}(f^{-1}(U)) = (f \circ \pi)^{-1}(U)$ is open in X. Again, by definition of the quotient topology, $f^{-1}(U)$ is open in Y.
$\square$

The following would provide an useful tool in identifying quotient spaces in nice situations.

Proposition 10.3 *Let X be compact and let $f : X \to Z$ be a continuous map onto another topological space Z. Then the injective factor $\tilde{f} : X/\sim \to Z$ is a homeomorphism, where $\sim$ is the equivalence relation given by $x \sim y$ if $f(x) = f(y)$.*

10.1 Examples

1. $X = [0,1]$, $A = \{0,1\}$. Then $X/A \cong S^1$.

2. $X = [0,1] \times [0,1]$, $A_t = \{(0,t),(1,t)\}$, $t \in [0,1]$. Then $X/\{A_t\}_t \cong$ cylinder over S^1 ($S^1 \times [0,1]$).

   Another way to describe this example would be the following. Let $\alpha : [0,1] \to [0,1]$ be given by $\alpha(t) = t$. Then $X/\{A_t\}_t = [0,1] \times [0,1]/\alpha$.

3. $X = [0,1]$, but this time take α to be $\alpha(t) = 1 - t$. Then $X \times [0,1]/\alpha$ is the **Möbius strip**.

4. $X = S^1$, $\alpha(z) = z$. The $X \times [0,1]/\alpha \cong \mathbb{T}^2 \cong S^1 \times S^1$.

5. $X = S^1$, $\alpha(z) = \bar{z}$. The $X \times [0,1]/\alpha$ is the **Klein's bottle**.

6. $X = \mathbb{D}^2$, $A = S^1$. Then $X/A \cong S^2$.

7. $X = \mathbb{D}^n$, $A = S^{n-1}$. Then $X/A \cong S^n$.

8. $X = S^1 \times [0,1]$, $A = S^1 \times \{0\}$. then $X/A \cong \mathbb{D}^2$.

9. $X = S^{n-1} \times [0,1]$, $A = S^{n-1} \times \{0\}$. then $X/A \cong \mathbb{D}^n$.
(look at the map $f : X \to \mathbb{D}^n$ given by $f(x,t) = tx$. One has $f^{-1}\{0\} = S^1 \times \{0\}$)

10. $X = \mathbb{R}^2$. Define

$$\begin{aligned} A_s &= \{(x, \tan x - s) : x \in (-\pi/2, \pi/2)\}, \quad s \in \mathbb{R}, \\ B_t &= \{(t, x) : x \in \mathbb{R}\}, \quad t \in \mathbb{R} \backslash (-\pi/2, \pi/2). \end{aligned}$$

Let $\sim$ be the equivalence relation coming from the partition $\{A_s\}_s \cup \{B_t\}_t$.

Exercise 10.4 Show that the quotient space in the above example is hemeomorphic to $\mathbb{R}$.
Hint: Define $f : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} t & \text{if } x \in B_t, \\ \frac{s}{1+|s|} & \text{if } x \in A_s. \end{cases}$$

11. $X = \mathbb{R}^2$. Define

$$\begin{aligned} A_s &= \{(x, |\tan x| - s) : x \in (-\pi/2, \pi/2)\}, \quad s \in \mathbb{R}, \\ B_t &= \{(t, x) : x \in \mathbb{R}\}, \quad t \in \mathbb{R} \backslash (-\pi/2, \pi/2). \end{aligned}$$

Let $\sim$ be the equivalence relation coming from the partition $\{A_s\}_s \cup \{B_t\}_t$.

Exercise 10.5 Show that the space in example 10 is Hausdorff while that in example 11 is not.
(Hint: one can not separate $B_{-\pi/2}$ and $B_{\pi/2}$).

12. $X = S^n$, $x \sim -x$. The space $S^n/\sim$ is known as the **real projective plane** of dimension n and is denoted by $\mathbb{R}P^n$.

13. $X = \mathbb{R}^{n+1} \backslash \{0\}$, $L_x =$line passing through x, $x \in X$. Then $X/\{L_x\}_x \cong \mathbb{R}P^n$.

14. $X = \mathbb{D}^n$, $\phi : S^{n-1} \to S^{n-1}$ is the identity map. Then $X \cup_\phi X \cong S^n$.

10.2 Properties

Proposition 10.6 *If X is connected then Y is also connected.*

Proof: Because the projection map is continuous. □

Same proof works for the next two statement also.

Proposition 10.7 *If X is path connected then Y is also path connected.*

Proposition 10.8 *If X is compact then Y is also compact.*

Theorem 10.9 *A homogeneous space G/H is Hausdorff if and only if H is closed in G.*

Proof: If H is closed, we have already seen that G/H is Hausdorff. Let us prove the converse.

Since G/H is Hausdorff, given any point xH of G/H, there is an open set U in G/H such that $xH \in U$ but $H \notin U$. By definition of quotient topology, UH is open in G and $xH \subseteq UH$, $H \subseteq (UH)^c$. Union of all the UH's is H^c. Hence H is closed. $\square$

Proposition 10.10 *If Y is Hausdorff, then each equivalence class in X is closed.*

Proof: This is essentially the second part of the proof of the previous theorem. $\square$

LECTURE 16

11 Examples of topological spaces

11.1 Problems

1. Let $X = [0,1]$ and let $\mathscr{E} = \{E_1, \dots, E_n\}$ be the partition given by

$$E_1 = [0, 1/n), E_2 = [1/n, 2/n), \dots, E_{n-1} = [1 - 2/n, 1 - 1/n), E_n = [1 - 1/n, 1].$$

   Describe the quotient space $X/\mathscr{E}$.

2. Prove that $\mathbb{R}/\mathbb{Z}$ is homeomorphic to S^1.

3. In each of the following cases, show that $S^1/\sim$ is homeomorphic to S^1:

   (a) $z \sim \exp(2\pi i/3)z$,

   (b) $z \sim -z$.

4. In each of the following cases, show that $\mathbb{D}/\sim$ is homeomorphic to $\mathbb{D}$:

   (a) $(x, y) \sim (x, -y)$,

   (b) $(x, y) \sim (-y, x)$,

   (c) $(x, y) \sim (-x, -y)$.

5. Describe the following spaces as quotient spaces of the interval $[0, 1]$:

   (a) the figure 'A',

   (b) the figure 'α',

   (c) the figure '8'.

6. Let $X = \{T \in M_{n+1}(\mathbb{R}) : T^t = T^2 = T, \operatorname{trace} T = 1\}$. Give it the subspace topology coming from $M_{n+1}(\mathbb{R})$. Define a map $q : S^n \to X$ by $q(x_1, \dots, x_n) = ((x_i x_j))$.

   (a) Show that q is continuous and onto.

   (b) Show that $q(x)z = \langle x, z \rangle x$.

   (c) Use this to prove that $q(x) = q(y)$ if and only if $y = -x$.

(d) Show that $\mathbb{R}P^n \cong X$.

7. (a) Let X and Y be two topological spaces and E be a subspace of X. Let f and g be two continuous maps from E to Y. If there is a homeomorphism $\phi : Y \to Y$ such that $\phi \circ f = g$, then $X \cup_f Y$ and $X \cup_g Y$ are homeomorphic.

   (b) Show that for any homeomorphism $\phi : S^{n-1} \to S^{n-1}$, the space $\mathbb{D}^n \cup_\phi \mathbb{D}^n$ is homeomorphic to S^n.

8. Describe Klein's bottle as

   (a) a quotient space of $[0,1]^2$.

   (b) a quotient space of $\mathbb{D}^2$.

   (c) a quotient space of the Mobius strip.

9. (a) Express S^2 as a quotient space of $[0,1]^2$.

   (b) Express S^n as a quotient space of $[0,1]^n$.

10. Prove that the following pairs of spaces are homeomorphic:

    (a) $\mathbb{R}^2 \backslash \{(0,0)\}$ and $S^1 \times \mathbb{R}$,

    (b) $\mathbb{R}^n \backslash \mathbb{R}^k = \{(x_1, \ldots, x_n) : (x_{k+1}, \ldots, x_n) \neq (0, \ldots, 0)\}$ and $S^{n-k-1} \times \mathbb{R}^{k+1}$,

    (c) $\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum x_i^2 = 1,\ x_1^2 + \ldots + x_k^2 \geq x_{k+1}^2 + \ldots + x_{n+1}^2\}$ and $S^{k-1} \times \mathbb{D}^{n-k+1}$,

    (d) $O(n)$ and $SO(n) \times O(1)$,

    (e) $GL(n, \mathbb{R})$ and $SL(n, \mathbb{R}) \times GL(1, \mathbb{R})$,

    (f) $GL_+(n, \mathbb{R}) = \{T \in GL(n, \mathbb{R}) : \det T > 0\}$ and $SO(n) \times \mathbb{R}^{\frac{n(n+1)}{2}}$.

11. (a) Prove that $S^1 \times S^1$ can be topologically embedded in $\mathbb{R}^3$.

    (b) Prove that $\underbrace{S^1 \times \ldots \times S^1}_{\text{n times}}$ can be topologically embedded in $\mathbb{R}^{n+1}$.

12. We have already seen that $S^1 \times S^1$ can be written as a quotient space of the unit square $[0,1]^2$. Draw a curve on the torus such that if you cut it along that curve, you get back the unit square.

13. Identify opposite points on the boundary of $\mathbb{D}^2$ (i. e. $x \sim -x$ for $\|x\| = 1$). Show that the resulting quotient space is homeomorphic to $\mathbb{R}P^2$.

11.2 Solutions

1. $Y = \{1, 2, \ldots, n\}$, $\mathscr{T} = \{\{1\}, \{1,2\}, \ldots, \{1, 2, \ldots, n-1\}, \{1, 2, \ldots, n\}\}$.

