Stochastic dilation of a quantum dynamical semigroup on a separable unital C^* algebra.

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Abstract

Given a uniformly continuous quantum dynamical semigroup on a separable unital C^* algebra, we construct a canonical Evans-Hudson (E-H) dilation. Such a result was already proved by Goswami and Sinha ([GS]) in the von-Neumann algebra set-up, which has been extended to the C^* algebraic framework in the present article. The authors make use of the coordinate-free calculus and results of [GS], but the proof of the existence of structute maps differs form that of [GS].

0. Introduction

Given a quantum dynamical semigroup (q.d.s.) of bounded linear maps on an operator algebra an important problem is to obtain a dilation of it, that is, to obtain a time-indexed family j_t of $*$-homomorphisms from $\mathcal{A}$ to a bigger algebra $\mathcal{B}$ with a conditional expectation $I\!E : \mathcal{B} \to \mathcal{A}$ such that $T_t = I\!E \circ j_t$. The notion of such a dilation was introduced by Acardi-Frigerio-Lewis ([AFL]). Various notions of such a dilation were studied by many authors, and among them the approach of Evans and Hudson ([Ev]) concerens us in the present article. For a q.d.s. T_t acting on a C^* or von Neumann algebra $\mathcal{A}$ ($\subseteq \mathcal{B}(h)$ where h is a Hilbert space) with generator θ_0^0, an Evans-Hudson dilation (E-H dilation for short) is a time-indexed family j_t of $*$-homomorphisms from $\mathcal{A}$ into $\mathcal{B}(h \otimes \Gamma(L^2(I\!R_+, k_0))$ for some Hilbert spcae k_0, called the noise or multiplicity space, such that j_t satisfies a quantum stochastic flow equations of the form $dj_t(x) = \sum_{\alpha,\beta \geq 0} j_t(\theta^\alpha_\beta(x)) d\Lambda^\beta_\alpha(t)$ with the initial value

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$j_0 = id$, where θ^α_β are (possibly unbounded) maps from $\mathcal{A}$ to itself known as the structure maps and $d\Lambda^\beta_\alpha(t)$ are the quantum stochastic differentials in the Fock space $\Gamma(L^2(I\!R_+, k_0))$ as constructed by Hudson and Parthasarathy (see [HP],[Par]). It is an interesting question : given a q.d.s. T_t, when can one construct an E-H dilation of it ? In a recent paper, Goswami and Sinha ([GS]) have been able to construct an E-H dilation for an arbitrary uniformly continuous normal q.d.s. on a von Neumann algebra. Here we want to extend the main result of that paper to the case when T_t is a unifomly continuous q.d.s. on a separable unital C^* algebra.

1 Preliminaries and notations

Let us first briefly discuss the coordinate-free language of quantum stochastic calculus developed in [GS], since it will be useful for us in the present context also.

Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces and A be a (possibly unbounded) linear operator from $\mathcal{H}_1$ to $\mathcal{H}_1 \otimes \mathcal{H}_2$ with domain $\mathcal{D}$. For each $f \in \mathcal{H}_2$, we define a linear operator $\langle f, A\rangle$ with domain $\mathcal{D}$ and taking value in $\mathcal{H}_1$ such that,

$$\langle\langle f, A\rangle u, v\rangle = \langle Au, v \otimes f\rangle \tag{1.1}$$

for $u \in \mathcal{D},\ v \in \mathcal{H}_1$. This definition makes sense because we have, $|\langle Au, v \otimes f\rangle| \leq \|Au\|\ \|f\|\ \|v\|$, and thus $\mathcal{H}_1 \ni v \rightarrow \langle Au, v \otimes f\rangle$ is a bounded linear functional. Moreover, $\|\langle f, A\rangle u\| \leq \|Au\|\ \|f\|$, for all $u \in \mathcal{D},\ f \in \mathcal{H}_2$. Similarly, for each fixed $u \in \mathcal{D}, v \in \mathcal{H}_1, f \rightarrow \langle Au, v \otimes f\rangle$ is bounded linear functional on $\mathcal{H}_2$, and hence there exists a unique element of $\mathcal{H}_2$, to be denoted by $A_{v,u}$, satisfying

$$\langle A_{v,u}, f\rangle = \langle Au, v \otimes f\rangle = \langle\langle f, A\rangle u, v\rangle. \tag{1.2}$$

