q-Analogues of Graf's Identities from the Regular Representation of $E_q(2)$

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One of the nicest features of quantum groups is its connections with various other apparently diverse areas of mathematics. The theory of special functions is one such. They are related to quantum groups exactly as their classical counterparts are to the classical Lie groups. q-special functions appear as matrix entries of representations of quantum groups and seem to play an important role in controlling the symmetry of noncommutative spaces. This fact has been exploited by several people to give new interpretations to q-functions, to prove new identities involving them, and to give new and, in a sense more natural, proofs of identities already proven (see, for example, [2], [3], [5], [6], [8]). In this article, we will prove a few identities that can be called q-analogues of Graf's identities involving classical Bessel functions. Some of these identities come in handy while doing computations involving other quantum groups, like the double group built over the quantum $E(2)$ group.

We start with a very brief description of the quantum $E(2)$ group in the first section. In the next section, we introduce the q-Bessel function, which is, up to a slight change in scale, essentially the q-analogue of Bessel function introduced by Exton ([1]). In section 3, some calculations involving the comultiplication are presented that we need subsequently. The (right) regular representation is introduced and its relation with the right convolution operator is given in section 4. Using this and the computations in the third section, we prove a whole class of identities in the last section.

We retain the notations used in [7].

1 Preliminaries

Let us first describe the Hopf-algebra of coordinate functions on $E_q(2)$. Let A be the unital $*$-algebra generated by two elements $\mathbf{v}$ and $\mathbf{n}$ satisfying the following relations:

$$\mathbf{v}^*\mathbf{v} = \mathbf{v}\mathbf{v}^* = I, \quad \mathbf{n}^*\mathbf{n} = \mathbf{n}\mathbf{n}^*, \quad \mathbf{v}\mathbf{n}\mathbf{v}^* = q\mathbf{n}. \tag{1.1}$$

The comultiplication map μ, the counit ϵ and the antipode κ are given on the generating elements as follows:

$$\mu(\mathbf{v}) = \mathbf{v} \otimes \mathbf{v}, \quad \mu(\mathbf{n}) = \mathbf{v} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{v}^*.$$

$$\epsilon(\mathbf{v}) = 1, \quad \epsilon(\mathbf{n}) = 0,$$
$$\kappa(\mathbf{v}) = \mathbf{v}^*, \quad \kappa(\mathbf{v}^*) = \mathbf{v}, \quad \kappa(\mathbf{n}) = -q^{-1}\mathbf{n}, \quad \kappa(\mathbf{n}^*) = -q\mathbf{n}^*.$$

To describe the group at the C^*-algebra level, one also assumes that

$$\sigma(\mathbf{n}) \subseteq \mathbb{C}^q := \{q^k z : z \in S^1, k \in \mathbb{Z}\} \cup \{0\}. \tag{1.2}$$

It is easy to see that the following list gives all the irreducible representations of the pair $(\mathbf{v}, \mathbf{n})$ once we assume (1.2):

$$\left.\begin{array}{lll} \pi_z & : & \left\{\begin{array}{l} v \mapsto \ell \\ n \mapsto zq^N \end{array}\right. \quad \text{on } L_2(\mathbb{Z}), \\ \epsilon_z & : & \left\{\begin{array}{l} v \mapsto z \\ n \mapsto 0 \end{array}\right. \quad \text{on } \mathbb{C}, \end{array}\right\} \quad z \in S^1. \tag{1.3}$$

Take $\mathbf{v} = \ell \otimes I$, $\mathbf{n} = q^N \otimes \ell^*$ on $L_2(\mathbb{Z}) \otimes L_2(\mathbb{Z})$, where $\ell : e_k \mapsto e_{k-1}$ and $N : e_k \mapsto ke_k$ are operators on $L_2(\mathbb{Z})$. The C^*-algebra $C_0(E_q(2))$ of 'continuous functions vanishing at infinity' on $E_q(2)$ is the norm closure of all finite sums of the form $\sum_k \mathbf{v}^k f_k(\mathbf{n})$, where $f_k \in C_0(\mathbb{C}^q)$. It is easy to check that $\mathbf{v}$ and $\mathbf{n}$ are affiliated to $C_0(E_q(2))$, and moreover, it has the following 'universality property'.

