Haar Measure on $E_q(2)$

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The quantum $E(2)$ group is one of the simplest known examples so far of a locally compact noncompact quantum group. The existence and uniqueness of an 'invariant measure' on this group has been proved in this article. Using the invariant measure, we compute certain orthogonality relations, which then tells us that any unitary representation can have both square-integrable and non square-integrable matrix entries.

1. Introduction. The notion of a compact quantum group has now reached more or less a final form after extensive investigations by several people. This, however, is far from true in the case of noncompact quantum groups, where one is yet to arrive at a satisfactory definition. In order to be able to give an appropriate definition of a noncompact locally compact quantum group, specific examples are being investigated. $E_q(2)$, the q-deformation of the group of motions of the Euclidean plane, is one example that has been studied by various authors ([2], [6], [7]). It is known to have many features not exhibited by any classical noncompact locally compact group. It will be shown in this article (see section 2) that, like any locally compact group, $E_q(2)$ also has an invariant 'measure'. As we shall see, the form of the haar weight is quite easy to guess, if we know the haar state for the group $SU_q(2)$ from which $E_q(2)$ comes via the contraction procedure([7]). But the proof of its invariance properties and uniqueness is quite involved. In the third section, we list all the irreducible unitary representations of $E_q(2)$, and compute the orthogonality relations between their matrix entries. As a consequence, it is observed that any unitary representation of $E_q(2)$ can have matrix entries which are square-integrable as well as those which are not. Such a situation can never arise for a locally compact unimodular group. The modular operator associated with the haar weight is written down explicitly in the last section. This enables us to use the Radon-Nikodym theorem for weights due to Pedersen and

Takesaki ([4]) in order to prove the uniqueness of the invariant weight.

We retain most of the notations in [7]. Places where they differ are the following. The deformation parameter is denoted by q here; μ denotes the comultiplication map for the group $E_q(2)$; $\mathbb{C}^q$ denotes the closure of the set $\{q^k z : k \in \mathbb{Z}, z \in S^1\}$; $C_0(E_q(2))$ denotes the algebra of continuous vanishing-at-infinity functions on $E_q(2)$, and $C_b(E_q(2))$ denotes the multiplier algebra of $C_0(E_q(2))$. For an element a in $C_0(E_q(2))$ and a bounded functional ρ on $C_0(E_q(2))$, $\rho * a$ denotes $(\rho \otimes id)\mu(a)$ and $a * \rho$ denotes $(id \otimes \rho)\mu(a)$.

Let $\{e_i\}$ be the canonical orthonormal basis for $\ell_2(\mathbb{Z})$. We denote $e_i \otimes e_j$ by e_{ij}, $e_i \otimes e_j \otimes e_k$ by e_{ijk}, and $e_i \otimes e_j \otimes e_k \otimes e_l$ by e_{ijkl}. Let ℓ and N denote the operators $e_k \mapsto e_{k-1}$ and $e_k \mapsto ke_k$ respectively. $\boldsymbol{v}$ and $\boldsymbol{n}$ are the operators introduced in [7]. Denote by τ the automorphism $a \mapsto \boldsymbol{v}a\boldsymbol{v}^*$ of $C_0(E_q(2))$.

Let us denote by $\mathcal{A}_r$ the C^*-algebra $\tau^r(C(SU_q(2)))$, where $r \in \mathbb{Z}$. Let $\mathcal{A}$ denote $C_0(E_q(2))$. For any $a \in \mathcal{A}$, define $p_r(a) = \tau^r(I_{SU}\tau^{-r}(a)I_{SU})$. Then $p_r(a)$ is a projection onto $\mathcal{A}_r$, i.e. p_r maps $\mathcal{A}$ onto $\mathcal{A}_r$, and satisfies ${p_r}^2 = p_r$, and $\|p_r(a)\| \leq \|a\|$ $\forall a \in \mathcal{A}$. Clearly, $0 \leq p_r(a) \leq p_{r+1}(a) \leq a$ for any positive a of the form $f(\boldsymbol{n})$.

Call an element $a \in \mathcal{A}$ *compactly supported* if $a = p_r(a)$ for some r, i.e. if $a \in \cup \mathcal{A}_r$. A continuous functional ρ on $\mathcal{A}$ is said to be *compactly supported* if there is an $r \in \mathbb{Z}$ such that $p_r(a) = 0$ implies $\rho(a) = 0$.

2. The Haar Weight. Define a weight h on $\mathcal{A}$ as follows:

$$h(a) = \sum_{i \in \mathbb{Z}} q^{2i} \langle e_{i0}, a e_{i0} \rangle, \quad a \in \mathcal{A}_+. \tag{2.1}$$

Let h_{SU} be the haar state for $SU_q(2)$. It is easy to see that

$$h(a) = \lim_{r \to \infty} (1 - q^2)^{-1} q^{-2r} h_{SU}(I_{SU}\tau^{-r}(a)I_{SU}). \tag{2.2}$$

Let $\mathcal{A}_+^h = \{a \in \mathcal{A}_+ : h(a) < \infty\}$. $\mathcal{A}_+^h$ contains all compactly supported positive elements, and hence is dense in $\mathcal{A}_+$. Therefore the linear span $\mathcal{A}^h$ of $\mathcal{A}_+^h$ is dense in $\mathcal{A}$ and contains $\cup_r \mathcal{A}_r$. For each $r \in \mathbb{Z}$, hp_r is a bounded positive functional, and $h(a) = \sup_r hp_r(a)$ whenever $a \geq 0$. Therefore h is a *lower semicontinuous* weight.

If we define h by (2.1) on the von-Neuman algebra $\mathcal{M} = \mathcal{A}''$ generated by $\mathcal{A}$, then

i. h is *semifinite*, i.e. $\mathcal{M}_+^h$ is weakly dense in $\mathcal{M}_+$,

ii. $p_r(I) \in \mathcal{M}_+^h \ \forall r$, $p_r(I)$ increases to I strongly (*σ-finiteness*),

iii. h is *σ-normal*, i.e. is a countable sum of positive normal functionals.

It is immediate from (2.1) that $\tau^k(\mathcal{A}^h) \subseteq \mathcal{A}^h$ for all $k \in \mathbb{Z}$, and

$$h\tau^k(a) = q^{-2k} h(a) \quad \forall a \in \mathcal{A}^h. \tag{2.3}$$

The next theorem describes the invariance properties of this functional under right and left convolutions.

Theorem 2.1 *For any $a \in \mathcal{A}^h$ and any bounded functional ρ on $\mathcal{A}$, both $a * \rho$ and $\rho * a$ are in $\mathcal{A}^h$, and the following equalities hold:*

$$h(a * \rho) = h(a)\rho(I) = h(\rho * a). \tag{2.4}$$

Remark: Notice that although the C^*-algebra $\mathcal{A}$ does not have identity, (2.4) makes sense because any continuous functional on $\mathcal{A}$ admits an extension to the multiplier algebra $M(\mathcal{A})$.