2. Define $f : \mathbb{R} \to S^1$ by $f(t) = \exp(2\pi i t)$. Prove that it lifts to a homeomorphism.

3. Look at the maps $f(z) = z^3$ and $f(z) = z^2$ respectively.

4. Define $f : \mathbb{D} \to Y := \{(x, y) \in \mathbb{D} : y \geq 0\}$ by $f(x, y) = (x, |y|)$. Lift it to a homeomorphism from $\mathbb{D}/\sim$ to Y. Then show that Y and $\mathbb{D}$ are homeomorphic.

   Other two parts are similar.

5. (a) collapse $\{1/3, 2/3\}$ to a point,

   (b) collapse $\{0, 1\}$ and $\{1/3, 2/3\}$.

6. Continuity is clear. If one fixes a basis for an $(n + 1)$-dimensional vector space, X is nothing but the space of all rank one projections, and $q(x)$ is the projection onto the span of x (part (b) says precisely this). Surjectivity of q is then immediate. And so is part (c).

7. (a) Denote by ψ the map from $X \sqcup Y$ to $X \sqcup \phi(Y) = X \sqcup Y$ given by

   $$\psi(z) = \begin{cases} z & \text{if } z \in X, \\ \phi(z) & \text{if } z \in Y. \end{cases}$$

   Let π be the projection map from $X \sqcup \phi(Y)$ to $X \sqcup \phi(Y)/\{x \sim \phi \circ f(x)\}$. For the composition $\pi \circ \psi$, inverse images of singletons are either singletons or of the form $\{x, f(x)\}$. Hence $\pi \circ \psi$ lifts to an injective continuous map from $X \cup_f Y$ to $X \cup_g Y$. Similarly construct a map from $X \cup_g Y$ to $X \cup_f Y$ and show they are inverses of each other.

   (b) Define $g : \mathbb{D}^n \sqcup \mathbb{D}^n \to S^n$ as follows:

   $$g(\mathbf{x}) = \begin{cases} (x_1, \ldots, x_n, \sqrt{1 - \sum x_i^2}) & \text{if } \mathbf{x} \text{ comes from the first copy of } \mathbb{D}^n, \\ (x_1, \ldots, x_n, -\sqrt{1 - \sum x_i^2}) & \text{if } \mathbf{x} \text{ comes from the second copy of } \mathbb{D}^n. \end{cases}$$

   Prove that $g(\mathbf{x}) = g(\mathbf{y})$ iff $\mathbf{x} = \mathbf{y} \in S^{n-1}$. Hence prove that $\mathbb{D}^n \cup_f \mathbb{D}^n \cong S^n$, where $f : S^{n-1} \to S^{n-1}$ is the identity map.

   Now by part (a) the result follows.

8.

9. For both parts, identify all points on the boundary.

10. (a) Take the map

    $$(x, y) \mapsto (\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, \log(x^2 + y^2)).$$

    (b) Write $r = \sqrt{\sum_{i=k+1}^n x_i^2}$. Now look at the map

    $$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k, \log r, \frac{x_{k+1}}{r}, \ldots, \frac{x_n}{r})$$

(c) Write $r = \sqrt{\sum_{i=1}^k x_i^2}$. Take the map

$$(x_1, \ldots, x_{n+1}) \mapsto (\frac{x_1}{r}, \ldots, \frac{x_k}{r}, x_{k+1}, \ldots, x_{n+1}).$$

(d) Look at the map $T \mapsto ((\det T)T, \det T)$.

(e) Look at the map $T \mapsto ((\det T)^{-1}T, \det T)$.

(f) Look at the map $T \mapsto ((TT^t)^{-1/2}T, TT^t)$ from $GL_+(n, \mathbb{R})$ to $SO(n) \times Y$ where Y is the space of nonnegative matrices. Now show that $Y \cong \mathbb{R}^{\frac{n(n+1)}{2}}$.

11. (a) Write $X = S^1 \times [-1, 1]$. Define two maps $f_\pm : X \to \mathbb{R}^3$ by

$$f_\pm(x, y, t) = (x(1 \pm \frac{1}{2}\sqrt{1 - t^2}), y(1 \pm \frac{1}{2}\sqrt{1 - t^2}), t).$$

Note that $f_+(x, y, t) = f_-(z, w, s)$ if and only if $s = t = \pm 1$, $x = z$ and $y = w$. Now use the fact that $S^1 \times S^1 \cong X \cup_g X$, where g is the identity map on ∂X.

(b) Write $X = \underbrace{S^1 \times \ldots \times S^1}_{\text{n times}}$. Suppose ϕ_n stands for a topological embedding of X in $\mathbb{R}^{n+1}$. Write $\phi_n(\mathbf{z}) = (x_1, \ldots, x_{n+1})$. Now define two maps $\phi_{n+1}^\pm : X \times [-1, 1] \to \mathbb{R}^{n+2}$ as follows:

$$\phi_{n+1}^\pm(\mathbf{z}, t) = (x_1(1 \pm r), \ldots, x_{n+1}(1 \pm r), t), \quad \text{where } r = \sqrt{1 - t^2}.$$

Now proceed as before.

12.

13. Define $f : \mathbb{D}^2 \to S^2/\sim$ (where $\mathbf{z} \sim -\mathbf{z}$) by

$$f(x, y) = \begin{cases} (x, y, -\sqrt{1 - x^2 - y^2}) & \text{if } x^2 + y^2 < 1, \\ (x, |y|, 0) & \text{if } x^2 + y^2 = 1 \text{ and } x \neq -1, \\ (1, 0, 0) & \text{if } x = -1, y = 0. \end{cases}$$

Prove that this is continuous and onto.

Alternately, Take the map $f : (x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2})$ that will give a homeomorphism between $\mathbb{D}^2$ and the upper hemisphere. Now compose this with the map q considered in problem 6.

Lecture 17

11.3 Manifolds

A topological space X is said to be a **manifold** of dimension n if it is Hausdorff and if every point x in X has an open nbd U_x homeomorphic to $\mathbb{R}^n$. Manifolds of dimension 2 are called **surfaces**.

Theorem 11.1 *A connected manifold is path connected.*

Proof: Let S be a manifold of dimension n. Choose and fix a point x in S. Denote by P the path component of S containing the point x. Take any point y in P. Since S is a manifold, there is a nbd U of y that is homeomorphic to $\mathbb{R}^n$. Therefore U is path-connected. It follows that $P \cup U$ is path-connected. Since P is a path-component, one must have $U \subseteq P$. Thus P is open. Since each path-component is open and S decomposes as a disjoint union of path-components, P is closed in S. Thus P is both open and closed in S. If S is connected, it follows that $P = S$. □

11.4 Embeddability in $\mathbb{R}^n$

Theorem 11.2 *Let X be a compact manifold of dimension n. Then X can be embedded in $\mathbb{R}^m$ for some $m \geq n$.*

Since X is compact, there is a finite open cover $\mathscr{U} = \{U_1, \ldots, U_k\}$ such that each U_i is homeomorphic to $\mathbb{R}^n$. Let $g_i : U_i \to \mathbb{R}^n$ be homeomorphisms. By results in the previous section, there is a partition of unity $\{\phi_1, \ldots, \phi_k\}$ subordinate to $\mathscr{U}$. Define functions h_i on X as follows:

$$h_i(x) = \begin{cases} g_i(x)\phi_i(x) & \text{if } x \in U_i, \\ 0 & \text{if } x \notin \operatorname{supp} \phi_i. \end{cases}$$

Check that each h_i is continuous. Now define a function

$$\psi : X \to \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{k \text{ times}} \times \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k \text{ times}}$$

by $\psi(x) = (\phi_1(x), \ldots, \phi_k(x), h_1(x), \ldots, h_k(x))$. It is easy to show that this is continuous and injective. Since X is compact, it follows that ψ is a homeomorphism between X and its image. □

11.5 Examples

Let us now have a look at examples of manifolds. Some of them arise naturally in various contexts. Some are parts of a topologist/geometer's toolbox.

1. **Graphs of functions**. Let U be an open set in $\mathbb{R}^2$ and let $f : U \to \mathbb{R}$ be a continuous map. Then $M = \{(x, y, f(x, y)) : (x, y) \in U\}$ is a surface (**Prove**!).

2. **Surface of revolution**. Take a curve lying in the upper half plane of the xy-plane. Rotate it around the x-axis. The trajectory is a surface in $\mathbb{R}^3$. (proving this is essentially same as proving that the cylinder $S^1 \times \mathbb{R}$ is a surface)

3. **Level sets**. Let U be an open set in $\mathbb{R}^3$, and $f : U \to \mathbb{R}$ be a smooth function. Call a real r a regular value of f if $Df(x)$ has rank one (i. e. $Df(x) \neq (0, 0, 0)$) for all $x \in f^{-1}\{r\}$. If r is a regular value of f, then $M := f^{-1}\{r\}$ is a surface.

   More generally, if U is open in $\mathbb{R}^{m+n}$ and $f : U \to \mathbb{R}^m$ is smooth, one says $r \in \mathbb{R}^m$ is a regular value of f if $Df(x)$ is of rank m for all $x \in f^{-1}\{r\}$. In such a situation, if r is a regular value of f, then $M := f^{-1}\{r\}$ is a manifold of dimension n.

   The proof of the above involves the following theorem.

   Theorem 11.3 (Inverse function theorem) *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, f be C^1 on $E \subseteq \mathbb{R}^n$. $a \in E$, $f'(a)$ is invertible, $b = f(a)$. Then*

   (a) there exist open sets U and V in $\mathbb{R}^n$ such that $a \in U$, $b \in V$, $f|_U$ is bijective and $f(U) = V$.

   (b) if g is the inverse of f on U, then $g \in C^1(V)$.

