A Counterexample on Idempotent States on a Compact Quantum Group

By

ARUPKUMAR PAL

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi 110 016
e-mail: arup@isid.ernet.in

Abstract. A simple example is given to illustrate that an idempotent state may not be the haar state of any subgroup in the case of compact quantum groups.

Suppose G is a locally compact group and ν is a probability measure on G. If ν is idempotent, i.e. if it satisfies the equation $\nu * \nu = \nu$, then it is a well-known fact that the support of ν is a compact subgroup H of G and ν is the haar measure of H. We present here a simple counterexample to show that the same thing cannot be said for measures on compact quantum groups.

1. Let $\mathcal{A}$ be the C^*-algebra $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$. Let

$$e_k = \delta_{1k} \oplus \delta_{2k} \oplus \delta_{3k} \oplus \delta_{4k} \oplus \begin{pmatrix} \delta_{5k} & \delta_{8k} \\ \delta_{7k} & \delta_{6k} \end{pmatrix}, \quad k = 1, 2, \ldots, 8,$$

where δ denotes the Kronecker delta. Then $\{e_1, \ldots, e_8\}$ form a basis for $\mathcal{A}$. Define a map $\mu : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ as follows:

$$\begin{aligned}
\mu(e_1) &= e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 + e_4 \otimes e_4 + \frac{1}{2}(e_5 \otimes e_5 + e_6 \otimes e_6 + e_7 \otimes e_7 + e_8 \otimes e_8), \\
\mu(e_2) &= e_1 \otimes e_2 + e_2 \otimes e_1 + e_3 \otimes e_4 + e_4 \otimes e_3 + \frac{1}{2}(e_5 \otimes e_6 + e_6 \otimes e_5 + ie_7 \otimes e_8 - ie_8 \otimes e_7), \\
\mu(e_3) &= e_1 \otimes e_3 + e_3 \otimes e_1 + e_2 \otimes e_4 + e_4 \otimes e_2 + \frac{1}{2}(e_5 \otimes e_6 + e_6 \otimes e_5 - ie_7 \otimes e_8 + ie_8 \otimes e_7), \\
\mu(e_4) &= e_1 \otimes e_4 + e_4 \otimes e_1 + e_2 \otimes e_3 + e_3 \otimes e_2 + \frac{1}{2}(e_5 \otimes e_5 + e_6 \otimes e_6 - e_7 \otimes e_7 - e_8 \otimes e_8), \\
\mu(e_5) &= e_1 \otimes e_5 + e_5 \otimes e_1 + e_2 \otimes e_6 + e_6 \otimes e_2 + e_3 \otimes e_6 + e_6 \otimes e_3 + e_4 \otimes e_5 + e_5 \otimes e_4, \\
\mu(e_6) &= e_1 \otimes e_6 + e_6 \otimes e_1 + e_2 \otimes e_5 + e_5 \otimes e_2 + e_3 \otimes e_5 + e_5 \otimes e_3 + e_4 \otimes e_6 + e_6 \otimes e_4, \\
\mu(e_7) &= e_1 \otimes e_7 + e_7 \otimes e_1 - ie_2 \otimes e_8 + ie_8 \otimes e_2 + ie_3 \otimes e_8 - ie_8 \otimes e_3 - e_4 \otimes e_7 - e_7 \otimes e_4, \\
\mu(e_8) &= e_1 \otimes e_8 + e_8 \otimes e_1 + ie_2 \otimes e_7 - ie_7 \otimes e_2 - ie_3 \otimes e_7 + ie_7 \otimes e_3 - e_4 \otimes e_8 - e_8 \otimes e_4.
\end{aligned}$$

It is a matter of straightforward verification that μ is a unital $*$-homomorphism and $G = (\mathcal{A}, \mu)$ is a compact quantum group.

2. Let ρ_k be the functional $\sum \alpha_i e_i \mapsto \alpha_k$. Then $\rho_1, \rho_2, \ldots, \rho_6$, along with $\psi_1 = \frac{1}{2}(\rho_5 + \rho_6 + \rho_7 + \rho_8)$ and $\psi_2 = \frac{1}{2}(\rho_5 + \rho_6 + i\rho_7 - i\rho_8)$ are all states, and they span the space of all functionals on $\mathcal{A}$. Therefore any state ρ will be of the form

$$\begin{aligned}
\rho &= \sum_{i=1}^{6} c_i \rho_i + c_7 \psi_1 + c_8 \psi_2 \\
&= \sum_{i=1}^{4} c_i \rho_i + (c_5 + \frac{1}{2}(c_7 + c_8))\rho_5 + (c_6 + \frac{1}{2}(c_7 + c_8))\rho_6 + \frac{1}{2}(c_7 + ic_8)\rho_7 + \frac{1}{2}(c_7 - ic_8)\rho_8,
\end{aligned}$$

where $c_i \geq 0$ for all i, and $\sum_{i=1}^{8} c_i = 1$. Evaluating the two functionals $\rho * \rho$ and ρ at the basis elements and equating them, we get a system of equations which lead to the following possibilities:

a) $\rho = \rho_1$,
b) $\rho = \frac{1}{2}(\rho_1 + \rho_2)$,
c) $\rho = \frac{1}{2}(\rho_1 + \rho_3)$,
d) $\rho = \frac{1}{2}(\rho_1 + \rho_4)$,
e) $\rho = \frac{1}{4}(\rho_1 + \rho_2 + \rho_3 + \rho_4)$,
f) $\rho = \frac{1}{8}(\rho_1 + \rho_2 + \rho_3 + \rho_4) + \frac{1}{4}(\rho_5 + \rho_6)$,
g) $\rho = \frac{1}{4}(\rho_1 + \rho_4) + \frac{1}{2}\rho_5$,
h) $\rho = \frac{1}{4}(\rho_1 + \rho_4) + \frac{1}{2}\rho_6$.

These, then, are all the idempotent states on $\mathcal{A}$.

3. We shall now show that the state $\rho = \frac{1}{4}(\rho_1 + \rho_4) + \frac{1}{2}\rho_6$ is not the haar state of any subgroup of G.

Suppose, if possible, $H = (C(H), \mu_H)$ is a subgroup of G and ρ is the haar state of H. This means that there is a unital $*$-homomorphism ϕ from $\mathcal{A}$ *onto* $C(H)$ obeying $(\phi \otimes \phi)\mu = \mu_H \phi$, and $\rho = h\phi$, where h is the haar state of $C(H)$. Let $\mathcal{I} = \{a \in \mathcal{A} : \rho(a^* a) = 0\}$. Using the modular properties of the haar state (see theorem 5.6 of [1]), we find that $\mathcal{I}$ is a closed two-sided ideal. Now, observe that $e_7 \in \mathcal{I}$, but $e_7^* \notin \mathcal{I}$, which contradicts the fact that $\mathcal{I}$ is an ideal.

References

1. Woronowicz, S.L. : Compact Matrix Pseudogroups, *Comm. Math. Phys.*, 111(1987), 613–665.
2. Woronowicz, S.L. : Compact Quantum Groups, *Preprint*, 1992.