We break the proof into several propositions. Let us begin with the following proposition.

Proposition 2.2 *Let $a \in \mathcal{A}$ and ρ be a continuous functional on $\mathcal{A}$. If both a and ρ are compactly supported, then a and $a * \rho$ are both in $\mathcal{A}^h$, and*

$$h(a * \rho) = h(a)\rho(I).$$

Proof: Observe that $C_0(E_q(2))$ is a type I C^*-algebra, so that any representation is a direct integral of the irreducible ones. Therefore any representation of the C^*-algebra $C_0(E_q(2))$ can be written as a direct sum $\pi_U \oplus \epsilon_V$, where U and V are two unitary operators acting on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, and π_U and ϵ_V are representations acting on $\ell_2(\mathbb{Z}) \otimes \mathcal{H}$ and $\ell_2(\mathbb{Z}) \otimes \mathcal{K}$ given by:

$$\pi_U : \begin{cases} \boldsymbol{v} \mapsto \ell \otimes I \\ \boldsymbol{n} \mapsto q^N \otimes U \end{cases} \qquad \epsilon_V : \begin{cases} \boldsymbol{v} \mapsto V \\ \boldsymbol{n} \mapsto 0 \end{cases}$$

Therefore any positive functional ρ is of the form

$$a \mapsto \langle u, \pi_U(a)u\rangle + \langle v, \epsilon_V(a)v\rangle. \tag{2.5}$$

Denote by $\rho_{u,U}$ the first term on the right hand side above. If $\mathcal{H} = \ell_2(\mathbb{Z})$, and $U = \ell^*$, then we will simply write ρ_u instead of $\rho_{u,U}$. Let $\{f_k\}$ be an orthonormal basis for $\mathcal{H}$. Denote $e_k \otimes f_j$ by e_{kj}.

Step I : Take $a \in (\mathcal{A}_0)_+$, and $\rho = \rho_{e_{mn},U}$.

$$\begin{aligned}
&|hp_{3r}(a * \rho) - h((id \otimes \rho)(\tau^{3r} \otimes \tau^{3r})\mu_{SU}(\tau^{-3r}a))| \\
&= \ |\sum_{i \geq -3r} q^{2i}\langle e_{i0}, (a * \rho - (\tau^{3r} \otimes \rho\tau^{3r})\mu_{SU}(\tau^{-3r}a))e_{i0}\rangle| \\
&= \ |\sum_{i \geq -3r} q^{2i}\langle e_{i\,0\,m\,n}, (id \otimes \pi_U)(\mu(a) - (\tau^{3r} \otimes \tau^{3r})\mu_{SU}(\tau^{-3r}a))e_{i\,0\,m\,n}\rangle|.
\end{aligned} \tag{2.6}$$

Suppose for the time being that the right hand side above tends to zero as r goes to infinity. Now,

$$\begin{aligned}
& h((id \otimes \rho)(\tau^{3r} \otimes \tau^{3r})\mu_{SU}(\tau^{-3r}a)) \\
&= h\tau^{3r}((id \otimes \rho\tau^{3r})\mu_{SU}(\tau^{-3r}a)) \\
&= q^{-6r}h((id \otimes \rho\tau^{3r})\mu_{SU}(\tau^{-3r}a)) \\
&= (1-q^2)^{-1}q^{-6r}(h_{SU} \otimes \rho\tau^{3r})\mu_{SU}(\tau^{-3r}a) \\
&= (1-q^2)^{-1}q^{-6r}h_{SU}(\tau^{-3r}a)\rho\tau^{3r}(I_{SU}) \\
&= (1-q^2)^{-1}h_{SU}(a)\rho\tau^{3r}(I_{SU}) \\
&= h(a)\rho\tau^{3r}(I_{SU}).
\end{aligned}$$

Since $\rho\tau^{3r}(I_{SU})$ tends to $\rho(I)$ as $r \to \infty$, $\lim_{r\to\infty} hp_{3r}(a * \rho) = h(a)\rho(I)$. Therefore $a * \rho \in \mathcal{A}_+^h$ and

$$h(a * \rho) = h(a)\rho(I). \tag{2.7}$$

We now proceed to show that the right hand side of (2.6) indeed goes to zero as r tends to infinity.

Remark 2.3 *The contraction formula, proved in [7], merely tells us that*

$$\lim_{r\to\infty} \|\mu(a) - (\tau^{3r}\otimes\tau^{3r})\mu_{SU}(\tau^{-3r}a)\| = 0,$$

but it does not say anything about the rate of convergence, which is what we need here.

Let $\boldsymbol{t} = \prod_{k=1}^{\infty}(I_{SU} - q^{2k}\beta^*\beta)$ and $X = \sum_{k=0}^{\infty} c_k(-q\beta^*\otimes\beta)^k(\boldsymbol{v}\otimes\boldsymbol{v})^{-k}$, where $c_r = \prod_{k=1}^{r}(1-q^{2i})^{-1}$. Write Λ for $\mu_{SU}(\boldsymbol{t}^{-1/2}a\boldsymbol{t}^{-1/2})$. Then by equation (30) of [7], $\mu(a) = X^*(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\Lambda(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}X$. Therefore,

$$\langle e_{i\,0\,m\,n}, (id\otimes\pi_U)(\mu(a) - (\tau^{3r}\otimes\tau^{3r})\mu_{SU}(\tau^{-3r}a))e_{i\,0\,m\,n}\rangle = \sum_{\nu=1}^{6}\langle e_{i\,0\,m\,n}, (id\otimes\pi_U)E_\nu e_{i\,0\,m\,n}\rangle, \quad (2.8)$$