Theorem 1.1 ([10]) *If π is a representation of $C_0(E_q(2))$ on some Hilbert space $\mathcal{K}$, then $\pi(\mathbf{v})$ and $\pi(\mathbf{n})$ satisfy the conditions (1.1) and (1.2), with $\pi(\mathbf{v})$ replacing v and $\pi(\mathbf{n})$ replacing n. Conversely, if $\overline{\mathbf{v}}$ and $\overline{\mathbf{n}}$ are two closed operators on a Hilbert space $\mathcal{K}$ and satisfy (1.1) and (1.2), then there is a unique representation π of $C_0(E_q(2))$ such that $\pi(\mathbf{v}) = \overline{\mathbf{v}}$ and $\pi(\mathbf{n}) = \overline{\mathbf{n}}$.*

Moreover, in the above situation, if $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{K})$, then $\overline{\mathbf{v}}$ and $\overline{\mathbf{n}}$ are affiliated to $\mathcal{A}$ if and only if $\pi \in mor(C_0(E_q(2)), \mathcal{A})$.

2 q-Bessel Functions

Let F_q be the following function on $\mathbb{C}^q$ introduced by Woronowicz in [10]:

$$F_q(z) = \begin{cases} \prod_{r=0}^{\infty} \frac{1+q^{2r}\bar{z}}{1+q^{2r}z} & \text{if } z \in \mathbb{C}^q - \{-1, -q^{-2}, -q^{-4}, \ldots\}, \\ -1 & \text{if } z \in \{-1, -q^{-2}, -q^{-4}, \ldots\}. \end{cases} \tag{2.1}$$

This defines a bounded continuous function on $\mathbb{C}^q$. For a positive real t and for $q \neq 0$, let us denote by $(t)_q$ the number $(1-q^t)/(1-q)$. Let n be a nonnegative integer. Define the q-factorial $(n)_q!$ by:

$$(n)_q! = \begin{cases} \prod_{k=1}^{n}(k)_q & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

One can now define the q-exponential function as follows:

$$\exp_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k)_q!}.$$

This function can be shown to have the following infinite product expansion for $q > 1$:

$$\exp_q(x) = \prod_{k=1}^{\infty} \Big(1 - q^{-k}(1-q)x\Big),$$

from which it follows that

$$F_q(z) = \frac{\exp_{q^{-2}}(\frac{\bar{z}}{1-q^2})}{\exp_{q^{-2}}(\frac{z}{1-q^2})}. \tag{2.2}$$

Let us next define a family of functions $J_q(\cdot,\cdot)$ on $\mathbb{C}^q \times \mathbb{Z}$ as follows:

$$J_q(z,k) = \int_{S^1} F_q(zu)u^{-k}\,du, \quad z \in \mathbb{C}^q, k \in \mathbb{Z}. \tag{2.3}$$

We call these q-analogs of Bessel functions. From equation (2.2), we find that for real values of z, and for $u \in S^1$,

$$F_q(zu) = \frac{\exp_{q^{-2}}\Big(\frac{z}{1-q}(\frac{1}{2})_{q^2}u^{-1}\Big)}{\exp_{q^{-2}}\Big(\frac{z}{1-q}(\frac{1}{2})_{q^2}u\Big)},$$

which is an analog of the function $\exp(\frac{1}{2}z(u^{-1}-u))$. Recall that the classical Bessel function $J(z,k)$ is the coefficient of t^k in the expansion $\exp(\frac{1}{2}z(t-t^{-1}))$.