We shall denote by $\langle A, f\rangle$ the adjoint of $\langle f, A\rangle$, whenever it exists. Clearly, if A is bounded, then so is $\langle f, A\rangle$ and $\|\langle f, A\rangle\| \leq \|A\|\ \|f\|$. Similarly, for any $T \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and $f \in \mathcal{H}_2$, one can define $T_f \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1 \otimes \mathcal{H}_2)$ by setting $T_f u = T(u \otimes f)$. For any Hilbert space $\mathcal{H}$, we denote by $\Gamma(\mathcal{H})$ and $\Gamma^f(\mathcal{H})$ the symmetric Fock space and the full Fock space of $\mathcal{H}$. For a systematic discussion of such spaces, the reader may be referred to [Par], from which we shall borrow all the standard notations and results. Now, we define a map $S : \Gamma^f(\mathcal{H}_2) \rightarrow \Gamma(\mathcal{H}_2)$ by setting,

$$S(g_1 \otimes g_2 \otimes \cdots \otimes g_n) = \frac{1}{(n-1)!} \sum_{\sigma \in S_n} g_{\sigma(1)} \otimes \cdots \otimes g_{\sigma(n)}, \tag{1.3}$$

and linearly extending it to $\mathcal{H}_2^{\otimes^n}$, where S_n is the group of permutations of n objects. Clearly, $\|S|_{\mathcal{H}_2^{\otimes^n}}\| \leq n$. We denote by $\tilde{S}$ the operator $1_{\mathcal{H}_1} \otimes S$.

Let us now define the creation operator $a^\dagger(A)$ abstractly which will act on the linear span of vectors of the form $vg^{\otimes^n}$ and $ve(g)$ (where $g^{\otimes^n}$ denotes $\underbrace{g \otimes \cdots \otimes g}_{n \text{ times}}$), $n \geq 0$, with $v \in \mathcal{D},\ g \in \mathcal{H}_2$. It is to be noted that we shall often omit the tensor product symbol $\otimes$ between two or more vectors when there is no confusion. We define,

$$a^\dagger(A)(vg^{\otimes^n}) = \frac{1}{\sqrt{n+1}}\,\tilde{S}((Av) \otimes g^{\otimes^n}). \tag{1.4}$$

It is easy to observe that $\sum_{n\geq 0} \frac{1}{n!}\|a^\dagger(A)(vg^{\otimes^n})\|^2 < \infty$, which allows us to define $a^\dagger(A)(ve(g))$ as the direct sum $\bigoplus_{n\geq 0} \frac{1}{(n!)^{\frac{1}{2}}}\ a^\dagger(A)(vg^{\otimes^n})$. In the same way, one can define annihilation and number operators in $\mathcal{H}_1 \otimes \Gamma(\mathcal{H}_2)$ for $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1 \otimes \mathcal{H}_2)$ and $T \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ as :

$$a(A)ue(h) =< A, h > ue(h),$$

$$\Lambda(T)ue(h) = a^\dagger(T_h)ue(h).$$

One can also verify that in this case $a^\dagger(A)$ is the adjoint of $a(A)$ on $\mathcal{H}_1 \otimes \mathcal{E}(\mathcal{H}_2)$, where $\mathcal{E}(\mathcal{H}_2)$ is the linear span of exponential vectors $e(g), g \in \mathcal{H}_2$. Next, to define the basic processes, we need some more notations. Let k_0 be a Hilbert space, $k = L^2(I\!R_+, k_0),\ k_t = L^2([0,t]) \otimes k_0, k^t = L^2((t,\infty)) \otimes k_0,\ \Gamma_t = \Gamma(k_t),\ \Gamma^t = \Gamma(k^t),\ \Gamma = \Gamma(k)$. We assume that $R \in \mathcal{B}(h, h \otimes k_0)$ and define $R_t^\Delta : h \otimes \Gamma_t \to h \otimes \Gamma_t \otimes k^t$ for $t \geq 0$ and a bounded interval Δ in (t, ∞) by,