   Write $f = (f_1, \ldots, f_m)$. Let $p \in M$. Without any loss in generality, we can assume that the submatrix $((\frac{\partial f_i}{\partial x_j}(p)))_{1 \leq i,j \leq m}$ is invertible. Consider now the map $\phi : U \to \mathbb{R}^{m+n}$ given by $\phi(\mathbf{x}) = (f(\mathbf{x}), x_{m+1}, \ldots, x_{m+n})$. Then we have

$$D\phi(p) = \begin{pmatrix} \left(\left(\frac{\partial f_i}{\partial x_j}(p)\right)\right) & * \\ 0 & I_{n\times n} \end{pmatrix},$$

   a nonsingular matrix. Hence by the Inverse Function Theorem, there is a neighbourhood V of p in U such that ϕ is a diffeomorphism from V onto $\phi(V)$. Let us now introduce a new set of coordinates on V:

$$x_i'(\mathbf{x}) = \begin{cases} f_i(\mathbf{x}) & \text{if } i \leq m, \\ x_i & \text{if } i > m. \end{cases}$$

   Then with respect to these coordinates, we see that $M \cap V = \{\mathbf{x} \in V : x_i'(\mathbf{x}) = q_i \text{ for } i \leq m\}$. On $M \cap V$, we introduce a new coordinate system by $y_i(\mathbf{x}) = x_{i+m}'(\mathbf{x})$, $1 \leq i \leq n$. Thus the relative neighbourhoods $M \cap V$ in M locally look like nbds in $\mathbb{R}^n$.

4. **Grassmann manifolds.** Let $< .,. >$ be a positive definite inner product on $\mathbb{R}^n$. For every k-dimensional subspace E of $\mathbb{R}^n$, let $\rho_E : \mathbb{R}^n \to E$ and $\rho_E^\perp : \mathbb{R}^n \to E^\perp$ be the orthogonal projection. Consider the set $\mathcal{G}_\mathbb{R}(n;k)$ of all k-dimensional subspaces of $\mathbb{R}^n$. For $E \in \mathcal{G}_\mathbb{R}(n;k)$, set $U_E = \{F \in \mathcal{G}_\mathbb{R}(n;k) : F \cap E^\perp = 0\}$ and define $\phi_E : U_E \to L(E, E^\perp)$ by

$$\phi_E(F) = P_{E^\perp} \circ (P_{FE})^{-1}, \quad (P_{FE} = P_E|_F : F \to E).$$

Check that $\mathcal{G}_\mathbb{R}(n;k)$ is a smooth manifold of dimension $k(n-k)$ with atlas $\{(U_E, \phi_E)\}$.

$\mathcal{G}_\mathbb{R}(n+1;1)$ is the real projective plane $\mathbb{R}P^n$ that we have already encountered.

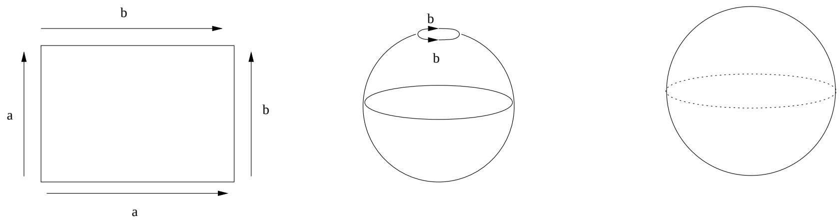
Examples 1 (for f smooth), 2, 3 and 4 occur in the context of differential geometry. Not all topological n-manifolds come naturally equipped with a smooth structure. Sometime one constructs and passes on to a smooth manifold that is homeomorphic to the given one in search of computable topological invariants.

Lecture 18

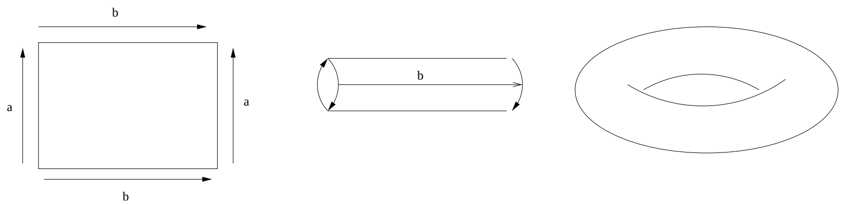
5. **Quotient spaces of polygons in $\mathbb{R}^2$.** Let P be a polygon in $\mathbb{R}^2$ with $2n$ edges $E_1, F_1, \ldots, E_n, F_n$. Let $\phi_i : E_i \to F_i$ be homeomorphisms. $\{(E_i, F_i, \phi_i) : 1 \leq i \leq n\}$ will be called a **gluing scheme** for the polygon P. Define an equivalence relation $\sim$ as follows: $x \sim y$ if either $x = y$ or $x = \phi_{i_1}^{\pm 1} \circ \ldots \circ \phi_{i_r}^{\pm 1}(y)$. Then $P/\sim$ gives a surface. Of course, this requires a proof. But for the moment, we will not bother about the proof. Instead, we will look at several examples of surfaces arising in this manner, and try and visualise them as far as possible.

   (a) Rectangle.

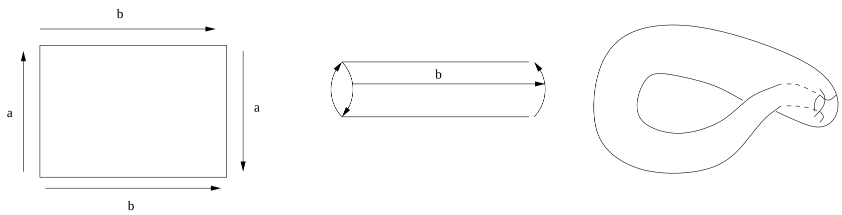
      i. Sphere:



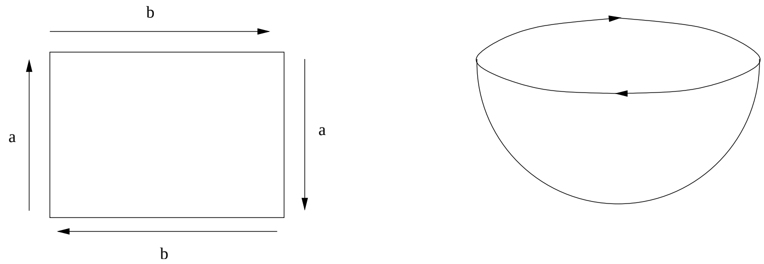
      ii. Torus:



      iii. Klein's bottle:



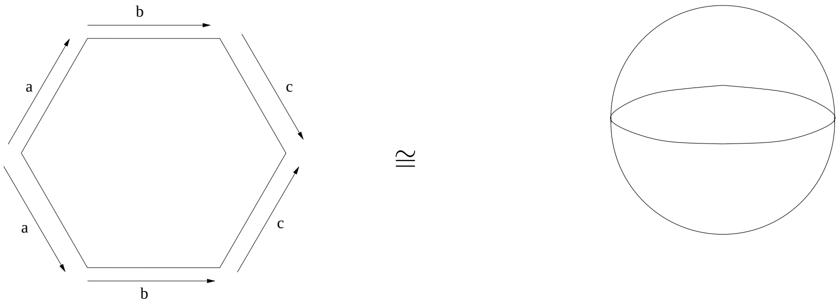
iv. Projective plane of dimension 2 ($\mathbb{R}P^2$):



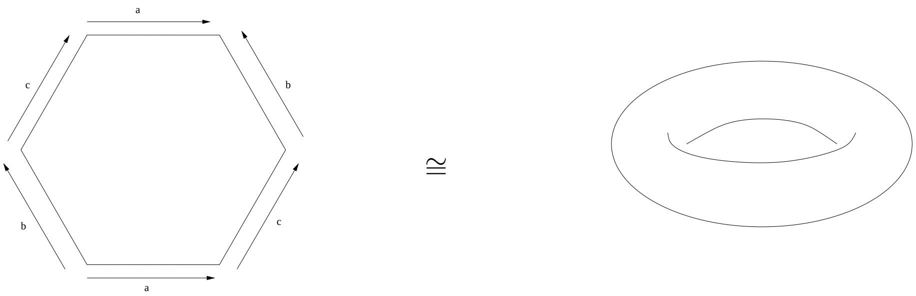
Exercise 11.4 Use the above descriptions of the Klein's bottle and $\mathbb{R}P^2$ to show that they both contain the Mobius strip.

(b) Hexagon.

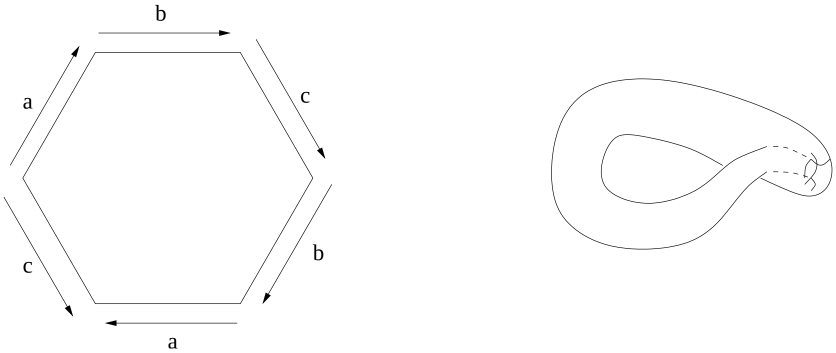
i. Sphere:



ii. Torus:

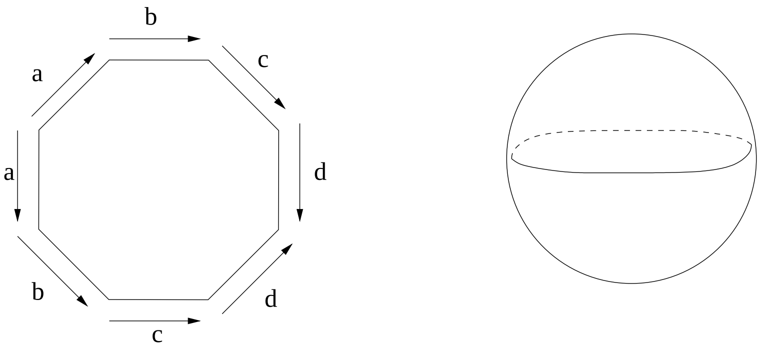


iii. Klein bottle:

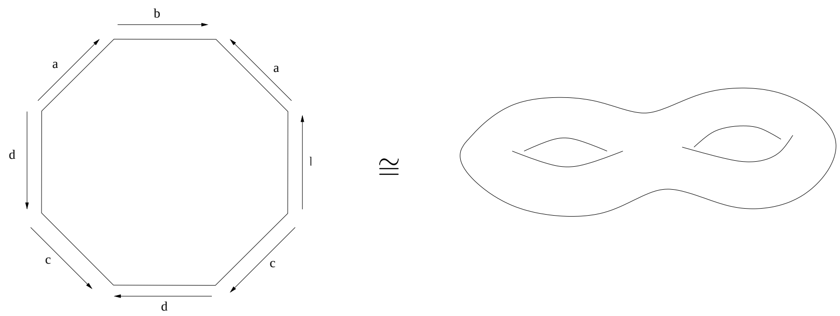


(c) Octagon.

i. Sphere:



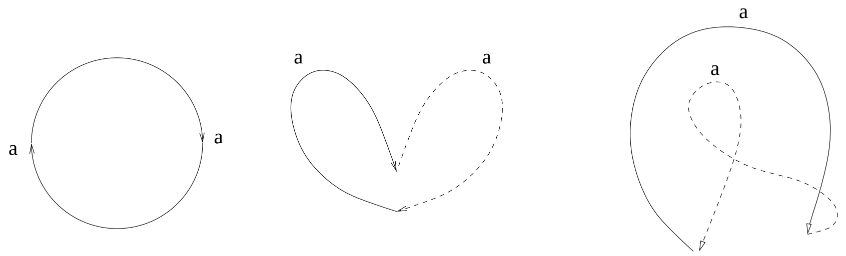
ii. Torus of genus 2:



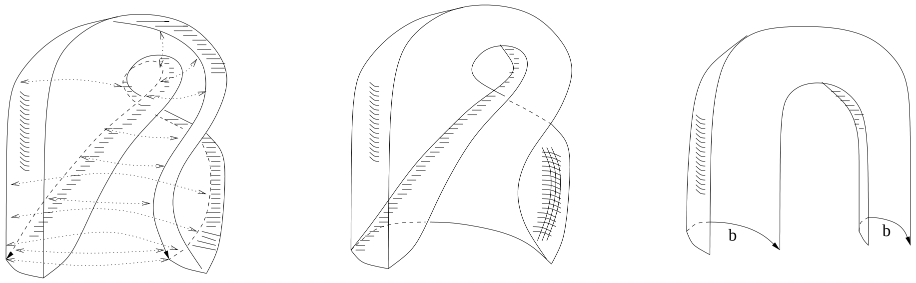
Exercise 11.5 In each of the above examples, draw curves on the surfaces such that if you cut the surface along those curves, you get back the polygons, with pairs of edges identified being along the cut.

LECTURE 19

We have seen that the projective plane (of dimension 2) is obtained from $\mathbb{D}^2$ by identifying diametrically opposite points on the boundary. Now, how does the part of $\mathbb{D}^2$ near the boundary look like after one makes this identification? Or, in other words, how does a punctured projective plane look like? First, let us just look at the boundary circle.

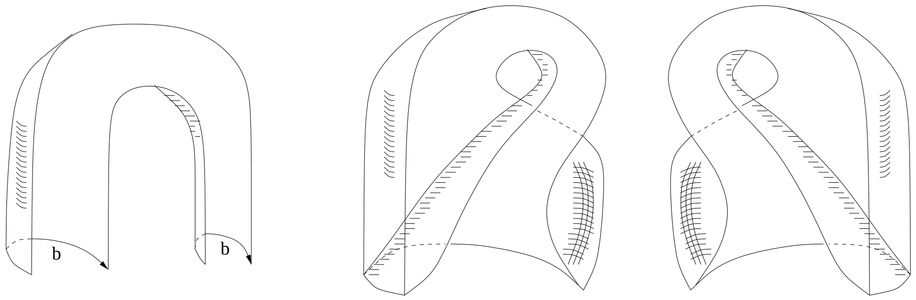


The diagram above shows just the boundary circle S^1, which after the identification will again look like a circle (see problem 3(b)). But there is also a strip along the original circle, and what shape this strip takes once this identification is made is perhaps slightly more difficult to visualize. The first picture in the next diagram shows the boundary circle along with the strip, and the next one shows how it looks like once the idenfication is made — one has to allow self-interaction in the intermediate stage, but it does not cause any serious problem because it can be gotten rid of if one was doing this in a higher dimensional Euclidean plane, rather than sitting in $\mathbb{R}^3$.



Another way to see that the punctured projective plane is indeed the Mobius strip is to go backward — take a Mobius strip and make a cut along the median circle, what you get is a simple cylinder where in one of the boundary circles, opposite points are being identified. (the cylinder will have two twists, but these twists can be made to disappear if the object is taken to $\mathbb{R}^4$).

Let M stand for the Mobius strip, and let $f : \partial M \to \partial M$ be the identity map. Next figure illustrates that the space $M \cup_f M$ is homeomorphic to the Klein's bottle.



The two figures above also illustrate that the connected sum of two projective planes is the Klein's bottle.

The question one might ask at this point is: why are we doing all this, why take this extra trouble and look for surfaces that are quotient spaces of polygons? The answer is, this provides us with a combinatorial object, a polygon with certain number of edges and vertices, that allows us to compute certain numbers associated with the surfaces. If we are clever, if we can get hold of a number that does not depend on the polygon used to construct the surface, but rather only on the homeomorphism class of the surface, then by looking at this quantity, it should be possible to distinguish between lots of surfaces. This is exactly what the Euler number does. Suppose S is a surface and it can be obtained from a polygon equipped with some gluing scheme. Then the **Euler number** $\chi(S)$ is defined to be

$$\chi(S) = \#\text{ vertices} - \#\text{ edges} + 1,$$

where the number of edges and vertices are counted after making the identifications. If it is a $2n$-gon that we have used, in which pairs of edges have been identified, then the formula above reduces to

$$\chi(S) = \#\text{ vertices} - n + 1.$$

Of course at this is stage it is not even clear that this quantity is well-defined, leave alone an invariant. We will give a more precise definition of the Euler number and prove all its properties later. For the moment, it would be a good exercise to compute $\chi(S)$ for all the gluing schemes mentioned earlier and try and discover a formula for oneself for the Euler number of a sphere with n handles and that of a sphere with n crosscaps.

Lecture 20

We will next prove that a polygon with even number of edges and a gluing scheme always gives rise to a surface.

Exercise 11.6 Let P be a polygon, E being an edge. Let ξ be a homeomorphism from the boundary of P to itself such that $\xi(x) = x$ for all points x not belonging to the interior of the edge E. Show that ξ can be extended to a homeomorphism from P to itself.

Lemma 11.7 *Let P be a polygon with $2n$ edges and $\{(E_i, F_i, \phi_i) : 1 \leq i \leq n\}$ and $\{(E_i, F_i, \psi_i) : 1 \leq i \leq n\}$ be two gluing schemes for P such that for each vertex v lying on E_i, one has $\phi_i(v) = \psi_i(v)$. Then the resulting quotient spaces are homeomorphic for these two gluing schemes.*

Proof: It is enough to prove the statement for the case where $\psi_i = \phi_i$ for all $i \geq 2$. Let us denote the equivalence relations corresponding to the two gluing schemes by $\sim_\phi$ and $\sim_\psi$ respectively. If there exists a homeomorphism $\xi : P \to P$ such that $\xi(x) \sim_\phi \xi(y)$ if and only if $x \sim_\psi y$, then the map $f : P \to P/\!\sim_\phi$ given by $f(x) = [\xi(x)]_\phi$ lifts to a homeomorphism from $P/\!\sim_\psi$ onto $P/\!\sim_\phi$. So it is enough to produce a homeomorphism ξ with the above property. Define ξ on the boundary of P as follows:

$$\xi(x) = \begin{cases} \phi_1^{-1} \circ \psi_1(x) & \text{if } x \in E_1, \\ x & \text{otherwise.} \end{cases}$$

By the exercise above, this can be extended to a homeomorphism on the whole of P, which we continue to denote by the same symbol ξ. It remains to show now that $\xi(x) \sim_\phi \xi(y)$ if and only if $x \sim_\psi y$. COMPLETE THE PROOF □

The above lemma ensures that while talking about a polygon with a gluing scheme, it is enough to specify the pairs of edges being identified along with the direction along which they are identified.

Lemma 11.8 *Let P be a polygon with $2n$ edges equipped with a gluing scheme. Label the vertices $V_1, V_2, \ldots, V_{2n}$ in such a way that each V_iV_{i+1} is an edge of the polygon. For a vertex v, denote by $[v]$ the equivalence class of v. Then*

1. *$[v]$ is singleton $(= \{V_i\})$ iff the edges V_iV_{i+1} and V_iV_{i-1} are identified,*
2. *if $\#[v] = k$, then the elements of $[v]$ and the edges containing them can be labelled in such a way that $v_1, \ldots, v_k$ are the elements in $[v]$, e_{i1} and e_{i2} the edges containing v_i and for each i, e_{i2} is identified with $e_{i+1,1}$.*

Proof: 1. Assume V_iV_{i+1} and V_iV_{i-1} are identified. Let ϕ be the homeomorphism from V_iV_{i+1} to V_iV_{i-1} in the gluing scheme. Then $\phi(V_i) = V_i$ and V_i does not belong to the domain or range of any other ϕ_j in the gluing scheme. Therefore $[V_i] = \{V_i\}$. For the converse, just note that if V_iV_{i+1} is identfied with any other edge, V_i would be identified with the starting point of that edge which would be different from V_i.