Let us describe here another similarity with the classical Bessel functions. Let Δ_q denote the q-differential operator given by

$$\Delta_q f(x) = \frac{f(x)-f(qx)}{(1-q)x}.$$

Define a function $B_q(x,n)$ as follows:

$$B_q(x,n) = J_q(q^{n/2}(1-q)x, n), \quad |x| < q^{-n/2}(1-q)^{-1}.$$

One can see that this function $B_q(x,n)$ obeys the following 'q-differential equation'

$$qx^2\Delta_q^2 f(x) + x\Delta_q f(x) + \left(x^2 - (n)_q^2 q^{-n}\right) f(qx) = 0,$$

which is a q-analog of the classical Bessel differential equation.

Remark. There are several q-analogs of Bessel functions in the literature, the earliest one dating back to Jackson. The q-Bessel functions defined here are very closely related to the ones studied by Exton ($B_q(x,n)$ is, upto a constant factor, equal to Exton's q-Bessel function J(q;n,x); see p. 181, [1]), and seem to be the most natural. We have already cited two 'reasons' above. See chapter 5, [8] for another quantum group-theoretic reason.

Let us list some properties of these functions.

Proposition 2.1 *The functions $J_q(\cdot,\cdot)$ obey the following identities:*
1. $\overline{J_q(z,k)} = J_q(\overline{z},k)$. In particular, $J_q(z,k)$ is real whenever z is real.
2. $J_q(z,k) = (z/|z|)^k J_q(|z|,k)$. More generally, $J_q(z,k) = w^k J_q(z\overline{w},k)$ for any $w \in S^1$.
3. $J_q(-z,k) = (-1)^k J_q(z,k)$.
4. $\sum_{k\in\mathbb{Z}} \overline{J_q(z,k)} J_q(z,k+j) = \delta_{j0}$.
5. $J_q(q^{-n},k) = J_q(q^{n+2}, n+k+1)$.

Proof: Proofs of 1, 2 and 3 are immediate. To prove 4, observe that for $u \in S^1$, $z \in \mathbb{C}^q$, $F_q(zu) = \sum_k J_q(z,k)u^k$, and $u^{-j}F_q(zu) = \sum_k J_q(z,k+j)u^k$. Also observe that both, as functions of u, are in $L_2(S^1)$; and $|F_q(zu)| = 1$. Now compute their inner product in $L_2(S^1)$. To prove 5, use (2.3) and the equality: $F_q(q^{-n}z) = z^{-n-1}F_q(q^{n+2}z)$ for all $z \in S^1$. □

3 The Comultiplication Map

Let $\{e_i\}$ be the canonical orthonormal basis for $L_2(\mathbb{Z})$. Denote $e_i \otimes e_j$ by e_{ij}, $e_i \otimes e_j \otimes e_k$ by e_{ijk} and so on.

Lemma 3.1 *Let V be the unitary operator on $L_2(\mathbb{Z})^{\otimes 4}$ given on the basis elements by $e_{i,j,k,l} \mapsto e_{i,j,i+j+k,l}$. Let $W = F_q(\mathbf{n}^{-1}\mathbf{v} \otimes \mathbf{vn})V$. Then $\mu(a) = W(a \otimes I)W^*$ for all $a \in C_0(E_q(2))$.*

Proof: For $a \in C_0(E_q(2))$, write $\nu(a) = W(a \otimes I)W^*$. Then both μ and ν are representations of $C_0(E_q(2))$ acting on the same space $L_2(\mathbb{Z})^{\otimes 4}$. Using theorem 2.1 of [11], it is easy to see that $\mu(\mathbf{v}) = \nu(\mathbf{v})$ and $\mu(\mathbf{n}) = \nu(\mathbf{n})$. Hence by theorem 1.1, $\mu = \nu$. □