$$R_t^\Delta(u\psi) = P((1_h \otimes \chi_\Delta)(Ru) \otimes \psi)$$

where $\chi_\Delta : k_0 \to k^t$ is the operator which takes α to $\chi_\Delta(\cdot)\alpha$ for $\alpha \in k_0$, and P is the canonical unitary isomorphism from $h \otimes k \otimes \Gamma$ to $h \otimes \Gamma \otimes k$. We define the creation field $a_R^\dagger(\Delta)$ on either of the domains consisting of the finite linear combinations of vectors of the form $u_t \otimes f^{t\otimes^n}$ or of $u_t \otimes e(f^t)$ for $u_t \in h \otimes \Gamma_t,\ f^t \in \Gamma^t,\ n \geq 0$, as :

$$a_R^\dagger(\Delta) = a^\dagger(R_t^\Delta), \tag{1.5}$$

where $a^\dagger(R_t^\Delta)$ carries the meaning discussed before, with $\mathcal{H}_1 = h \otimes \Gamma_t,\ \mathcal{H}_2 = k^t$. Similarly the other two fields $a_R(\Delta)$ and $\Lambda_T(\Delta)$ can be defined as :

$$a_R(\Delta)(u_t e(f^t)) = ((\int_\Delta \langle R, f(s)\rangle ds)u_t)e(f^t), \tag{1.6}$$

and for $T \in \mathcal{B}(h \otimes k_0)$,

$$\Lambda_T(\Delta)(u_t e(f^t)) = a^\dagger(T^\Delta_{f^t})(u_t e(f^t)). \tag{1.7}$$

In the above, $T^\Delta_{f^t} : h \otimes \Gamma_t \to h \otimes \Gamma_t \otimes k^t$ is defined as,

$$T^\Delta_{f^t}(u\alpha_t) = P(1 \otimes \hat{\chi}_\Delta)(\hat{T}(uf^t) \otimes \alpha_t), \tag{1.8}$$

and $\hat{T} \in \mathcal{B}(h \otimes L^2((t,\infty), k_0))$ is given by, $\hat{T}(u\varphi)(s) = T(u\varphi(s)),\ s > t$, and $\hat{\chi}_\Delta$ is the multiplication by $\chi_\Delta(\cdot)$ on $L^2((t,\infty), k_0)$. Clearly, $\|\hat{T}\| \leq \|T\|$, which makes $T^\Delta_{f^t}$ bounded. We shall often denote an operator B and its trivial extension $B \bigotimes I$ to some bigger space by the same notation, unless there is any confusion in doing so.

At this point we refer the reader to [GS] for a coordinate-free calculus using the above basic integrators. Now consider a von Neumann algebra $\mathcal{B}$ in $\mathcal{B}(h)$ for some Hilbert space h. Let k_0 be a Hilbert space. We consider $\mathcal{B} \otimes_{\text{alg}} k_0$ and denote its completion in the strong-operator topology, that is, the von Neumann module generated by $\mathcal{B} \otimes_{\text{alg}} k_0$ by $\mathcal{B} \otimes_s k_0$ or $\mathcal{B} \otimes k_0$ for short (see [GS] for details). In [GS] a stochastic calculus for map-valued processes in the Fock module $\mathcal{B} \otimes \Gamma$ has been developed. We briefly recall the definitions of basic processes and the main theorem.

Assume that we are given the *structure maps*, that is, the triple of normal maps $(\mathcal{L}, \delta, \sigma)$ where $\mathcal{L} \in \mathcal{B}(\mathcal{B}), \delta \in \mathcal{B}(\mathcal{B}, \mathcal{B} \otimes k_0)$ and $\sigma \in \mathcal{B}(\mathcal{B}, \mathcal{B} \otimes \mathcal{B}(k_0))$ satisfying :

(S1) $\sigma(x) = \pi(x) - x \otimes I_{k_0} \equiv \Sigma^*(x \otimes I_{k_0})\Sigma - x \otimes I_{k_0}$, where Σ is a partial isometry in $h \otimes k_0$ such that π is a $*$-representation on $\mathcal{B}$.

(S2) $\delta(x) = Rx - \pi(x)R$, where $R \in \mathcal{B}(h, h \otimes k_0)$ so that δ is a π-derivation, i.e. $\delta(xy) = \delta(x)y + \pi(x)\delta(y)$.