where

$$\begin{aligned}
E_1 &= X^*(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\Lambda(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}(X - \sum_{s=0}^{r} c_s(-q\beta^*\otimes\beta)^s(\boldsymbol{v}\otimes\boldsymbol{v})^{-s}),\\
E_2 &= (X^* - \sum_{s=0}^{r} c_s(\boldsymbol{v}\otimes\boldsymbol{v})^s(-q\beta\otimes\beta^*)^s)(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\Lambda(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\\
&\quad\times\sum_{s=0}^{r} c_s(-q\beta^*\otimes\beta)^s(\boldsymbol{v}\otimes\boldsymbol{v})^{-s},\\
E_3 &= (\sum_{s=0}^{r} c_s(\boldsymbol{v}\otimes\boldsymbol{v})^s(-q\beta\otimes\beta^*)^s)(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\Lambda(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}(\sum_{s=0}^{r} c_s(-q\beta^*\otimes\beta)^s(\boldsymbol{v}\otimes\boldsymbol{v})^{-s})\\
&\quad - (\sum_{s=0}^{r} d_{rs}(\boldsymbol{v}\otimes\boldsymbol{v})^s(-q\beta\otimes\beta^*)^s)(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\Lambda(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}(\sum_{s=0}^{r} d_{rs}(-q\beta^*\otimes\beta)^s(\boldsymbol{v}\otimes\boldsymbol{v})^{-s}),\\
E_4 &= (\sum_{s=0}^{r} d_{rs}(\boldsymbol{v}\otimes\boldsymbol{v})^s(-q\beta\otimes\beta^*)^s(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2} - (\boldsymbol{v}\otimes\boldsymbol{v})^{3r}\mu_{SU}(\alpha^{*3r}))\\
&\quad\times\Lambda(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\sum_{s=0}^{r} d_{rs}(-q\beta^*\otimes\beta)^s(\boldsymbol{v}\otimes\boldsymbol{v})^{-s},\\
E_5 &= (\boldsymbol{v}\otimes\boldsymbol{v})^{3r}\mu_{SU}(\alpha^{*3r})\Lambda\\
&\quad\times((\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\sum_{s=0}^{r} d_{rs}(-q\beta^*\otimes\beta)^s(\boldsymbol{v}\otimes\boldsymbol{v})^{-s} - \mu_{SU}(\alpha^{3r})(\boldsymbol{v}\otimes\boldsymbol{v})^{-3r}),\\
E_6 &= (\tau^{3r}\otimes\tau^{3r})(\mu_{SU}(\alpha^{*3r}\boldsymbol{t}^{-1/2}a\boldsymbol{t}^{-1/2}\alpha^{3r}) - \mu_{SU}(\boldsymbol{v}^{-3r}a\boldsymbol{v}^{3r})),\\
\text{and } d_{rs} &= (\prod_{i=1}^{3r}(1-q^{2i}))/(\prod_{i=1}^{3r-s}(1-q^{2i})\prod_{i=1}^{s}(1-q^{2i})).
\end{aligned}$$

Assume, for the time being, that

$$\lim_{r\to\infty} q^{-6r}\sup_{i\geq -3r}|\langle e_{i\,0\,m\,n}, (id\otimes\pi_U)E_\nu e_{i\,0\,m\,n}\rangle| = 0 \quad \text{for } \nu = 1,2, \quad (2.9)$$

$$\lim_{r\to\infty} | \sum_{i\geq -3r} q^{2i}\langle e_{i\,0\,m\,n}, (id\otimes\pi_U)E_\nu e_{i\,0\,m\,n}\rangle| = 0 \quad \text{for } \nu = 3,\ldots,6. \tag{2.10}$$

These, together with (2.8), will then ensure that the right hand side of (2.6) tends to zero as r approaches infinity. So let us now prove (2.9) and (2.10).

$\boldsymbol{\nu = 1}$. For any integer k,

$$\begin{aligned}
& q^{-kr}\sup_i \| \sum_{s\geq r+1} c_s(-q\beta^*\otimes\pi_U(\beta))^s(\boldsymbol{v}\otimes\pi_U(\boldsymbol{v}))^{-s} e_{i\,0\,m\,n}\| \\
&= q^{-kr}\sup_i \| \sum_{\substack{s\geq r+1\\ i+s\geq 0\\ m+s\geq 0}} c_s(-1)^s q^{s(i+m+2s+1)}(I\otimes I\otimes I\otimes U^*)^s e_{i+s\,s\,m+s\,n}\| \\
&\leq q^{-kr}\sup_i (\sum_{\substack{s\geq r+1\\ i+s\geq 0\\ m+s\geq 0}} c_s^2 q^{2s(i+s+1)+2s(m+s)})^{1/2} \\
&\leq (\sum_{s\geq r+1} c_s^2 q^{2s+s(2m+s)+s^2-2kr})^{1/2},
\end{aligned}$$

and now, clearly the right hand side tends to zero as r goes to infinity. Using this for $k = 6$, we get (2.9) for $\nu = 1$.

$\boldsymbol{\nu = 2}$. Similar to the previous case.

$\boldsymbol{\nu = 3}$. In this case,

$$\begin{aligned}
& | \sum_{i\geq -3r} q^{2i}\langle e_{i\,0\,m\,n}, (id\otimes\pi_U)E_3 e_{i\,0\,m\,n}\rangle| \\
&\leq \sum_{i\geq -r} q^{2i}|\langle e_{i\,0\,m\,n}, \sum_{s=0}^{r}\sum_{s'=0}^{r}(c_s c_{s'} - d_{rs}d_{rs'})(\boldsymbol{v}\otimes\pi_U(\boldsymbol{v}))^s(-q\beta\otimes\pi_U(\beta^*))^s \\
&\quad \times (\boldsymbol{t}\otimes\pi_U(\boldsymbol{t}))^{1/2}(id\otimes\pi_U)\Lambda(\boldsymbol{t}\otimes\pi_U(\boldsymbol{t}))^{1/2}(-q\beta^*\otimes\pi_U(\beta))^{s'}(\boldsymbol{v}\otimes\pi_U(\boldsymbol{v}))^{-s'} e_{i\,0\,m\,n}\rangle| \\
&\leq \sum_{i\geq -r} q^{2i}| \sum_{s=0\vee(-i)\vee(-m)}^{r} \sum_{s'=0\vee(-i)\vee(-m)}^{r} (c_s c_{s'} - d_{rs}d_{rs'})q^{s(i+m+2s+1)+s'(i+m+2s'+1)} \\
&\quad \times \langle (I\otimes I\otimes I\otimes U^*)^s e_{i+s\,s\,m+s\,n}, \\
&\qquad (id\otimes\pi_U)((\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\Lambda(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2})(I\otimes I\otimes I\otimes U^*)^{s'} e_{i+s'\,s'\,m+s'\,n}\rangle| \\
&\leq \text{const.}q^{-2r}\sup_{0\leq s\leq r}(c_s - d_{rs}),
\end{aligned}$$

and the right hand side here goes to zero as r approaches infinity.

$\boldsymbol{\nu = 4}$. We shall need the following lemma.

Lemma 2.4 *For any integer k,*

$$\lim_{r\to\infty} q^{-kr}\sup_i \|\sum_{s=r+1}^{3r} d_{rs}(-q\beta^*\otimes\pi_U(\beta))^s(\alpha\otimes\pi_U(\alpha))^{3r-s}(\boldsymbol{v}\otimes\pi_U(\boldsymbol{v}))^{-3r}e_{i\,0\,m\,n}\| = 0.$$

Proof: $q^{-kr}\|\sum_{s=r+1}^{3r} d_{rs}(-q\beta^*\otimes\pi_U(\beta))^s(\alpha\otimes\pi_U(\alpha))^{3r-s}(\boldsymbol{v}\otimes\pi_U(\boldsymbol{v}))^{-3r}e_{i\,0\,m\,n}\|$

$$\begin{aligned}
&\leq \text{const.}\,(\sum_{s=(r+1)\vee(-i)\vee(-m)}^{3r} d_{rs}^2 q^{2s(i+m+2s+1)-2kr})^{1/2}\\
&\leq \text{const.}\,(\sum_{s=(r+1)\vee(-i)\vee(-m)}^{3r} c_s^2 q^{2s(i+s+1)+s(2m+s)+s^2-2kr})^{1/2}\\
&\leq \text{const.}\,(\sum_{s\geq r+1} q^{2s+s(2m+s)+s^2-2kr})^{1/2}.
\end{aligned}$$