Define an operator U on $L_2(\mathbb{Z})^{\otimes 4}$ by $Ue_{i,j,k,l} = e_{k-i,j,k,l}$. It is easy to see that U is unitary, and $\mathbf{n}^{-1}\mathbf{v} \otimes \mathbf{vn} = U^*(q^{N+1} \otimes \ell \otimes \ell \otimes \ell^*)U$. Combining this observation with lemma 3.1, we find that for any $a \in C_0(E_q(2))$,

$$\mu(a) = U^* F_q(q^{N+1} \otimes \ell \otimes \ell \otimes \ell^*)UV(a \otimes I)V^*U^* F_q(q^{N+1} \otimes \ell^* \otimes \ell^* \otimes \ell)U.$$

Hence

$$\begin{aligned}&\langle e_{ijkl}, \mu(a)e_{rstu}\rangle = \\ &\langle V^*U^*F_q(q^{N+1} \otimes \ell^* \otimes \ell^* \otimes \ell)Ue_{i,j,k,l}, (a \otimes I)V^*U^*F_q(q^{N+1} \otimes \ell^* \otimes \ell^* \otimes \ell)Ue_{r,s,t,u}\rangle.\end{aligned}$$

Now $F_q(q^{N+1} \otimes \ell^* \otimes \ell^* \otimes \ell)Ue_{i,j,k,l}$ can very easily be shown to have the following expression:

$$\begin{aligned}&F_q(q^{N+1} \otimes \ell^* \otimes \ell^* \otimes \ell)Ue_{i,j,k,l} \\ &\quad = \sum_n J_q(q^{k-i+1}, n-j)e_{k-i,n,k-j+n,l+j-n}.\end{aligned}$$

Therefore

$$\begin{aligned}&V^*U^*F_q(q^{N+1} \otimes \ell^* \otimes \ell^* \otimes \ell)Ue_{i,j,k,l} \\ &\quad = \sum_n J_q(q^{k-i+1}, n-j)V^*e_{i-j+n,n,k-j-n,l+j-n} \\ &\quad = \sum_n J_q(q^{k-i+1}, n-j)e_{i-j+n,n,k-i-n,l+j-n}.\end{aligned}$$

From the above equation, we now get

$$\begin{aligned}&\langle e_{ijkl}, \mu(a)e_{rstu}\rangle \\ &\quad = \begin{cases} \sum_m J_q(q^{k-i+1}, m)J_q(q^{t-r+1}, m+u-l)\langle e_{i+m,j+m}, ae_{r+u-l+m,s+u-l+m}\rangle \\ \hfill \text{if } t-r-s-u = k-i-j-l, \\ 0 \hfill \text{otherwise.}\end{cases}\end{aligned} \tag{3.1}$$

4 The Regular Representation

For a closed operator T, let V_T denote the partial isometry appearing in the polar decomposition of T. Let (b,T) be a pair of closed operators acting on some Hilbert space $\mathcal{H}$ such that the following conditions hold:

$$\left.\begin{array}{rl} \text{i.} & T \text{ is self-adjoint}, \\ \text{ii.} & b \text{ is normal}, \\ \text{iii.} & T \text{ and } |b| \text{ commute strongly}, \\ \text{iv.} & V_b{}^* T V_b = T + 2I \text{ on } (\ker b)^\perp, \\ \text{v.} & \sigma(T,|b|) \subseteq \overline{\Sigma_q}, \text{ where } \Sigma_q = \{(r, q^{s+r/2}) : r,s \in \mathbb{Z}\}, \\ & \sigma(T,|b|) \text{ being the joint spectrum of } T \text{ and } |b|. \end{array}\right\} \tag{4.1}$$

It has been proved in [10] that if (b,T) is such a pair, then $F_q(q^{T/2}b \otimes \mathbf{vn})(I \otimes \mathbf{v})^{T\otimes I}$ is a unitary representation of $E_q(2)$ acting on $\mathcal{H}$, and conversely, given any unitary representation w of $E_q(2)$ acting on a Hilbert space $\mathcal{H}$, there is a pair (b,T) of operators on $\mathcal{H}$ satisfying the requirements above such that $w = F_q(q^{T/2}b \otimes \mathbf{vn})(I \otimes \mathbf{v})^{T\otimes I}$.