(S3) $\mathcal{L}(x) = R^*\pi(x)R + lx + xl^*$, where $l \in \mathcal{B}$ with the condition $\mathcal{L}(1) = 0$ so that $\mathcal{L}$ satisfies the second order cocycle relation with δ as coboundary, i.e.

$$\mathcal{L}(x^*y) - x^*\mathcal{L}(y) - \mathcal{L}(x)^*y = \delta(x)^*\delta(y)\ \forall x, y \in \mathcal{B}.$$

We now introduce the basic processes. Fix $t \geq 0$, a bounded interval $\Delta \subseteq (t, \infty)$, elements $x_1, x_2, \ldots, x_n \in \mathcal{B}$ and vectors $f_1, f_2, \ldots, f_n \in k; u \in h$. We define the followings :

$$\begin{aligned}
\left(a_\delta(\Delta)(\sum_{i=1}^n x_i \otimes e(f_i)) \right) u &= \sum_{i=1}^n a_{\delta(x_i^*)}(\Delta)(ue(f_i)), \\
\left(a_\delta^\dagger(\Delta)(\sum_{i=1}^n x_i \otimes e(f_i)) \right) u &= \sum_{i=1}^n a^\dagger_{\delta(x_i)}(\Delta)(ue(f_i)),
\end{aligned}$$

$$\left(\Lambda_\sigma(\Delta)(\sum_{i=1}^n x_i \otimes e(f_i))\right) u = \sum_{i=1}^n \Lambda_{\sigma(x_i)}(\Delta)(ue(f_i)),$$
$$\left(\mathcal{I}_{\mathcal{L}}(\Delta)(\sum_{i=1}^n x_i \otimes e(f_i))\right) u = \sum_{i=1}^n |\Delta|(\mathcal{L}(x_i)u) \otimes e(f_i)), \tag{1.9}$$

where $|\Delta|$ denotes the length of Δ.

We can define $\int_0^t Y(s) \circ (a_\delta^\dagger + a_\delta + \Lambda_\sigma + \mathcal{I}_{\mathcal{L}})(ds)$ where $Y(s) : \mathcal{B} \otimes_{\text{alg}} \mathcal{E}(k) \to \mathcal{B} \otimes \Gamma(k)$ is an adapted strongly continuous process satisfying the estimate

$$\sup_{o \le t \le t_0} ||Y(t)(x \otimes e(f))u|| \le ||(x \otimes 1_{\mathcal{H}''})ru||, \tag{1.10}$$

for $x \in \mathcal{B}, f \in \mathcal{C}$, where $\mathcal{C}$ is the set of bounded continuous k_0-valued square-integrable functions on $[0, \infty)$, $\mathcal{H}''$ is a Hilbert space and $r \in \mathcal{B}(h, h \otimes \mathcal{H}'')$.

Now we are ready to state the main result concerning existence-uniqueness and homomorphism property of E-H flow equation.

Theorem 1.1 *(i) There exists a unique solution J_t of equation*

$$dJ_t = J_t \circ (a_\delta^\dagger(dt) + a_\delta(dt) + \Lambda_\sigma(dt) + \mathcal{I}_{\mathcal{L}}(dt), \ J_0 = id, \tag{1.11}$$

which is an adapted regular process mapping $\mathcal{B} \otimes \mathcal{E}(\mathcal{C})$ into $\mathcal{B} \otimes \Gamma$. Furthermore, one has an estimate

$$\sup_{0 \le t \le t_0} ||J_t(x \otimes e(g))u|| \le C'(g)||(x \otimes 1_{\Gamma^f(\hat{k})})E_{t_0}u||,$$

where $g \in \mathcal{C}, \hat{k} = L^2([0, t_0], \hat{k_0}), E_t \in \mathcal{B}(h, h \otimes \Gamma^f(\hat{k}))$, $C'(g)$ is some constant and $\Gamma^f(\hat{k})$ is the full Fock space over $\hat{k}$.
(ii) Setting $j_t(x)(ue(g)) = J_t(x \otimes e(g))u$, we have
(a) $\langle j_t(x)ue(g), j_t(y)ve(f)\rangle = \langle ue(g), j_t(x^*y)ve(f)\rangle \ \forall g, f \in \mathcal{C}$, and
(b) j_t extends uniquely to a normal $*$-homomorphism from $\mathcal{B}$ into $\mathcal{B} \otimes \mathcal{B}(\Gamma)$,
(iii) If $\mathcal{B}$ is commutative, then the algebra generated by $\{j_t(x)|x \in \mathcal{B}, 0 \le t \le t_0\}$ is commutative.
(iv) $j_t(1) = 1 \ \forall t \in [0, t_0]$ if and only if $\Sigma^*\Sigma = 1_{h \otimes k_0}$.