It is clear now that the required limit is zero. □

Now, using the binomial expansion for $\mu_{SU}(\alpha^{*3r})$, we get

$$\begin{aligned}
&|\sum_{i\geq -3r} q^{2i}\langle e_{i\,0\,m\,n}, (id\otimes\pi_U)E_4 e_{i\,0\,m\,n}\rangle|\\
&= \ |\sum_{i\geq -r} q^{2i}\langle e_{i\,0\,m\,n}, (id\otimes\pi_U)(\sum_{s=0}^{r} d_{rs}(\boldsymbol{v}\otimes\boldsymbol{v})^s((\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}-(\boldsymbol{v}\otimes\boldsymbol{v})^{3r-s}(\alpha^*\otimes\alpha^*)^{3r-s})\\
&\qquad\times(-q\beta\otimes\beta^*)^s\Lambda(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\sum_{s'=0}^{r} d_{rs'}(-q\beta^*\otimes\beta)^{s'}(\boldsymbol{v}\otimes\boldsymbol{v})^{-s'})e_{i\,0\,m\,n}\rangle|\\
&\quad+|\sum_{i\geq -3r} q^{2i}\langle e_{i\,0\,m\,n}, (id\otimes\pi_U)(\sum_{s=r+1}^{3r} d_{rs}(\boldsymbol{v}\otimes\boldsymbol{v})^{3r}(\alpha^*\otimes\alpha^*)^{3r-s}(-q\beta\otimes\beta^*)^s\\
&\qquad\times\Lambda(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\sum_{s'=0}^{r} d_{rs'}(-q\beta^*\otimes\beta)^{s'}(\boldsymbol{v}\otimes\boldsymbol{v})^{-s'})e_{i\,0\,m\,n}\rangle|\\
&\leq\ \text{const.}\, q^{-2r}\sup_{0\leq s\leq r}(1-\prod_{k\geq 3r-s+1}(1-q^{2k})^{1/2})\\
&\quad+\text{const.}\, q^{-6r}\sup_i\|(id\otimes\pi_U)(\sum_{s=r+1}^{3r} d_{rs}(-q\beta^*\otimes\beta)^s(\alpha\otimes\alpha)^{3r-s}(\boldsymbol{v}\otimes\boldsymbol{v})^{-3r}e_{i\,0\,m\,n}\|.
\end{aligned}$$

The first term obviously goes to zero as r approaches infinity. By lemma 2.4, the same conclusion holds for the second term also. Therefore (2.10) holds for $\nu = 4$.

$\boldsymbol{\nu = 5}$. Similar to the previous case.

$\boldsymbol{\nu = 6}$. Let us denote by P_r the operator $\prod_{k\geq 3r+1}(1-q^{2N+2k})^{-1/2}\otimes I$. Then

$$|\sum_{i\geq -3r} q^{2i}\langle e_{i\,0\,m\,n}, (id\otimes\pi_U)E_6 e_{i\,0\,m\,n}\rangle|$$

$$
\begin{aligned}
&= \Big|\sum_{i\geq -3r} q^{2i}\langle e_{i+3r\,0\,m+3r\,n}, (id\otimes\pi_U)(\mu_{SU}(\alpha^{*3r})\Lambda\mu_{SU}(\alpha^{3r}) - \mu_{SU}(\boldsymbol{v}^{-3r}a\boldsymbol{v}^{3r}))e_{i+3r\,0\,m+3r\,n}\rangle\Big| \\
&= \Big|\sum_{i\geq -3r} q^{2i}\langle e_{i+3r\,0\,m+3r\,n}, (id\otimes\pi_U)\mu_{SU}(\tau^{-3r}(P_r a P_r - a))e_{i+3r\,0\,m+3r\,n}\rangle\Big| \\
&= q^{-6r}(1-q^2)^{-1}|h_{SU}(\tau^{-3r}(P_r a P_r - a))| \\
&= (1-q^2)^{-1}|h_{SU}(P_r a P_r - a)|.
\end{aligned}
$$

Therefore (2.10) holds for $\nu = 6$.

Observe that in all the estimates above, we have used crucially the fact that m is not allowed to go too near minus infinity. Essentially the same calculations can therefore be used to show that the conclusion holds even when ρ is of the form $\rho_{u,U}$, where

$$
u = \sum_{i\geq m} \lambda_{ij} e_{ij}, \quad m \in \mathbb{Z}. \tag{2.11}
$$

Step II : Take $a \in (\mathcal{A}_0)_+$, and ρ compactly supported.

In this case, it can be shown that ρ must be of the form $\rho_{u,U} + \langle w, \epsilon_V(\cdot)w\rangle$, where u is as in (2.11). For $\rho = \rho_{u,U}$, the proof is already done in step I. Let us now prove the equality for $\rho = \langle w, \epsilon_V(\cdot)w\rangle$. It is easy to see that in this case, $a * \rho \in \mathcal{A}_0$ and $a * \rho = (id \otimes \rho)\mu_{SU}(a)$. Therefore $h(a*\rho) = (1-q^2)^{-1}(h_{SU}\otimes\rho)\mu_{SU}(a) = (1-q^2)^{-1}h_{SU}(a)\rho(I) = h(a)\rho(I)$.

Step III : Take $a \in (\mathcal{A}_r)_+$, and ρ to be any compactly supported state. Observe that $\tau^{-r}a * \rho\tau^r = \tau^{-r}(a*\rho)$. Since $\tau^{-r}a \in (\mathcal{A}_0)_+$ and $\rho\tau^r$ is compactly supported, we have $h(\tau^{-r}a * \rho\tau^r) = h(\tau^{-r}a)\rho\tau^r(I) = q^{2r}h(a)\rho(I)$. On the other hand $h(\tau^{-r}(a*\rho)) = q^{2r}h(a*\rho)$. Therefore $h(a*\rho) = h(a)\rho(I)$. As $\cup_r\mathcal{A}_r$ is just the linear span of the $(\mathcal{A}_r)_+$'s, and any compactly supported contnuous functional is a linear combination of compactly supported states, the equality above holds for any compactly supported a and any compactly supported continuous functional ρ. $\square$

Let F_q be the function introduced by Woronowicz in [6]. Denote by f_k^n the k$^{\text{th}}$ Fourier coefficient of the function $z \mapsto F_q(q^n z)$, i.e. $f_k^n = \int_{S^1} F_q(q^n z)z^{-k}\,dz$. The identity below involving these Fourier coefficients follow from the above proposition.