We call a pair (b,T) satisfying (4.1) irreducible if the Hilbert space $\mathcal{H}$ on which they act does not have any nonzero proper closed subspace that is kept invariant by b, b^*,and T. Now, thanks to the following proposition, finding irreducible representations boils down to finding irreducible copies of the pair (b,T).

Proposition 4.1 ([7]) *Let w be a unitary representation of $E_q(2)$. Then w is irreducible if and only if the associated pair (b,T) is irreducible.*

Using this, it has been shown in [7] that the infinite dimensional irreducible unitaries are indexed by $\frac{1}{2}\mathbb{Z}$, and the matrix entries are given by

$$w_{rs}^{(m)} = \begin{cases} \mathbf{v}^{r+s} J_q(q^{m-s+1}\mathbf{n}, r-s) & \text{if } m \in \mathbb{Z}, \\ \mathbf{v}^{r+s+1} J_q(q^{m-s+\frac{1}{2}}\mathbf{n}, r-s) & \text{if } m \in \mathbb{Z}+\frac{1}{2}. \end{cases} \tag{4.2}$$

The following proposition gives the orthogonality relations between the matrix entries.

Proposition 4.2 ([7]) *The matrix entries $w_{rs}^{(m)}$ satisfy the following:*

i. $w_{rs}^{(m)} \in L_2(h) \quad \forall\, r,s \in \mathbb{Z}, \forall m \in \frac{1}{2}\mathbb{Z}$.

ii. (**orthogonality relations**) $\langle w_{rs}^{(m)}, w_{r's'}^{(m')}\rangle = \delta_{mm'}\delta_{rr'}\delta_{ss'}q^{2(r-[m])}$.

iii. $\{q^{[m]-r}w_{rs}^{(m)} : r,s \in \mathbb{Z}, m \in \frac{1}{2}\mathbb{Z}\}$ *form an orthonormal basis for $L_2(h)$.*

Denote $q^{[m]-r}w_{rs}^{(m)}$ by $\xi_{rs}^{(m)}$. Define two operators $\tilde{b}$ and $\tilde{T}$ on $L_2(h)$ as follows:

$$\left.\begin{array}{rcl} \tilde{b}\xi_{rs}^{(m)} & = & q^m \xi_{r,s+1}^{(m)}, \\ \tilde{T}\xi_{rs}^{(m)} & = & \begin{cases} 2s\xi_{rs}^{(m)} & \text{if } m \in \mathbb{Z}, \\ (2s+1)\xi_{rs}^{(m)} & \text{if } m \in \mathbb{Z} + \frac{1}{2}. \end{cases} \end{array}\right\} \tag{4.3}$$

$\tilde{b}$ and $\tilde{T}$ are then closed operators on $L_2(h)$ and they satisfy (4.1). Therefore $w(\tilde{b}, \tilde{T}) := F_q(q^{\tilde{T}/2}\tilde{b} \otimes \mathbf{vn})(I \otimes \mathbf{v})^{\tilde{T} \otimes I}$ is a unitary representation of $E_q(2)$ acting on $L_2(h)$. We shall denote this representation by $\Re$. Notice that the restriction of $\Re$ to the closed span of $\{\xi_{rs}^{(m)} : s \in \mathbb{Z}\}$ is equivalent to $w^{(m)}$.