2. Pick an element in $[v]$, and call it v_1. Denote the two (directed) edges emanating from v_1 by e_{11} and e_{12}. Denote the edge that is identified with e_{12} by e_{21} and the vertex from which e_{21} emanates by v_2. If $e_{21} = e_{11}$ (and hence $v_2 = v_1$) then by part 1, $[v]$ is singleton and we are through. If not, call the other edge starting at v_2 as e_{22}. Let e_{31} be the edge that is identified with this and v_3 be the edge where it starts. One has either $v_3 = v_1$, in which case e_{31} must be e_{11}, or $v_3 \not\in \{v_1, v_2\}$. In the latter case, continue the process. At some stage k, the process has to terminate, i. e. one must have $v_{k+1} = v_1$ so that e_{k2} is being identified with e_{11}. Now we claim that $[v] = \{v_1, \ldots, v_k\}$. That each v_i is equivalent is clear from the way we chose them. Suppose then that w is a vertex equivalent to the v_i's. This means $v_1 = \phi_{i_1}^{\pm 1} \circ \ldots \circ \phi_{i_r}^{\pm 1}(w)$. Now it is easy to argue recursively that w must be one of the v_i's. □

Theorem 11.9 *Let D be a polygon equipped with a gluing scheme for its edges. Then the resulting quotient space is a compact connected surface.*

Proof: Compactness and connectedness follow because we are quotienting a polygon which is compact and connected. In order to show that the quotient is a manifold, note that points in the space can be divided into three subclasses: those in the interior of the polygon, those lying on the interior of the edges, and the remaining ones (i. e. the vertices). For points in the interior of the polygon, if we take a small open nbd around it that does not intersect any edge, then quotienting procedure does not affect points in this nbd, and hence it is homeomorphic to $\mathbb{R}^2$. COMPLETE THE PROOF □

Lecture 21

12 Classification of compact surfaces

Complete classification of compact connected surfaces (without boundary) involves three major steps:

Step 1. prove that any such surface comes necessarily from a polygon with a gluing scheme,

Step 2. start with a polygon with a gluing scheme and prove that it must be of a certain type.

Step 3. prove that various types of surfaces obtained in step 2 are nonhomeomorphic.

We will give a rough outline of the proof of step 1 in the next section. The proof of steps 1 and 3 will then be completed once we have developed the notion of a simplicial complex in some detail. In this section we will proceed with the second step. The technique employed will be largely algebraic/combinatorial. We will normally deal with polygons with $2n$ sides, but occasionally will work with topological polygons (for example while talking about a 2-gon). First, we will describe the gluing scheme algebraically, as a word of length $2n$ in a group with n generators $a_1, \dots, a_n$, such that it has exactly two occurences of letters from each $\{a_i, a_i^{-1}\}$. We will call such words **admissible words**. Let the gluing scheme be given by the homeomorphisms $\phi_i : E_i \to F_i$, $E_1, F_1, \dots, E_n, F_n$ being the edges. Choose any vertex v. List the edges cyclically starting from this vertex. This will give us a listing of the $2n$ edges $E_1, \dots, E_n$ and $F_1, \dots, F_n$ in some order. Replace each E_i with a_i and each F_i with either with a_i (in case the orientation of F_i is same as that of E_i) or with a_i^{-1} (if the orientation of F_i is different from that of E_i). This will give us a word that has exactly two occurences of $a_i^{\pm}$ for each i.

Conversely, given a word satisfying the above restriction, here is how to recover the gluing scheme: if $a_i^{\pm 1}$ occur at positions j and k, then the edges $|V_j V_{j+1}|$ and $|V_k V_{k+1}|$ are to be identified by a homeomorphism ϕ_i for which

$$\phi_i(V_j) = \begin{cases} V_k & \text{if exponents of } a_i \text{ are the same,} \\ V_{k+1} & \text{otherwise.} \end{cases}$$

In what follows, we will use capital letter (U, V, W etc) to denote a word or a string (i. e. part of a word) and small letters to denote individual letters in a word.

Lemma 12.1 *Let W be a word. Let $T_i(W)$ be the word obtained from W by replacing each occurence of a_i by a_i^{-1} and vice versa, and $C_\sigma(W)$ denote the word obtained by applying a cyclic permutation σ. Then $T_i(W)$ and $C_\sigma(W)$ both give rise to the same gluing scheme as W.*

Proof: For $T_i(w)$, it is obvious, and for the above prescription for constructing the gluing scheme from $C_\sigma(w)$ amounts to a relabelling of the vertices of the form $v_k \mapsto V_{k+r}$ for some r (addition modulo $2n$), which again does not change the scheme. □

Lemma 12.2 *Let S be the surface coming from a polygon with gluing scheme W and S' be the surface coming from a polygon with gluing scheme W^{-1}. Then S and S' are homeomorphic.*

Proof: The proof in this case follows from the observation that under the reflection map $(x, y) \mapsto (x, -y)$, a $2n$-gon with gluing scheme W is mapped to a $2n$-gon with gluing scheme W^{-1}, and the reflection map is a homeomorphism of the plane. □

Combining the two lemmas above, one concludes that if W' is the word obtained from W by reversing the order of the letters, and S and S' are the surfaces corresponding to W and W', then S and S' are homeomorphic.

Our strategy next would be as follows. We will look for certain 'transformations' of the polygon and the gluing scheme that do not affect the quotient space. Then we will use these transformations to reduce an arbitrary word coming from a gluing scheme to one of a specific form, for which we know/describe the quotient space.

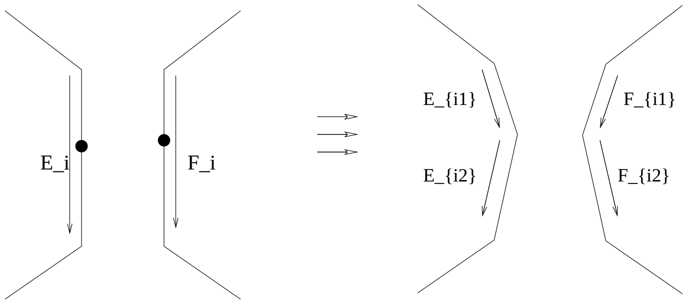
12.1 Elementary transformations

Before introducing the transformations, we need to make the following definition. By **a system of polygons with a gluing scheme**, we mean $\mathscr{G} = \{(E_i, F_i, \phi_i) : 1 \leq i \leq n\}$, where $E_1, \ldots, E_n$ and $F_1, \ldots, F_n$ are the edges of a system of r polygons $P_1, \ldots, P_r$, and $\phi_i : E_i \to F_i$ is a homeomorphism for each i. Let $\sim$ be the equivalence relation coming from identifying a point x on E_i with $\phi_i(x)$ on F_i. One associates the space $S(\mathscr{G}) := P_1 \sqcup \ldots \sqcup P_r/\sim$ with this system.

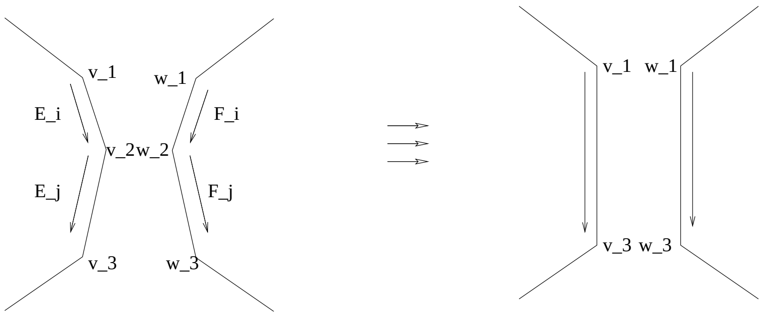
We will use the following four types of transformations.

1. Suppose $\mathscr{G}$ is a system of polygons with a gluing scheme. Let us label the vertices of E_i as v_1^i and v_2^i and those of F_i as w_1^i and w_2^i so that $\phi_i(v_1^i) = w_1^i$ and $\phi_i(v_2^i) = w_2^i$. Let v be an interior point of E_i and let $w = \phi_i(v)$. Denote the edges $v_1^i v$, vv_2^i, $w_1^i w$ and ww_2^i by E_{i1}, E_{i2}, F_{i1} and F_{i2} respectively, and the restrictions of ϕ_i to E_{i1} and E_{i2} by ϕ_{i1} and ϕ_{i2} respectively. If we remove the triple (E_i, F_i, ϕ_i) from $\mathscr{G}$ and plug in the two triples $(E_{i1}, F_{i1}, \phi_{i1})$ and $(E_{i2}, F_{i2}, \phi_{i2})$, then we get another system of polygons with a gluing scheme, which we denote by $\mathscr{G}'$. What we are doing in effect is introducing a new pair of vertices.

It is clear that the collection of polygons will remain the same, and the equivalence relation coming from the gluing scheme will also not change. Hence $S(\mathscr{G})$ is same as $S(\mathscr{G}')$.

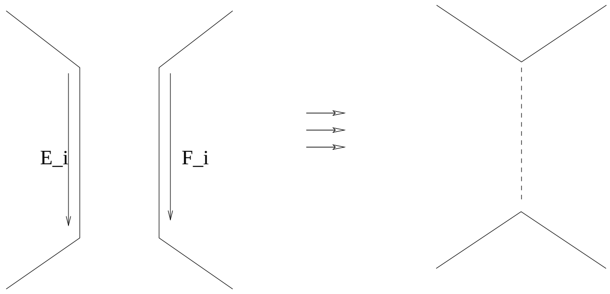


2. The next transformation would be the inverse of the earlier one. More specifically, suppose $E_i = v_1v_2$ and $E_j = v_2v_3$ be adjacent edges of one polygon, $F_i = w_1w_2$ and $F_j = w_2w_3$ are also adjacent edges of one (not necessarily same as the earlier one) polygon, and $\phi_i(v_1) = w_1$, $\phi_i(v_2) = w_2$, $\phi_j(v_2) = w_2$ and $\phi_j(v_3) = w_3$. Now from $\mathscr{G}$, remove the triples (E_i, F_i, ϕ_i) and (E_j, F_j, ϕ_j) and plug in the triple $(v_1v_3, w_1w_3, \phi_i')$, where ϕ_i' is some homeomorphism that takes v_1 to w_1 and v_3 to w_3. Call the resulting gluing scheme $\mathscr{G}'$.