2 E-H dilation for uniformly continuous q.d.s. on separable unital C^* algebra.

We first quote a basic theorem due to Christensen and Evans [CE]

Theorem 2.1 *Let* $(T_t)_{t\geq 0}$ *be a uniformly continuous q.d.s. on a unital* C^* *algebra* $\mathcal{A} \subseteq \mathcal{B}(h)$ *with* $\mathcal{L}$ *as its generator. Then there is a quintuple* $(\rho, \mathcal{K}, \alpha, H, R)$ *where* ρ *is a unital* $*$*-representation of* $\mathcal{A}$ *in a Hilbert space* $\mathcal{K}$ *and a* ρ*-derivation* $\alpha : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ *such that the set* $\mathcal{D} \equiv \{\alpha(x)u | x \in \mathcal{A}, u \in h\}$ *is total in* $\mathcal{K}$*,* H *is a self-adjoint element of* $\mathcal{A}''$*, and* $R \in \mathcal{B}(h, \mathcal{K})$ *such that* $\alpha(x) = Rx - \rho(x)R$*, and* $\mathcal{L}(x) = R^*\rho(x)R - \frac{1}{2}(R^*R - \mathcal{L}(1))x - \frac{1}{2}x(R^*R - \mathcal{L}(1)) + i[H, x]\ \forall x \in \mathcal{A}$*. Furthermore,* $\mathcal{L}$ *satisfies the cocycle relation with* ρ *as coboundary, namely,*

$$\mathcal{L}(x^*y) - \mathcal{L}(x^*)y - x^*\mathcal{L}(y) + x^*\mathcal{L}(1)y = \alpha(x)^*\alpha(y).$$

Moreover, R *can be chosen from the ultraweak closure of* $\{\alpha(x)y : x, y \in \mathcal{A}\}$ *and hence in particular* $R^*\rho(x)R \in \mathcal{A}''$*.*

Assume for the remainder of this section that $\mathcal{A}$ is a separable unital C^*-algebra acting *nondegenerately* on the Hilbert space h. The universal enveloping von Neumann algebra of $\mathcal{A}$ in such a case can be identified with $\mathcal{A}''$ in $\mathcal{B}(h)$. Let k_0 be a separable Hilbert space with an orthonormal basis $\{e_1, e_2, \ldots\}$. Denote by F the Hilbert $\mathcal{A}$-module $\mathcal{A} \otimes k_0$ and by G the Hilbert $\mathcal{A}$-module $\mathcal{A} \otimes \Gamma(L_2(\mathbb{R}_+) \otimes k_0)$ or $\mathcal{A} \otimes \Gamma$ for short. In the following, we shall use the notations of [Lan]. We denote by $\mathfrak{L}(E_1, E_2)$ the space of all $\mathcal{A}$-linear, adjointable, everywhere defined maps from E_1 to E_2 for two Hilbert $\mathcal{A}$-modules E_1 and E_2. When $E_1 = E_2 = E$, we denote $\mathfrak{L}(E_1, E_2)$ by $\mathfrak{L}(E)$ for short and $\mathfrak{K}(E)$ will denote the norm-closure of finite-rank $\mathcal{A}$-linear maps in E. Then we have the following :

Lemma 2.2 *If* $\eta \in \mathfrak{L}(F)$ *then* $\eta_f \in F$ *for* $f \in k$*.*

Proof: Clearly it is enough to prove the lemma for $f \neq 0$. First we claim that for any nonzero f, there exists a $\xi \in \mathfrak{K}(F)$ such that

$$\|\eta_f\| \leq \|\eta\xi\| \|f\| \tag{2.12}$$

for all $\eta \in \mathcal{B}(h \otimes k_0)$. To see this, choose $\xi = (I \otimes |f\rangle\langle f|)/\|f\|^2$. Then for any $u \in h$, we have $\eta\xi(u \otimes f) = \eta(u \otimes f) = \eta_f(u)$. so that $\|\eta_f(u)\| = \|\eta\xi(u \otimes f)\| \leq \|\eta\xi\| \|u\| \|f\|$, and the required inequality follows.