Lemma 4.3 *Let ρ be a bounded linear functional on $C_0(E_q(2))$. Then the map $a \mapsto (id \otimes \rho)\mu(a)$ defined on $C_0(E_q(2)) \cap L_2(h)$ extends uniquely to a bounded operator from $L_2(h)$ into itself.*

Proof: Let us first prove the following inequality:

$$\rho_1(((id \otimes \rho)c)^*(id \otimes \rho)c) \leq (\rho_1 \otimes \rho_2)(c^*c), \quad c \in C_0(E_q(2)) \otimes C_0(E_q(2)). \tag{4.4}$$

Take $c = \sum a_i \otimes b_i \in C_0(E_q(2)) \otimes_{alg} C_0(E_q(2))$. The matrix $(\!(\rho_1({a_i}^* a_j))\!)$ is positive. Hence for any real t, $\sum(b_i - t\rho_2(b_i)I)^* \rho_1({a_i}^* a_j)(b_j - t\rho_2(b_j)I) \geq 0$. Applying ρ_2, we get

$$\sum \rho_1({a_i}^* a_j)\rho_2({b_i}^* b_j) + t^2 \sum \overline{\rho_2(b_i)}\rho_2(b_j)\rho_1({a_i}^* a_j) - 2t \sum \overline{\rho_2(b_i)}\rho_2(b_j)\rho_1({a_i}^* a_j) \geq 0$$

for all real t. Therefore $\sum \overline{\rho_2(b_i)}\rho_2(b_j)\rho_1({a_i}^* a_j) \leq \sum \rho_1({a_i}^* a_j)\rho_2({b_i}^* b_j)$ which means (4.4) holds for $c \in C_0(E_q(2)) \otimes_{alg} C_0(E_q(2))$. By continuity, the same thing holds for all $c \in C_0(E_q(2)) \otimes C_0(E_q(2))$.

Putting $c = \mu(a)$ in (4.4), we get the following:

$$\rho_1((a * \rho_2)^*(a * \rho_2)) \leq \rho_1 * \rho_2(a^*a) \quad \forall a \in C_0(E_q(2)).$$

Now take $\rho_1 = hp_r$, $\rho_2 = \rho$, where p_r is as in [7]. This gives

$$hp_r((a * \rho)^*(a * \rho)) \leq hp_r * \rho(a^*a) = hp_r(a * \rho).$$

Taking limit as r goes to infinity and using the invariance of the haar weight, we get $\|a * \rho\|_2^2 \leq \|a\|_2^2$, which proves the lemma. □

Let us denote the operator in the forgoing lemma by R_ρ. Observe that $\Re$ is a unitary element of $M(\mathcal{B}_0(L_2(h)) \otimes C_0(E_q(2)))$. Any continuous functional on $C_0(E_q(2))$ extends uniquely to a strictly continuous functional on $M(C_0(E_q(2)))$. Denoting by ρ such a linear functional, proposition 8.3 of [4] tells us that the expression $\Re_\rho := (id\otimes\rho)\Re$ makes sense, and is a bounded operator on $L_2(h)$. We will now prove that this $\Re_\rho$ is in fact the right convolution operator R_ρ described in the previous lemma.

It follows from 1.3 that any continuous functional ρ on $C_0(E_q(2))$ is of the form

$$\rho(a) = \langle u_1, \pi_{U_0}(a)u_2\rangle + \langle v_1, \epsilon_{V_0}(a)v_2\rangle, \tag{4.5}$$

where U_0 and V_0 are two unitary operators acting on the spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, and $u_1, u_2 \in L_2(\mathbb{Z})\otimes\mathcal{H}$, $v_1, v_2 \in \mathcal{K}$. Let us first show that if $\rho = \langle v_1, \epsilon_{V_0}(\cdot)v_2\rangle$, then $(id\otimes\rho)\Re$ is same as the operator R_ρ. In this case, $(id\otimes\epsilon_{V_0})\Re = (I\otimes V_0)^{\tilde{T}\otimes I}$. Therefore

$$\begin{aligned} (id\otimes\rho)\Re\xi_{rs}^{(m)} &= (I\otimes\langle v_1|)((id\otimes\epsilon_{V_0})\Re)(\xi_{rs}^{(m)}\otimes v_2)\\ &= \begin{cases} \langle v_1, V_0^{2s}v_2\rangle\xi_{rs}^{(m)} & \text{if } m\in\mathbb{Z},\\ \langle v_1, V_0^{2s+1}v_2\rangle\xi_{rs}^{(m)} & \text{if } m\in\mathbb{Z}+\frac{1}{2}. \end{cases} \end{aligned}$$