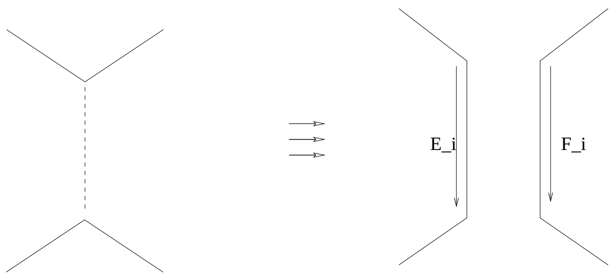


Exercise 12.3 Let S be the surface coming from a polygon with gluing scheme $W = abUb^{-1}a^{-1}V$ (resp. $W = abUabV$) and S' be the surface coming from a polygon with gluing scheme $W' = aUa^{-1}V$ (resp. $W' = aUaV$). Then S and S' are homeomorphic.

3. Suppose $E_i = vw$ and $F_i = v'w'$ are edges of two different polygons P_j and P_k and $\phi_i(v) = v'$, $\phi_i(w) = w'$. If we join these two polygons along the edges E_i and F_i making the identification coming from ϕ_i, we get a new system of polygons with gluing scheme in which P_j and P_k will be replaced by one single polygon, the triple (E_i, F_i, ϕ_i) gets removed, the vertex v is identified with v', w with w' and a corresponding modification takes place in the triples involving edges that contain these vertices.

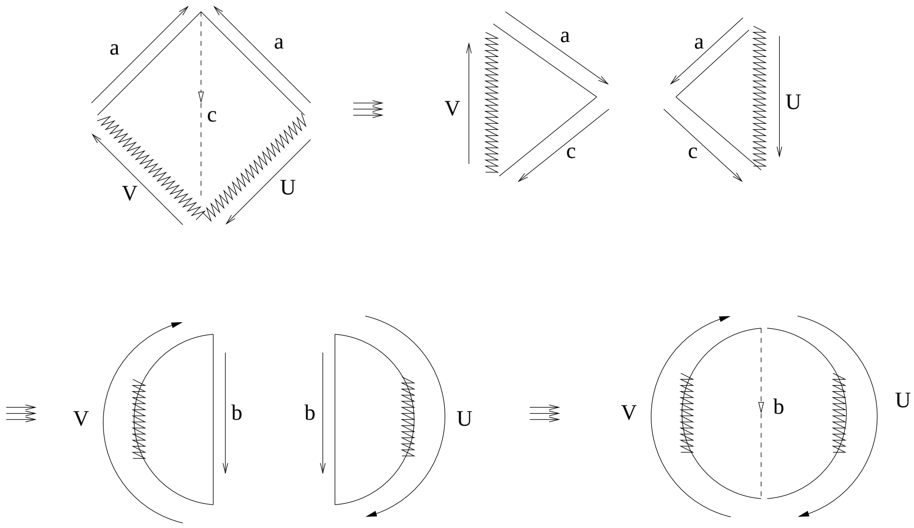


4. Let v and w be two vertices of a polygon that are not adjacent. If we cut the polygon along the diagonal vw, it will give rise to a new system of polygons with a gluing scheme.



Lemma 12.4 $S(aa^{-1}UV) \cong S(UV)$.

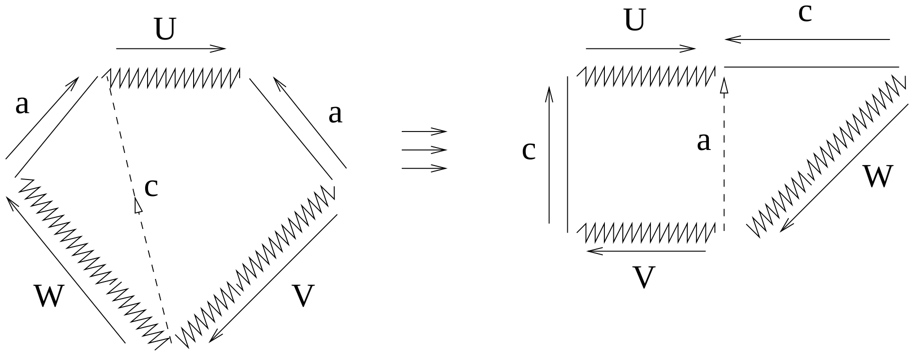
Proof:



□

Lemma 12.5 $S(aUa^{-1}VW) \cong S(aUa^{-1}WV)$.

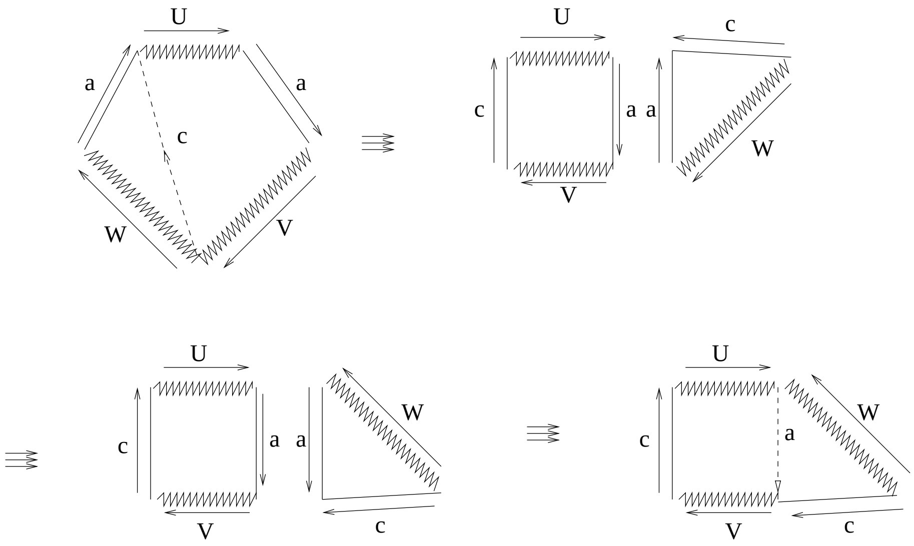
Proof:



□

Lemma 12.6 $S(aUaVW) \cong S(aUW^{-1}aV)$.

Proof:



□

Lecture 22

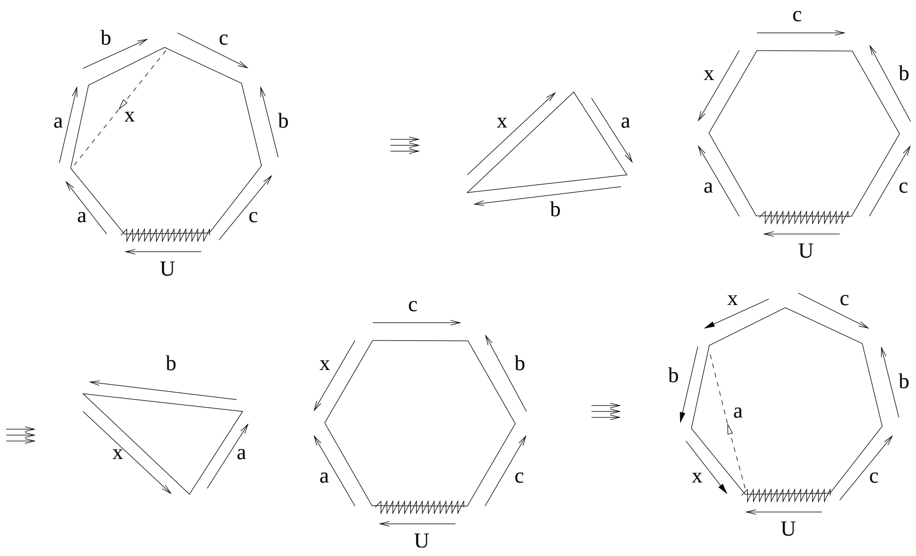
Corollary 12.7 $S(aUaV) \cong S(aaVU^{-1})$.

Proof: By the previous lemma, $S(aUaV) \cong S(aVaU) \cong S(aVU^{-1}a) \cong S(aaVU^{-1})$. $\square$

Exercise 12.8 Give a direct proof of the above lemma by using the elementary transformations.

Corollary 12.9 $S(aabcb^{-1}c^{-1}U) \cong S(aabbccU^{-1})$.

Proof: The following diagram gives $S(aabcb^{-1}c^{-1}U) \cong S(x^{-1}b^{-1}x^{-1}cb^{-1}c^{-1}U)$.