Next observe that for any η of the form

$$\eta = \sum_{i,j=1}^{n} a_{ij} \otimes |e_i\rangle\langle e_j|,$$

n varying over $\mathbb{N}$, the element $\eta_f = \sum_{i,j} \langle e_j, f\rangle a_{ij} \otimes e_i$ belongs to $\mathcal{A} \otimes_{alg} k_0 \subseteq F$. Using now the norm inequality (2.12) and the fact that such η's are dense in $\mathfrak{L}(F)$ in the strict topology, we conclude that $\eta_f \in F$ for all $\eta \in \mathfrak{L}(F)$. $\square$

Notice that for $\eta \in \mathfrak{L}(F)$, η_f can actually be identified with the element $\eta(I \otimes f)$ in F. For a bounded linear map $\sigma : \mathcal{A} \to \mathfrak{L}(F)$ and $f \in k_0$ we define a map $\sigma_f : \mathcal{A} \to F$ by $\sigma_f(x) = \sigma(x)_f$. We are now ready for the following existence theorem for a canonical E-H dilation.

Theorem 2.3 *Let T_t be a uniformly continuous, conservative quantum dynamical semigroup on a separable unital C^*-algebra $\mathcal{A}$ with generator $\mathcal{L}$. Then there exists a separable Hilbert space k_0 and a $*$-homomorphism $\pi : \mathcal{A} \to \mathfrak{L}(F)$, a π-derivation $\delta : \mathcal{A} \to F$ such that $\mathcal{L}(x^*y) - \mathcal{L}(x^*)y - x^*\mathcal{L}(y) = \delta(x)^*\delta(y)$.*

Furthermore, we can extend the maps $\mathcal{L}$, δ and π to $\hat{\mathcal{L}}$, $\hat{\delta}$ and $\hat{\pi}$ respectively on the universal enveloping von Neumann algebra $\mathcal{A}''$ of $\mathcal{A}$ such that the E-H type flow equation $d\hat{J}_t = \hat{J}_t \circ (a_{\hat{\delta}}(dt) + a^\dagger_{\hat{\delta}}(dt) + \Lambda_{\hat{\pi}-id}(dt) + \mathcal{I}_{\hat{\mathcal{L}}}(dt))$ with the initial condition $\hat{J}_0 \equiv id$ admits a unique solution as a map from $\mathcal{A}'' \otimes_s \mathcal{E}(k)$ to itself. The restricton J_t of $\hat{J}_t$ on $\mathcal{A} \otimes \mathcal{E}(k)$ takes values in $\mathcal{A} \otimes \Gamma$ and similarly, the restriction j_t of the $*$-homomorphism $\hat{j}_t : \mathcal{A}'' \to \mathcal{A}'' \otimes \mathcal{B}(\Gamma)$ as defined in the theorem 1.1(ii) on $\mathcal{A}$ takes values in $\mathfrak{L}(\mathcal{A} \otimes \Gamma)$.

Proof: By the theorem 2.1, we obtain a Hilbert space $\mathcal{K}$, a $*$-homomorphism $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{K})$, a ρ-derivation $\alpha : \mathcal{A} \to \mathcal{B}(h, \mathcal{K})$ such that

$$\mathcal{L}(a^*b) - \mathcal{L}(a^*)b - a^*\mathcal{L}(b) = \alpha(a)^*\alpha(b), \quad a, b \in \mathcal{A}. \tag{2.13}$$

Let E be the completion of the algebraic linear span of elements of the form $\alpha(a)b$, where $a, b \in \mathcal{A}$, with respect to the operator norm of $\mathcal{B}(h, \mathcal{K})$. E has an inner product inherited from $\mathcal{B}(h, \mathcal{K})$, namely, $\langle L, M\rangle = L^*M$ for $L, M \in E$. Using equation 2.13, we find that $\langle \alpha(a)b, \alpha(a')b'\rangle = b^*\alpha(a)^*\alpha(a')b' \in \mathcal{A}$. It follows then that $\langle x, y\rangle \in \mathcal{A}$ for all x and y in E. $\mathcal{A}$ has a natural right action on E (as composition of operators in $\mathcal{B}(h, \mathcal{K})$). Thus E is indeed a Hilbert $\mathcal{A}$-module. We identify ρ with a left action $\hat{\rho}$ given by, $\hat{\rho}(a)(\alpha(b)c) = \alpha(ab)c - \alpha(a)bc$. Furthermore, since $\mathcal{A}$ is separable, E is countably generated as a Hilbert $\mathcal{A}$-module. To see this, one can choose any countable dense subset $\{x_1, x_2, \ldots\}$ of $\mathcal{A}$ and note that E is the closed $\mathcal{A}$-linear span of $\{\alpha(x_1), \alpha(x_2), \ldots\}$. Now, Kasparov's stabilisation theorem (see [Lan]) yields a separable Hilbert space k_0 and an isometric $\mathcal{A}$-linear map $t \in \mathfrak{L}(E, F)$ that imbeds