Corollary 2.5 $\sum_{i\in\mathbb{Z}} q^{2i} f_{-r-i}^{k-i} f_{-i}^{k-i+r} = \delta_{r0} \quad \forall\, r,k \in \mathbb{Z}.$

Proof: Take ρ to be the functional $a \mapsto \langle e_{k-1\,0}, a e_{r+k-1\,r}\rangle$ and let $b = \boldsymbol{v}^r g(\boldsymbol{n})$, where g is the function $q^k z \mapsto I_{\{0\}}(k) z^{-r}$, $k \in \mathbb{Z}$, $z \in S^1$. Using equation (12) of [7], it can be shown that

$$\begin{aligned} &\langle e_{ij\,k\,l}, \mu(a) e_{r\,s\,t\,u}\rangle \\ &= \begin{cases} \sum_n f_n^{k-i+1} f_{n+u-l}^{t-r+1} \langle e_{i+n,j+n}, a\, e_{r+u-l+n,s+u-l+n}\rangle & \text{if } t-r-s-u = \\ & \qquad k-i-j-l, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.12}$$

Use this to evaluate $b * \rho$. Now, use the relation $h(b*\rho) = h(b)\rho(I)$. $\square$

Proposition 2.6 *For any $a \in \mathcal{A}_+^h$, and state ρ, we have $h(a*\rho) \leq h(a)$.*

Proof: For any $a \in \mathcal{A}$, $\lim \|p_r(a) - a\| = 0$, and for any state ρ on $\mathcal{A}$, $\lim \rho p_r(a) = \rho(a)$ for all a. This, along with the fact that $\mu(\mathcal{A}) \subseteq \mathcal{A} \otimes \mathcal{A}$, yields that $p_r(a) * \rho p_r$ converges to $a * \rho$ in norm. Also, for a and ρ positive, all these quantities remain positive. Since $p_r(a)$ and ρp_r are compactly supported, we have $h(p_r(a) * \rho p_r) = h p_r(a) \rho p_r(I)$. Therefore, combining all these observations together, we get

$$\begin{aligned} h(a*\rho) &\leq \lim_{r\to\infty} h p_r(a) \rho p_r(I) \\ &= h(a) \lim \rho p_r(I) \\ &= h(a). \end{aligned}$$

$\square$

Let $(L_2(h), \eta_h, \pi_h)$ and $(\mathcal{H}_\rho, \eta_\rho, \pi_\rho)$ be the GNS triples associated with the weights h and $a \mapsto h(a*\rho)$ respectively. We shall very often identify a and $\eta_h(a)$ for $a \in \{b \in \mathcal{A} : h(b^*b) < \infty\}$.

Proposition 2.7 *Let g_{jk} be the function on $\mathbb{C}^q$ defined by $q^r z \mapsto I_{\{j\}}(r) z^k$, $r \in \mathbb{Z}$, $z \in S^1$. Then $\{q^{-j} \boldsymbol{v}^i g_{jk}(\boldsymbol{n}) : i,j,k \in \mathbb{Z}\}$ is a complete orthonormal basis in $L_2(h)$ and $\{q^{-j} \eta_\rho(\boldsymbol{v}^i g_{jk}(\boldsymbol{n})) : i,j,k \in \mathbb{Z}\}$ is an orthonormal system of vectors in $\mathcal{H}_\rho$.*

Proof: The first part is easy. We prove the second part here. Write a_1 for $\boldsymbol{v}^i g_{jk}(\boldsymbol{n})$ and a_2 for $\boldsymbol{v}^r g_{st}(\boldsymbol{n})$. Then from the first part, $h(a_i^* a_i) < \infty$, $i = 1, 2$. Now, using proposition 2.6, we get

$$|\langle \eta_\rho(a_1), \eta_\rho(a_2)\rangle| = |h(a_1^* a_2 * \rho)|$$

$$
\begin{aligned}
&\leq |h(a_1^* a_1 * \rho)|^{1/2} |h(a_2^* a_2 * \rho)|^{1/2} \\
&\leq |h(a_1^* a_1)|^{1/2} |h(a_2^* a_2)|^{1/2} \\
&< \infty.
\end{aligned}
$$

Next,

$$
\begin{aligned}
\langle \eta_\rho(a_1), \eta_\rho(a_2) \rangle &= h(a_1^* a_2 * \rho) \\
&= \lim_{m\to\infty} hp_m(a_1^* a_2 * \rho) \\
&= \lim_m \sum_{l\geq -m} q^{2l} \langle e_{l0}, (a_1^* a_2 * \rho) e_{l0} \rangle \\
&= \lim_m \rho\Big(\sum_{l\geq -m} q^{2l} \rho_{e_{l0}} * a_1^* a_2 \Big) \\
&= \lim_m \rho(T_m),
\end{aligned}
$$

where $T_m = \sum_{l\geq -m} q^{2l} \rho_{e_{l0}} * a_1^* a_2$. Now using (2.12), we get

$$
\rho_{e_{l0}} * a_1^* a_2 = \delta_{i,r+t-k} \delta_{j,s+t-k} \boldsymbol{v}^{k-t} V_{\boldsymbol{n}}^{t-k} (f_{t-k+s-l}^{N+t-k+1-l} f_{s-l}^{N+1-l} \otimes I),
$$

where $V_{\boldsymbol{n}}$ is the unitary appearing in the polar decomposition of $\boldsymbol{n}$, and f_j^{N+i} denote the operator $e_k \mapsto f_j^{k+i} e_k$. Therefore

$$
T_m = \delta_{i,r+t-k} \delta_{j,s+t-k} \boldsymbol{v}^{k-t} V_{\boldsymbol{n}}^{t-k} \Big(\sum_{l\geq -m} q^{2l} f_{t-k+s-l}^{N+t-k+1-l} f_{s-l}^{N+1-l} \otimes I \Big).
$$

Now using corollary 2.5 one can show that for any positive functional ρ of the form $\rho_{u,U}$, $\lim \rho(T_m) = \delta_{ir}\delta_{js}\delta_{kt} q^{2j} \|u\|^2$. Since $\epsilon_V(T_m) = \delta_{ir}\delta_{js}\delta_{kt} q^{2j}$ for all $m > -j$, it follows that

$$
\langle \eta_\rho(\boldsymbol{v}^i g_{jk}(\boldsymbol{n})), \eta_\rho(\boldsymbol{v}^r g_{st}(\boldsymbol{n})) \rangle = \delta_{ir}\delta_{js}\delta_{kt} q^{2s},
$$

which proves the assertion. □

Proof of theorem 2.1: Take $a \in \mathcal{A}_+^h$, and ρ to be a state. Define a map S from $L_2(h)$ to $\mathcal{H}_\rho$ by the prescription $a \mapsto \eta_\rho(a)$. Then by the above proposition S extends as an isometry to the whole of $L_2(h)$. Therefore, for any $a \in \mathcal{A}_+^h$, $\|Sa^{1/2}\| = \|a^{1/2}\|$, which means $h(a * \rho) = h(a) = h(a)\rho(I)$. By taking linear combinations, the same conclusion holds for any $a \in \mathcal{A}^h$ and any continuous functional ρ.