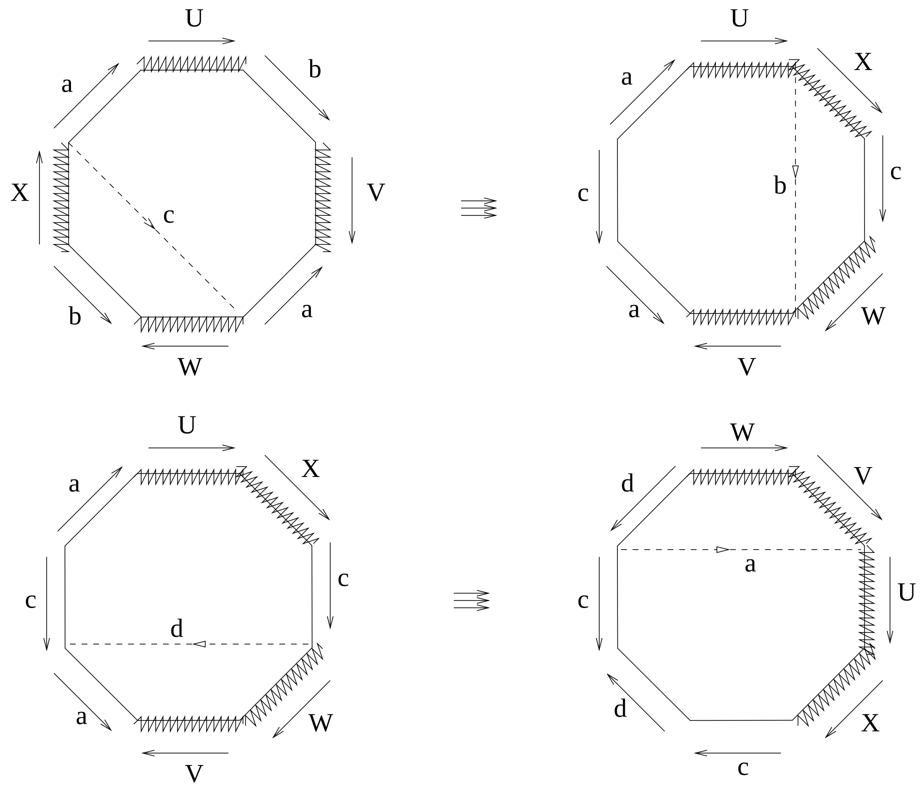
Now from the previous lemma, we get $S(UaVaW) \cong S(aVaWU) \cong S(aaWUV^{-1})$. Apply this repeatedly to get

$$\begin{aligned}
S(x^{-1}b^{-1}x^{-1}cb^{-1}c^{-1}U) &\cong S(b^{-1}b^{-1}c^{-1}Ux^{-1}c^{-1}x) \\
&\cong S(c^{-1}c^{-1}xb^{-1}b^{-1}xU^{-1}) \\
&\cong S(xxU^{-1}c^{-1}c^{-1}bb) \\
&\cong S(c^{-1}c^{-1}bbxxU^{-1}) \\
&\cong S(aabbccU^{-1}).
\end{aligned}$$

$\square$

Lemma 12.10 $S(aUbVa^{-1}Wb^{-1}X) \cong S(aba^{-1}b^{-1}WVUX)$.

Proof:

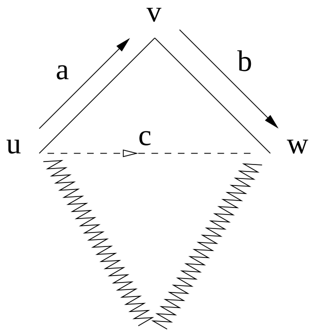


□

Lemma 12.11 *Let S be the surface coming from a polygon with gluing scheme W. Then there is a word W' such that the surface given by the gluing scheme W' is homeomorphic to S and gluing scheme W' has only one equivalence class of vertices.*

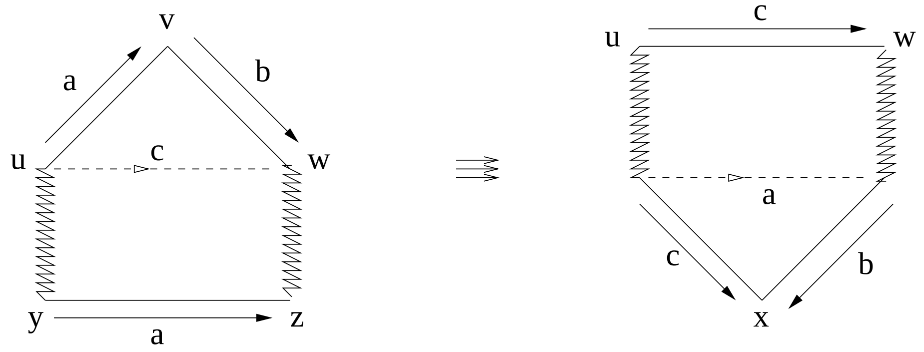
Proof: We will describe a mechanism below that will reduce the number of equivalence classes by one if there are more than one classes present. First observe that if there an equivalence class containing just one vertex, then the edges containing this vertex will contribute a string of the form aa^{-1} to the word which can be cancelled to yield a word with one less equivalence class of vertices. Assume then that V is an equivalence class of vertices such that not all vertices of the polygon belong to V, and all the equivalence classes contain more than one element. We will show that $\#V$ can be reduced by one while keeping the total number of equivalence classes the same. This along with the observation above will then complete the proof.

One can find two adjacent vertices v and w (as in the folowing diagram) such that $v \in V$ but $w \notin V$.



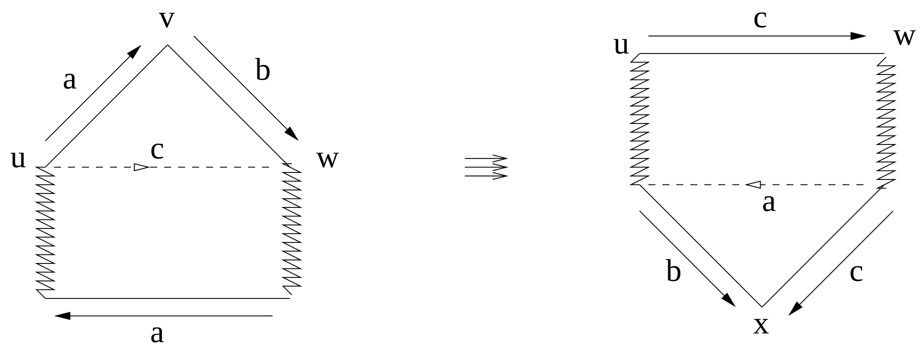
Observe that we can not have $a = b$, because then v would be equivalent to w, which belongs to a different equivalence class. If $a = b^{-1}$, then $V = \{v\}$. Since $\#V > 1$, this can not happen either. Let us now consider the following two cases separately:

Case 1. *a appears in the word with different exponents.*
In this case, if we write y and z the two vertices of the edge to be paired with uv, then $y \neq u$ because then $[u]$ would be a singleton, and $z \neq w$ because $w \notin [v] = V$.



Cut the polygon along c and paste the triangle back along the edge a.

Case 2. *a appears in the word with the same exponent.*
Again cut the polygon along c and paste the triangle back along the edge a.



In both the cases, it is now straightforward (**how?**) to see that v would be replaced by x in the set of vertices, number of elements in V would reduce by one, that in $[w]$ would go up by one, and none of the other equivalence classes would be touched. $\square$

Remark 12.12 Observe that once we have performed the reduction given by the above lemma to a word W,

1. the word will not contain any string of the form aa^{-1},

2. there will not exist any substring U (with length bigger than 1) such that the word is of the form $UVUW$ or $UVU^{-1}W$.

We will require the above lemma along with the following homeomorphisms for the proof of the next theorem.

$$\begin{aligned} S(aa^{-1}UV) &\cong S(UV), && (12.1)\\ S(aUaV) &\cong S(aaVU^{-1}), && (12.2)\\ S(aUbVa^{-1}Wb^{-1}X) &\cong S(aba^{-1}b^{-1}WVUX), && (12.3)\\ S(aabcb^{-1}c^{-1}U) &\cong S(aabbccU^{-1}). && (12.4) \end{aligned}$$

Theorem 12.13 *Let $W_0 = aa^{-1}$, $W_r = a_1b_1a_1^{-1}b_1^{-1}\dots a_rb_ra_r^{-1}b_r^{-1}$, and let $W'_s = a_1a_1\dots a_sa_s$, where $r, s \geq 1$. Then for any admissible word W, $S(W)$ is homeomorphic to some $S(W_r)$ or $S(W'_s)$.*

Proof: Use the first homeomorphism above to cancel all letters that appear with different exponents and adjacent to each other. In doing so, if we reach a 2-gon, it has to be either of the form aa^{-1} or aa and we will be through. So assume we have an admissible word of length $2n$ ($n > 1$) where all adjacent pairs of the form aa^{-1} have been removed. Now employ the previous lemma to get a word such that all the vertices are equivalent. This in particular implies that even in this new word, adjacent pairs of the form aa^{-1} do not appear.

Next, use the homeomorphism (12.2) above to club together every pair of letters that appear with the same exponent. Observe that since all the vertices are equivalent to begin with, they will continue to be equivalent after the process (**explain why**!). If there is no letter that appears with different exponents, then we will end up with a word of the form W'_s and we will be through. So let us assume that there exists a letter a that appears with different exponents. Then we claim that *the word must be of the form $aUbVa^{-1}Wb^{-1}X$ for some other letter b.* If not, then the string between a and a^{-1} would itself be an admissible word, and consequently there has to exist more than one equivalence class of vertices. Now apply the homeomorphism (12.3). This will give us a word of the form $aba^{-1}b^{-1}U$, where in the string U, letters appearing with the same exponent continue to be together. Next look for another pair of letters appearing with different exponents and intertwined as earlier. use (12.3) to bring them together. Continuing this process, one would end up with a word that consists of strings of the form aa and $bcb^{-1}c^{-1}$ placed one after another. If strings of only one of the above forms are present, we are through. Otherwise use the homeomorphism (12.4) repeatedly to remove any occurence of strings of the form $bcb^{-1}c^{-1}$ and end up with a word of the form W'_s. □

Lecture 23

13 Triangulation of surfaces

Let us next focus our attention to Step 1 of the proof of classification theorem. In other words, we want to express every compact connected 2-manifold as a polygon with a gluing scheme. For this, we need to be able to make some cuts along some curves on the surface, so as to end up with several polygons. If we keep track of all the pairs of points that need to be identified in order to undo the cuts, we get the gluing scheme necessary to get back the manifold. Also it is not too difficult to check that if we start with a connected manifold and get a system of polygons with a gluing scheme, one can actually reduce this to *one* polygon with a gluing scheme.

From the definition of a 2-manifold, it is clear that we can cover a surface M with the interiors of finitely many polygons. What about cutting these polygons out one at a time. The problem is, the intersection of two polygons in this collection, if nonempty, will have nonempty interior. So when you cut one polygon out, it would affect the other ones, and it is difficult to keep track of what is going to happen to these other polygons in a systematic manner. Therefore what we would be looking for is an extra structure, some kind of 'tiling' of the surface with polygons, or, to keep matters simple and manageable, with triangles. Notice at this point two desirable properties that this 'tiling' must possess: any two triangles should have disjoint interiors, and the intersection of two triangles must be either disjoint or a common edge of the two. Also, we have already seen that any compact manifold can be embedded in an Euclidean space. So we will do the whole exercise sitting in some $\mathbb{R}^m$.