E as a complemented closed submodule of $F = \mathcal{A} \otimes k_0$. We set $\delta(x) = t(\alpha(x))$ and $\pi(x) = t\hat{\rho}(x))t^*$. Then $\delta \in \mathcal{B}(\mathcal{A}, F)$ and $\pi \in \mathcal{B}(\mathcal{A}, \mathfrak{L}(F))$ and this completes the first part of the proof.

For the second part, note that π being a $*$-homomorphism of $\mathcal{A}$ into $\mathfrak{L}(F)$, admits an extension as a normal $*$-homomorphism $\hat{\pi}$ from $\mathcal{A}''$ into $(\mathfrak{L}(F))'' = \mathcal{A}'' \otimes_s \mathcal{B}(k_0)$ (see [Dix]). Observe also that by [CE], we obtain an element R in the ultraweak closure of $\{\delta(a)b : a, b \in \mathcal{A}\}$ in $\mathcal{B}(h, h \otimes k_0)$ such that

$$\mathcal{L}(a) = R^*\pi(a)R - \frac{1}{2}R^*Ra - \frac{1}{2}aR^*R + i[H, a],$$

and

$$\delta(a) = Ra - \pi(a)R,$$

for all $a \in \mathcal{A}$ and for some self-adjoint $H \in \mathcal{A}''$. We extend $\mathcal{L}$ and δ by the same expressions as above by replacing π by $\hat{\pi}$. Since R is in the ultraweak closure of F, which is $\mathcal{A}'' \otimes_s k_0$, we see that the extended maps $\mathcal{L}$ and δ map $\mathcal{A}''$ into $\mathcal{A}''$ and $\mathcal{A}'' \otimes_s k_0$ respectively. Now, by the theorem 1.1, we obtain $\hat{J}_t$ and $\hat{j}_t$. It remains to show that $J_t(x \otimes e(f)) \in \mathcal{A} \otimes \Gamma$ for $x \in \mathcal{A}$, and $j_t(x) \in \mathfrak{L}(\mathcal{A} \otimes \Gamma)$ for $x \in \mathcal{A}$ and $f \in k$.

If $\beta \in \mathcal{B}(\mathcal{A}'', \mathcal{A}'' \otimes_s k_0)$ is such that $\beta(x) \in F$ for all $x \in \mathcal{A}$, then by (1.9) $a^\dagger_\beta(\Delta)(x \otimes f^{\otimes^n})$ and hence $a^\dagger_\beta(\Delta)(x \otimes e(f))$ belongs to $\mathcal{A} \otimes \Gamma$ for any bounded subinterval Δ of $\mathbb{R}_+$ and $x \in \mathcal{A}$, $f \in L^2(\mathbb{R}_+, k_0)$. Then from the definition of $\Lambda(.)$, it follows that $\Lambda_{\pi - id}(.)$ maps $x \otimes e(f)$ into $\mathcal{A} \otimes \Gamma$ for $x \in \mathcal{A}$. That $a_\delta(.)$ and $\mathcal{I}_\mathcal{L}(.)$ have the same property is still simpler to see.

In [GS], the solution $\hat{J}_t$ was constructed by an iteration procedure and from the above, it is clear that each iterate $\hat{J}^{(i)}_t$ maps $x \otimes e(f)$ into $\mathcal{A} \otimes \Gamma$ for $x \in \mathcal{A}$ and $f \in k$. By the estimates in [GS], one has

$$\|(\hat{J}_t(x \otimes e(f)) - \sum_{i \le n} \hat{J}^{(i)}_t(x \otimes e(f)))u\| \le \|u\|\|x\|\|e(f)\|\|E_t\| \sum_{i=n+1}^{\infty} C^{\frac{i}{2}}(i!)^{-1/4}$$

for some constant C, and thus, $\|\hat{J}_t(x \otimes e(f)) - \sum_{i \le n} \hat{J}^{(i)}_t(x \otimes e(f))\|$ converges to zero. Thus J_t maps $\mathcal{A} \otimes \mathcal{E}(k)$ into $\mathcal{A} \otimes \Gamma$.