Proof of the other equality, namely, $h(\rho * a) = h(a)\rho(I)$, is exactly similar. □

Uniqueness of this weight will be proved in the last section.

Remark 2.8 For a positive, theorem 2.1 tells us that if $h(a)$ is finite, then so also is $h(a*\rho)$, and $h(a*\rho) = h(a)$. The equality actually holds always, i.e. if $h(a) = \infty$, then $h(a*\rho)$ also is infinity. To see this, take $a_r = a - (I - \tau^r(I_{SU}))a(I - \tau^r(I_{SU}))$. Then $h(a_r) = hp_r(a)$, so that $h(a_r)$ increases to infinity. On the other hand, since $h(a_r) < \infty$, $h(a_r) = h(a_r * \rho)$, and since $a - a_r \geq 0$, $h(a*\rho) \geq h(a_r * \rho)$. Therefore we must have $h(a*\rho) = \infty$.

Remark 2.9 We call h the *haar weight* for the group $E_q(2)$. It is easy to see that h *is faithful.*

Let us prove here another identity using the invariance of the weight h that will be needed in the next section.

Corollary 2.10 $\sum_{i\in\mathbb{Z}} q^{2i} f_s^{i+r} f_s^{i+r'} = \delta_{rr'} q^{2(1-r+s)} \quad \forall\, r, r', s \in \mathbb{Z}$.

Proof: Let g be the following function on $\mathbb{C}^q$: $g(q^k z) = I_{\{1-r'+s\}}(k) z^{r-r'}$, $k \in \mathbb{Z}$, $z \in S^1$, and let $b = \boldsymbol{v}^{r-r'} g(\boldsymbol{n})$. Take ρ to be the functional $a \mapsto \langle e_{1-r,0}, a e_{1-r',r'-r}\rangle$. Now use (2.11) to compute $h(\rho * b)$, and use the equation $h(\rho * b) = h(b)\rho(I)$. $\square$

3. Orthogonality Relations. For a closed operator T, let V_T denote the partial isometry appearing in the polar decomposition of T. Let (b, T) be a pair of closed operators acting on some Hilbert space $\mathcal{H}$ such that the following conditions hold:

$$\left.\begin{array}{rl} \text{i.} & T \text{ is self-adjoint}, \\ \text{ii.} & b \text{ is normal}, \\ \text{iii.} & T \text{ and } |b| \text{ commute strongly}, \\ \text{iv.} & V_b{}^* T V_b = T + 2I \text{ on } (\ker b)^{\perp}, \\ \text{v.} & \sigma(T, |b|) \subseteq \overline{\Sigma_q}, \text{ where } \Sigma_q = \{(r, q^{s+r/2}) : r, s \in \mathbb{Z}\}, \\ & \sigma(T, |b|) \text{ being the joint spectrum of } T \text{ and } |b|. \end{array}\right\} \qquad (3.1)$$

It has been proved in [6] that if (b, T) is such a pair, then $F_q(q^{T/2} b \otimes \boldsymbol{vn})(I \otimes \boldsymbol{v})^{T\otimes I}$ is a unitary representation of $E_q(2)$ acting on $\mathcal{H}$, and conversely, given any unitary representation w of $E_q(2)$ acting on a Hilbert space $\mathcal{H}$, there is a pair (b, T) of operators on $\mathcal{H}$ satisfying the requirements above such that $w = F_q(q^{T/2} b \otimes \boldsymbol{vn})(I \otimes \boldsymbol{v})^{T\otimes I}$.

We call a pair (b, T) satisfying (3.1) *irreducible* if the Hilbert space $\mathcal{H}$ on which they act does not have any nonzero proper closed subspace that is kept invariant by b, b^*,and T.

Proposition 3.1 *Let w be a unitary representation of $E_q(2)$. Then w is irreducible if and only if the associated pair (b,T) is irreducible.*

Proof: If the associated pair (b,T) is not irreducible, then clearly w cannot be irreducible. We now prove the converse.

Simple computations yield that for each $m \in \frac{1}{2}\mathbb{Z}$, there is an irreducible copy $(b^{(m)}, T^{(m)})$ acting on $\ell_2(\mathbb{Z})$ given by $b^{(m)} = q^m \ell^*$, $T^{(m)} = 2N$, if $m \in \mathbb{Z}$ and $b^{(m)} = q^m \ell^*$, $T^{(m)} = 2N+1$, if $m \in \mathbb{Z} + \frac{1}{2}$. It is easy to see that these are *all* the infinite dimensional irreducible copies of (b,T). Finite dimensional irreducible copies are all one dimensional, and they are $(0,m)$ where $m \in \mathbb{Z}$. Now $w(0,m) = \boldsymbol{v}^m \in C_b(E_q(2))$, which means it is a one dimensional representation and hence obviously irreducible. Denote by $w^{(m)}$ the representation corresponding to the pair $(b^{(m)}, T^{(m)})$, where $m \in \frac{1}{2}\mathbb{Z}$. We now show that each $w^{(m)}$ is irreducible. For this, let us first compute the quantity $\langle e_{rij}, w^{(m)} e_{skl}\rangle$. For $m \in \mathbb{Z}$ we have

$$\langle e_{rij}, w^{(m)} e_{skl}\rangle = \begin{cases} f_{j-l}^{m+1+k-s} & \text{if } i = k-r-s, j = l+r-s, \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

and for $m \in \mathbb{Z} + \frac{1}{2}$,

$$\langle e_{rij}, w^{(m)} e_{skl}\rangle = \begin{cases} f_{j-l}^{m+\frac{1}{2}+k-s} & \text{if } i = k-r-s-1,\ j = l+r-s, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Let P be a nonzero projection on $\ell_2(\mathbb{Z})$ such that $w^{(m)}(P \otimes I) = (P \otimes I) w^{(m)}$. Then for any continuous functional ρ on $C_0(E_q(2))$, $(id \otimes \rho) w^{(m)}$ commutes with P.