13.1 Simplicial complex

A **hyperplane** in $\mathbb{R}^m$ is a set of the form $V + v$, where V is some subspace of $\mathbb{R}^m$ and $v \in \mathbb{R}^m$. Dimension of the hyperplane is the dimension of V. A subset S of $\mathbb{R}^m$ is said to be **in general position** if for each $k < m$, any k-dimensional hyperplane contains at most $k+1$ points of S.

Exercise 13.1 A finite subset $\{v_0, \ldots, v_k\}$ of $\mathbb{R}^m$, where $k \leq m$, is in general position if and only if the set $\{v_1 - v_0, \ldots, v_k - v_0\}$ is independent.

Let $V = \{v_0, \ldots, v_n\}$ be a set of n points in general position in $\mathbb{R}^m$ $(n \leq m)$. The convex hull σ^n of V is called an n**-simplex**. Convex hull of nonempty subsets of V are called faces of

σ^n. A **k-face** is the convex hull of a subset of size $k+1$. Note that any k-face is a k-simplex in its own right. 1-simplex are also called **edges** and 0-simplexes are called **vertices**. The **boundary** $\operatorname{bdy}\sigma^n$ of an n-simplex σ^n is the union of all its $(n-1)$-faces. The **interior** $\operatorname{int}\sigma^n$ is the set of points in σ^n not on its boundary.

Let us list a few elementary properties of simplices. Let $\sigma = (v_0 v_1 \dots v_n)$ be an n-simplex. Then

1. σ is the union of all line segments joining v_0 and the face opposite v_0.
2. σ is a compact convex subset of $\mathbb{R}^m$ and is the of all convex sets containing the points $v_0, \dots, v_n$.
3. there is a unique set of points in general position that span σ.
4. $\operatorname{int}\sigma$ is a convex open subset of the plane spanned by the vertices of σ.
5. $\operatorname{int}\sigma$ is the union of all open line segments joining v_0 and interior of the the face opposite v_0.
6. there is a homeomorphism $\phi : \sigma \to \mathbb{D}^n$ that takes $\operatorname{bdy}\sigma$ to S^{n-1}.
7. if $\tau = (w_0 \dots w_n)$ is an n-simplex in some other $\mathbb{R}^k$, then there is a homeomorphism $f : \sigma \to \tau$ such that $f(\sum_{i=0}^n \alpha_i v_i) = \sum_{i=0}^n \alpha_i w_i$.

Proofs are left as exercises.

An **Euclidean simplicial complex** is a collection K of simplexes in (some) $\mathbb{R}^m$ such that
1. if $\sigma \in K$ then all the faces of σ are also in $\mathbb{R}^m$,
2. if $\sigma, \tau \in K$ and $\sigma \cap \tau \neq \phi$, then $\sigma \cap \tau$ is a common face of the two,
3. for each $\sigma \in K$, there is an open set U in $\mathbb{R}^m$ such that $\sigma \subseteq U$ and U intersects only a finite number of elements of K.

Exercise 13.2 Show that the condition 2 above can be replaced by $\sigma, \tau \in K, \sigma \cap \tau \neq \phi \Rightarrow \operatorname{int}\sigma \cap \operatorname{int}\tau = \phi$ and that condition 3 can be replaced by *every point of $\mathbb{R}^m$ has a neighbourhood that intersects only finitely many simplices of K.*

Let K be a complex. A subset L of K is said to be a **subcomplex** if for any $\sigma \in L$, all the faces of σ also belong to L. In such a case, L is a complex in its own right.

Exercise 13.3 Show that if L and M are subcomplexes of K, then $L \cup M$ and $L \cap M$ are also subcomplexes of K.

The set K^i of all simplices in K of dimension less than or equal to i is called the **i-skeleton** of K. The space $|K| := \cup_{\sigma \in K}\sigma$, equipped with the subspace topology from $\mathbb{R}^m$, is called **the space underlying** K. If K is a complex for which $|K|$ is an n-manifold, then one calls K a **triangulated n-manifold**. If M is a manifold, and K is a complex such that $|K|$ is homeomorphic to M, then K is called a **triangulation** of M. Thus our aim in this section would be to start with a compact connected surface M and *get a triangulation of M consisting of finitely many simplices.*

Lecture 24

13.2 Abstract simplicial complex

Often it is necessary to not get distracted by which Euclidean space a compex is sitting in and focus just on the combinatorial structure of the complex. For this, one needs the notion of an abstract complex. An **abstract simplicial complex** $\mathscr{K}$ is a collection of finite subsets, to be called *simplices*, such that if $\sigma \in \mathscr{K}$ and $\tau \subseteq \sigma$, then $\tau \in \mathscr{K}$. Subsets of a simplex σ are called **faces** of σ. Elements of the union $\cup_{\sigma \in \mathscr{K}} \sigma$ are called *vertices*. If $\#\sigma = n+1$, then the dimension of σ is n. Dimension of $\mathscr{K}$ is the maximum dimension of its simplices.

Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be two abstract complexes. A map ϕ from the vertex set of $\mathscr{K}_1$ to that of $\mathscr{K}_2$ is called **simplicial** if for any simplex σ in $\mathscr{K}_1$, $\phi(\sigma) := \{\phi(v) : v \in \sigma\}$ is a simplex in $\mathscr{K}_2$.

Let $\mathscr{K}$ be a complex. Another complex $\mathscr{L}$ is called a **subcomplex** of $\mathscr{K}$ if $\mathscr{L} \subseteq \mathscr{K}$. Thus for example, $\mathscr{K}^n := \{\sigma \in \mathscr{K} : \#\sigma \leq n+1\}$ is a subcomplex of $\mathscr{K}$, called the n**-skeleton** of $\mathscr{K}$.

Exercise 13.4 Show that
1. a subset $\mathscr{L}$ of $\mathscr{K}$ is a subcomplex if and only if any $\sigma \in \mathscr{K}$ that is a face of a simplex $\tau \in \mathscr{L}$ is in $\mathscr{L}$.
2. if $\mathscr{L}$ and $\mathscr{M}$ are subcomplexes of $\mathscr{K}$, then so are $\mathscr{L} \cup \mathscr{M}$ and $\mathscr{L} \cap \mathscr{M}$.

Let $|\mathscr{K}|$ be the set of all maps f from the set of vertices of $\mathscr{K}$ to the interval $[0,1]$ such that

1. $\{v \in V : f(v) \neq 0\}$ is a simplex in $\mathscr{K}$,
2. $\sum_{v \in V} f(v) = 1$.

One can define a metric on $|\mathscr{K}|$ by $d(f,g) = \sqrt{\sum_v (f(v) - g(v))^2}$. Let $\sigma \in \mathscr{K}$. Define the closed simplex $|\sigma|$ to be the following subset of $|\mathscr{K}|$:

$$\{f \in |\mathscr{K}| : f(v) \neq 0 \Rightarrow v \in \sigma\}.$$

Exercise 13.5 Assume $\#\sigma = n+1$. Show that $|\sigma|$ with the topology coming from the above metric is homeomorphic to the simplex $(e_0 e_1 \dots e_n)$ in $\mathbb{R}^{n+1}$, where $e_0 = (0, \dots, 0)$ and for $i \geq 1$, e_i stands for the element that has 1 at the ith position and 0 elsewhere.

The coherent topology on $|\mathscr{K}|$ is the strongest (largest) topology on $|\mathscr{K}|$ such that the inclusion maps $|\sigma| \hookrightarrow |\mathscr{K}|$ are all continuous.

Exercise 13.6 Show that a subset E of $|\mathscr{K}|$ is closed in the coherent topology if and only if $E \cap |\sigma|$ is closed in $|\sigma|$ for all $\sigma \in \mathscr{K}$.

Exercise 13.7 Show that $|\mathscr{K}| = \cup_{\sigma \in \mathscr{K}} |\sigma|$ and $|\mathscr{K}^n| = \cup_{\substack{\sigma \in \mathscr{K} \\ \#\sigma \leq n+1}} |\sigma|$.

Proposition 13.8 *Let X be a topological space and $f : |\mathscr{K}| \to X$ be a function. Then the following conditions are equivalent:*

1. *f is continuous,*
2. *the restriction of f to $|\sigma|$ is continuous for all $\sigma \in \mathscr{K}$,*
3. *restriction of f to $|\mathscr{K}^n|$ is continuous for all n.*

Exercise 13.9 Show that the identity map on $|\mathscr{K}|$ is continuous with coherent topology for the domain space and metric topology for the range space. Hence show that $|\mathscr{K}|$ is Hausdorff.

Proposition 13.10 *$|\mathscr{K}|$ is normal.*

Let $\sigma \in \mathscr{K}$. The open simplex $\langle \sigma \rangle$ is defined to be the subset

$$\{f \in |\mathscr{K}| : f(v) \neq 0 \Leftrightarrow v \in \sigma\}$$

of $|\mathscr{K}|$.

Proposition 13.11 *Every compact subset of $|\mathscr{K}|$ is contained in a finite union of open simplices.*

Corollary 13.12 *An abstract complex $\mathscr{K}$ is finite if and only if the space $|\mathscr{K}|$ is compact.*

A complex $\mathscr{K}$ is said to be **locally finite** if every vertex v belongs to only finitely many simplices of $\mathscr{K}$. A **realization** of an abstract complex $\mathscr{K}$ in $\mathbb{R}^m$ is a linear embedding of $|\mathscr{K}|$ in $\mathbb{R}^m$.

Theorem 13.13 *If $\mathscr{K}$ has a realization in $\mathbb{R}^m$, then it is countable, locally finite with dimension $n \leq m$. Conversely, if $\mathscr{K}$ is countable and locally finite with dimension n, then it has a realization as a closed subset of $\mathbb{R}^{2n+1}$.*

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