On the other hand, since $(id\otimes\epsilon_{V_0})\mu(\mathbf{v}) = \mathbf{v}\otimes V_0$, and $(id\otimes\epsilon_{V_0})\mu(\mathbf{n}) = \mathbf{n}\otimes V_0^*$, we have, for $m\in\mathbb{Z}$,

$$\begin{aligned} R_\rho(\xi_{rs}^{(m)}) &= q^{m-r}(id\otimes\rho)\mu(w_{rs}^{(m)})\\ &= q^{m-r}(I\otimes\langle v_1|)\Big((id\otimes\epsilon_{V_0})\mu\left(\mathbf{v}^{r+s}J_q(q^{m-s+1}\mathbf{n}, r-s)\right)\Big)(\cdot\otimes|v_2\rangle)\\ &= q^{m-r}(I\otimes\langle v_1|)(\mathbf{v}\otimes V_0)^{r+s}J_q\left(q^{m-s+1}(\mathbf{n}\otimes V_0^*), r-s\right)(\cdot\otimes|v_2\rangle)\\ &= q^{m-r}\mathbf{v}^{r+s}J_q(q^{m-s+1}\mathbf{n}, r-s)\langle v_1, V_0^{2s}v_2\rangle\\ &= \langle v_1, V_0^{2s}v_2\rangle\xi_{rs}^{(m)}. \end{aligned}$$

Similarly, for $m\in\mathbb{Z}+\frac{1}{2}$, $R_\rho(\xi_{rs}^{(m)}) = \langle v_1, V_0^{2s+1}v_2\rangle\xi_{rs}^{(m)}$. Thus $\Re_\rho = R_\rho$ in this case.

Let $\{f_i\}$ be an orthonormal basis for the space $\mathcal{H}$ on which U_0 acts. Denote, as usual, $e_i\otimes f_j$ by e_{ij} on $L_2(\mathbb{Z})\otimes\mathcal{H}$. Take ρ to be the functional

$$\rho(a) = \langle e_{i'j'}, \pi_{U_0}(a)e_{ij}\rangle.$$

Now, $(id\otimes\pi_{U_0})\Re = F_q(q^{\tilde{T}/2}\tilde{b}\otimes\ell q^N\otimes U_0)(I\otimes\ell\otimes U_0)^{\tilde{T}\otimes I\otimes I}$. Therefore, denoting $\pi^{(m)}(\mathbf{T})$ and $\pi^{(m)}(\mathbf{b})$ by $\mathbf{T}^{(m)}$ and $\mathbf{b}^{(m)}$ respectively, we have

$$\begin{aligned} &\langle\xi_{r's'}^{(m')}\otimes e_{i'j'}, ((id\otimes\pi_{U_0})\Re)\xi_{rs}^{(m)}\otimes e_{ij}\rangle\\ &\quad= \delta_{mm'}\delta_{rr'}\left\langle e_{s'i'j'}, F_q(q^{\frac{1}{2}\mathbf{T}^{(m)}}\mathbf{b}^{(m)}\otimes\ell q^N\otimes U_0)(I\otimes\ell\otimes U_0)^{\mathbf{T}^{(m)}\otimes I\otimes I}e_{sij}\right\rangle \end{aligned}$$

$$= \begin{cases} \delta_{mm'}\delta_{rr'}\delta_{s,i-i'-s'}J_q(q^{m+1+i-s}, i-i'-2s)\langle e_{j'}, U_0^{i-i'-2s}e_j\rangle \\ \hfill \text{if } m \in \mathbb{Z}, \\ \delta_{mm'}\delta_{rr'}\delta_{s,i-i'-s'-1}J_q(q^{m+\frac{1}{2}+i-s}, i-i'-2s-1)\langle e_{j'}, U_0^{i-i'-2s-1}e_j\rangle \\ \hfill \text{if } m \in \mathbb{Z}+\frac{1}{2}. \end{cases}$$