Take a nonzero vector $u = \sum u_s e_s \in P(\ell_2(\mathbb{Z}))$. Then $u_t \neq 0$ for some t. Take any $p \in \mathbb{Z}$.
Case I: $m \in \mathbb{Z}$. Let ρ be the functional $a \mapsto \langle e_{00}, a e_{t+p\, t-p}\rangle$. Then $(id \otimes \rho) w^{(m)}(u) = u_t f_{p-t}^{m+p+1} e_p$.
Case II: $m \in \mathbb{Z} + \frac{1}{2}$. Take ρ to be the functional $a \mapsto \langle e_{0\,-1}, a e_{t+p+1\, t-p-1}\rangle$. Then $(id \otimes \rho) w^{(m)}(u) = u_t f_{p-t}^{m+p+\frac{3}{2}} e_p$.
Therefore in both the cases, $e_p \in P\ell_2(\mathbb{Z})$. Therefore $P = I$. □

Let $w_{rs}^{(m)}$, $r, s \in \mathbb{Z}$, denote the matrix entries of $w^{(m)}$ with respect to the basis $\{e_i\}$, i.e. $w_{rs}^{(m)} = (\rho \otimes id) w^{(m)}$, where ρ is the functional $b \mapsto \langle e_r, b e_s\rangle$. Denote by f_{ij} the following function on $\mathbb{C}^q$: $f_{ij}(z) = \int_{S^1} F_q(q^i z u) u^{-j}\, du, \quad z \in \mathbb{C}^q$. It is easy to see, from (3.2) and (3.3), that

$$w_{rs}^{(m)} = \begin{cases} \boldsymbol{v}^{r+s} f_{m-s+1,r-s}(\boldsymbol{n}) & \text{if } m \in \mathbb{Z}, \\ \boldsymbol{v}^{r+s+1} f_{m-s+\frac{1}{2},r-s}(\boldsymbol{n}) & \text{if } m \in \mathbb{Z}+\frac{1}{2}. \end{cases} \tag{3.4}$$

Since $f_{ij} \in C_0(\mathbb{C}^q)$ for all i,j, we have $w_{rs}^{(m)} \in C_0(E_q(2))$ for all $r,s \in \mathbb{Z}$, and for all $m \in \frac{1}{2}\mathbb{Z}$. We shall shortly see that they belong to $L_2(h)$ also.

The following lemma will be very useful in the sequel.

Lemma 3.2 *$\{\boldsymbol{v}^r f_{st}(\boldsymbol{n}) \, : \, r,s,t \in \mathbb{Z}\}$ is a complete set of orthogonal vectors in $L_2(h)$.*

Proof: Using corollary 2.10, one can easily compute that

$$h((\boldsymbol{v}^r f_{st}(\boldsymbol{n}))^* \boldsymbol{v}^{r'} f_{s't'}(\boldsymbol{n})) = \delta_{rr'}\delta_{ss'}\delta_{tt'} q^{2(1-s+t)}. \tag{3.5}$$

Therefore all we need to prove is $\{\boldsymbol{v}^r f_{st}(\boldsymbol{n}) \, : r,s,t \in \mathbb{Z}\}^\perp = \{0\}$.

Take an operator $a \in L_2(h)$ such that $h((\boldsymbol{v}^i f_{jk}(\boldsymbol{n}))^* a) = 0$ for all $i,j,k \in \mathbb{Z}$. This implies

$$\sum_r q^{2r} f_k^{j+r} \langle e_{r-i\,k}, a e_{r0} \rangle = 0 \;\; \forall i,j,k. \tag{3.6}$$

Write $u(z) = \sum_r q^r \langle e_{r-i\,k}, a e_{r0} \rangle z^r$, $\xi_j^k(z) = \sum_r q^{r+j-k-1} f_k^{j+r} z^r$. Then $u, \xi_j^k \in L_2(S^1)$. From (3.6), $\langle \xi_j^k, u \rangle = 0$ for all j and k. Therefore if we can show that $\{\xi_j^k\}_{j\in\mathbb{Z}}$ is complete in $L_2(S^1)$ for each fixed k, it follows that $a e_{r0} = 0$ for all $r \in \mathbb{Z}$, so that $h(a^* a) = 0$, which means a is 0 in $L_2(h)$.

Fix any $k \in \mathbb{Z}$. From corollary 2.10, it follows that $\{\xi_j^k\}_{j\in\mathbb{Z}}$ is an orthonormal set of vectors. Observe that $z^s \xi_j^k(z) = \xi_{j-s}^k(z)$. Therefore $\langle \xi_0^k, \xi_j^k \rangle = \delta_{j0}$ implies $\xi_0^k(z) \neq 0$ almost everywhere. If P is the projection onto $\{\xi_j^k : j \in \mathbb{Z}\}^\perp$, then P commutes with all the multiplication operators, and hence is multiplication by an indicator. Since $P\xi_0^k = 0$ and $\xi_0^k \neq 0$ almost everywhere, P must be zero. $\square$

We are now in a position to state the following proposition.

Proposition 3.3 *The matrix entries $w_{rs}^{(m)}$ satisfy the following:*

i. $w_{rs}^{(m)} \in L_2(h) \quad \forall\, m,r,s.$

ii. $\langle w_{rs}^{(m)}, w_{r's'}^{(m')} \rangle = \delta_{mm'}\delta_{rr'}\delta_{ss'} q^{2(r-[m])}$.

iii. $\{q^{[m]-r} w_{rs}^{(m)} : r,s \in \mathbb{Z}, m \in \frac{1}{2}\mathbb{Z}\}$ *form an orthonormal basis for $L_2(h)$.*

Proof: Follows from (3.4), (3.5) and lemma 3.2. □

Remark 3.4 Though the matrix entries in the given basis are all in $L_2(h)$, this is not, in general, true; that is, there are vectors u, v such that $(\langle u| \otimes id)w^{(m)}(v \otimes \cdot) \notin L_2(h)$. One could, for example, take $u = \sum_{n\geq 1} \frac{1}{n} e_{-n}$ and $v = e_0$. Thus each $w^{(m)}$ has both square-integrable and non square-integrable matrix entries.

4. Uniqueness of *h*. In this section we deviate a little bit from the C^*-algebraic setup in which we have worked so far. To be more specific, we deal with the von Neumann algebra $\mathcal{M}$ generated by $C_0(E_q(2))$. This does not pose any serious problem, as the comultiplication map μ, being unitarily implemented (cf. equation (12) of [7]), extends readily to $\mathcal{M}$.

To start with, let $\mathcal{U}$ be the $*$-subalgebra of $C_0(E_q(2))$ generated by $\{\boldsymbol{v}^r f_{st}(\boldsymbol{n}) : r, s, t \in \mathbb{Z}\}$. It is easy to see that $\mathcal{U}$ is contained in $L_2(h)$ and it follows from lemma 3.2 that it is dense there.

Let P_r denote the projection $\sum_s |e_{rs}\rangle\langle e_{rs}|$. For an operator a on $\ell_2(\mathbb{Z}) \otimes \ell_2(\mathbb{Z})$, denote by a^{rs} the operator $P_r a P_s$. Then any bounded operator can be written as a strong sum of the form $\sum_r \sum_s a^{r+s,s}$. Observe that for $a \in \mathcal{U}$, the first summation is finite, i.e. $a = \sum_{r\in F} \sum_s a^{r+s,s}$, where F is some finite subset of $\mathbb{Z}$. Define an operator Δ_0 on $\mathcal{U}$ as follows:

$$\Delta_0 a = \sum_r q^{2r} (\sum_s a^{r+s,s}).$$

One can check that the closure Δ of this operator is a positive self-adjoint operator, and is in fact the modular operator associated with the weight h. That is, we have $h(ab) = h(b\Delta a)$ for $a, b \in \mathcal{U}$. The corresponding modular automorphism group Δ^{it} is seen to be given by

$$\Delta^{it} a = \epsilon_{q^{-it}} * a * \epsilon_{q^{-it}}, \tag{4.1}$$

where ϵ_z is the functional $\boldsymbol{v} \mapsto z$, $\boldsymbol{n} \mapsto 0$, $z \in S^1$. From proposition 1.4 of [6], fixed point subalgebra of this automorphism group is $\{f(\boldsymbol{n}) : f \text{ bounded measurable function on } \mathbb{C}^q\}$.