By the theorem 1.1, $j_t \equiv \hat{j}_t|_\mathcal{A}$ is a $*$-homomorphism of $\mathcal{A}$ into $\mathcal{A}'' \otimes \mathcal{B}(\Gamma)$. Thus it remains to prove that $j_t(x) \in \mathfrak{L}(\mathcal{A} \otimes \Gamma)$ for $x \in \mathcal{A}$. Note that ([Lan]) the algebra $\mathfrak{L}(\mathcal{A} \otimes \Gamma)$ can be naturally identified with the multiplier algebra of $\mathcal{A} \otimes \mathcal{B}_0(\Gamma)$ and that the set $\{y \otimes |e(f)\rangle\langle e(g)| : y \in \mathcal{A}, f, g \in L^2(\mathbb{R}_+, k_0)\}$ is total in $\mathcal{A} \otimes \mathcal{B}_0(\Gamma)$ in the norm topology. Therefore it is enough to show that for fixed $x \in \mathcal{A}$ and $t \ge 0$, the operators $j_t(x)(y \otimes |e(f)\rangle\langle e(g)|)$ and $(y \otimes |e(f)\rangle\langle e(g)|)j_t(x)$ both are in $\mathcal{A} \otimes \mathcal{B}_0(\Gamma)$ for all $y \in \mathcal{A},\ f, g \in L^2(I\!R_+, k_0)$.

Since $J_t(x \otimes e(f)) \in \mathcal{A} \otimes \Gamma$, we can choose a sequence L_n of the form $\sum_{i=1}^{k_n} z_i^{(n)} \otimes \rho_i^{(n)}$ where $z_i^{(n)} \in \mathcal{A}$ and $\rho_i^{(n)} \in \Gamma$, such that L_n converges in the norm of $\mathcal{A} \otimes \Gamma$ to $J_t(x \otimes e(f))$. Now, observe that for $u \in h$ and $\eta \in \Gamma$, $j_t(x)(y \otimes |e(f)\rangle\langle e(g)|)(u \otimes \eta) = \langle e(g), \eta\rangle J_t(x \otimes e(f))yu = \lim_{n\to\infty}\langle e(g), \eta\rangle L_n yu$. Choose an orthonormal basis $\{\gamma_m\}$ of Γ and take a vector $w \equiv \sum_m w_m \otimes \gamma_m$ of $h \otimes \Gamma$. It is easy to see that

$$\left\|\Big(j_t(x)(y \otimes |e(f)\rangle\langle e(g)|) - \sum_{i=1}^{k_n} z_i^{(n)} y \otimes |\rho_i^{(n)}\rangle\langle e(g)|\Big)w\right\|$$

$e(f)) - L_n)w_l\|$

$)^{\frac{1}{2}}$

$$\begin{aligned}
&= \left\|\sum_m \langle e(g), \gamma_m\rangle \{J_t(x \otimes e(f)) - \sum_i z_i^{(n)} \otimes |\rho_i^{(n)}\rangle\} y w_m\right\| \\
&= \left\|\sum_m \{J_t(x \otimes e(f)) - L_n\}\langle e(g), \gamma_m\rangle y w_m\right\| \\
&\leq \|J_t(x \otimes e(f)) - L_n\| \|e(g)\| \|y\| (\sum_m \|w_m\|^2)^{\frac{1}{2}} \\
&= \|J_t(x \otimes e(f)) - L_n\| \|e(g)\| \|y\| \|w\|,
\end{aligned}$$

and hence $j_t(x)(y \otimes |e(f)\rangle\langle e(g)|)$ is the norm-limit of $\sum_{i=1}^{k_n} z_i^{(n)} y \otimes |\rho_i^{(n)}\rangle\langle e(g)| \in \mathcal{A} \otimes_{\text{alg}} \mathcal{B}_0(\Gamma)$. A similar proof works for $y \otimes |e(f)\rangle\langle e(g)| j_t(x)$. □

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