Suppose now that h_1 is a normal semifinite weight on $\mathcal{M}$ for which theorem 2.1 holds. Also assume that all compactly supported elements of $C_0(E_q(2))$ are in the domain of h_1. It is clear from (4.1) that $h_1\Delta^{it} = h_1$. Therefore by the Radon-Nikodym theorem for weights on a von

Neumann algebra (theorem 5.12 of [4]), there is a positive measurable function f on $\mathbb{C}^q$ such that

$$h_1(a) = h(f(\boldsymbol{n})a)$$

for all a in the domain of h_1. Using the invariance properties of h and h_1, faithfulness of h and the fact that $\epsilon_z * f(\boldsymbol{n}) = f(z\boldsymbol{n})$ for all $z \in S^1$, it is easy to see that $f(z\boldsymbol{n}) = f(\boldsymbol{n})$ for all $z \in S^1$. This means there is a positive measurable function c on $\{q^k : k \in \mathbb{Z}\}$ such that h_1 is given by

$$h_1(a) = \sum_{r\in\mathbb{Z}} q^{2r} c(q^r) \langle e_{r0}, a e_{r0} \rangle.$$

Notice that each $c(q^r)$ has to be strictly positive. Because, if $c(q^\nu) = 0$ for some ν, then from the relation $h_1(\rho_0 * a_\nu) = h_1(a_\nu)$, we get $\sum_r q^{2r} c(q^r) |f_\nu^{r+1}|^2 = 0$, which forces each $c(q^r)$ to be zero.

It follows from lemma 3.2 and the above observation that $\{\boldsymbol{v}^r f_{st}(\boldsymbol{n}) \ : r, s, t \in \mathbb{Z}\}$ form an orthogonal basis for $L_2(h_1)$ also. Simple computations give

$$\begin{aligned} h_1((\boldsymbol{v}^r f_{st}(\boldsymbol{n}))^* \boldsymbol{v}^{r'} f_{s't'}(\boldsymbol{n})) &= \delta_{rr'} \delta_{ss'} \delta_{tt'} c(q^{1-s+t}) q^{2(1-s+t)}, \\ h_1(\boldsymbol{v}^{r'} f_{s't'}(\boldsymbol{n}) (\boldsymbol{v}^r f_{st}(\boldsymbol{n}))^*) &= \delta_{rr'} \delta_{ss'} \delta_{tt'} c(q^{1-r-s+t}) q^{2(1-r-s+t)}. \end{aligned}$$

Therefore, denoting by $\Delta_{(1)}$ the modular operator for the weight h_1, we get $\Delta_{(1)}(\boldsymbol{v}^r f_{st}(\boldsymbol{n})) = \frac{c(q^{1-r-s+t})}{c(q^{1-s+t})} q^{-2r} \boldsymbol{v}^r f_{st}(\boldsymbol{n})$. On the other hand, it can easily be verified that $\Delta_{(1)} g(\boldsymbol{n}) = g(\boldsymbol{n})$ for any compactly supported function g on $\mathbb{C}^q$. Therefore $\frac{c(q^{1-r-s+t})}{c(q^{1-s+t})}$ must be independent of s and t, which means there is a positive real d such that $c(q^r) = c(1) d^r$. Now it remains only to show that $d = 1$.

If we use the weight h_1 instead of h in the proof of corollary 2.5, we get the following identity:

$$\sum_r q^{2r} d^r f_{-r-s}^{k-r} f_{-r}^{k-r+s} = \delta_{s0} \ \ \forall\, k \in \mathbb{Z}. \tag{4.2}$$

Suppose now that $d < q^{-1}$. Let $\xi_k(z) = \sum_r q^{r-k} f_{k-r}^{k-r} z^r$. One can see that $z^s \xi_k(z) = \xi_{k+s}(z)$ and $\langle \xi_k, \xi_{k'} \rangle = \delta_{kk'}$. Therefore $\{\xi_k\}_{k\in\mathbb{Z}}$ form a complete orthonormal basis for $L_2(S^1)$. Write $u(z) = \sum_r q^r d^r f_{-r}^{-r} z^r$. From (4.2), $\langle u, \xi_k \rangle = \delta_{k0}$. Hence $u \in \mathbb{C}.\xi_0$, which implies that d is 1. If $d \geq q^{-1}$, then taking $\xi_k(z) = \sum_r q^{r-k} \sqrt{d^{r-k}} f_{k-r}^{k-r} z^r$ and $u(z) = \sum_r q^r \sqrt{d^{-r}} f_{-r}^{-r} z^r$ and using the same arguments as in the earlier case, we get $u \in \mathbb{C}.\xi_0$, which is impossible since $d > 1$. Thus, up to a scalar multiple, an invariant weight is unique.

Remark. After the first version of this paper was submitted, the author came to know of the paper ([1]) by Baaj, in which the invariance of the haar measure for the quantum $E(2)$ group has been proved. His method of proof, however, is different. He essentially uses the cross product structure of the C^*-algebra $C_0(E_q(2))$, and makes use of some q-identities, whereas in our case, the close relatioship between $E_q(2)$ and $SU_q(2)$ has been exploited, and the q-identities are derived as by-products.

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References

1. Baaj, S. : Representation Reguliere du groupe quantique des deplacements de Woronowicz, *Asterisque*, 232(1995), 11–49.
2. Celeghini, E., Giachetti, R., Sorace, E. & Tarlini, M. : Three-dimensional Quantum Groups from Contractions of $SU(2)_q$, *J. Math. Phys.*, 31(11), 1990, 2548–2551.
3. Pedersen, G.K. : C^*-algebras and Their Automorphism Groups, *Academic Press*, 1979.
4. Pedersen, G.K. & Takesaki, M. : The Radon-Nikodym Theorem for Von Neumann Algebras, *Acta Mathematica*, 130(1973), 53–87.
5. Woronowicz, S.L. : Unbounded Elements Affiliated With C^*-algebras and Noncompact Quantum Groups, *Comm. Math. Phys.*, 136(1991), 399–432.
6. ——— : Quantum $E(2)$ Group and its Pontryagin Dual, *Lett. Math. Phys.*, 23(1991), 251-263.
7. ——— : Quantum $SU(2)$ and $E(2)$ Groups – Contraction Procedure, *Comm. Math. Phys.*, 149(1992), 637–652.