That is,

$$\Re_\rho(\xi_{rs}^{(m)}) = \begin{cases} J_q(q^{m+1+i-s}, i-i'-2s)\langle e_{j'}, U_0^{i-i'-2s}e_j\rangle \xi^{(m)}_{r,i-i'-s} \\ \hfill \text{if } m \in \mathbb{Z}, \\ J_q(q^{m+\frac{1}{2}+i-s}, i-i'-2s-1)\langle e_{j'}, U_0^{i-i'-2s-1}e_j\rangle \xi^{(m)}_{r,i-i'-s-1} \\ \hfill \text{if } m \in \mathbb{Z}+\frac{1}{2}. \end{cases}$$

On the other hand, observe that $(id\otimes\mu)(w^{(m)}) = \phi_{12}(w^{(m)})\phi_{13}(w^{(m)})$, where ϕ_{12} and ϕ_{13} are the morphisms

$$\begin{aligned} \sum a_i\otimes b_i &\mapsto \sum a_i\otimes b_i\otimes I, \\ \sum a_i\otimes b_i &\mapsto \sum a_i\otimes I\otimes b_i, \end{aligned}$$

from $M(\mathcal{B}_0(L_2(\mathbb{Z}))\otimes C_0(E_q(2)))$ to $M(\mathcal{B}_0(L_2(\mathbb{Z}))\otimes C_0(E_q(2))\otimes C_0(E_q(2)))$. Hence, if $m\in\mathbb{Z}$, then

$$\begin{aligned} &\langle e_{k'l'}, \xi_{rs}^{(m)} * \rho\, e_{kl}\rangle \\ &= q^{m-r}\left\langle e_{k'l'i'j'}, (id\otimes\pi_{U_0})\mu(w_{rs}^{(m)})e_{klij}\right\rangle \\ &= q^{m-r}\left\langle e_{rk'l'i'j'}, (id\otimes\pi_{U_0})(id\otimes\mu)(w^{(m)})e_{sklij}\right\rangle \\ &= q^{m-r}\left\langle e_{rk'l'i'j'}, (id\otimes id\otimes\pi_{U_0})\phi_{12}(w^{(m)})\phi_{13}(w^{(m)})e_{sklij}\right\rangle \\ &= q^{m-r}\sum_p \langle e_{rk'l'}, w^{(m)}e_{pkl}\rangle \left\langle e_{pi'j'}, (id\otimes\pi_{U_0})(w^{(m)})e_{sij}\right\rangle \\ &= q^{m-r}\sum_p \langle e_{k'l'}, w_{rp}^{(m)}e_{kl}\rangle \left\langle e_{i'j'}, (id\otimes\pi_{U_0})(w_{ps}^{(m)})e_{ij}\right\rangle \\ &= \sum_p \langle e_{k'l'}, \xi_{rp}^{(m)}e_{kl}\rangle\left\langle e_{i'j'}, (\ell^{p+s}\otimes I)J_q\left(q^{m-s+1}(q^N\otimes U_0), p-s\right)e_{ij}\right\rangle \\ &= \sum_p \langle e_{k'l'}, \xi_{rp}^{(m)}e_{kl}\rangle\delta_{p,i-i'-s}\langle e_{j'}, U_0^{p-s}e_j\rangle J_q(q^{m+1+i-s}, p-s) \\ &= \langle e_{k'l'}, \xi^{(m)}_{r,i-i'-s}e_{kl}\rangle\langle e_{j'}, U_0^{i-i'-2s}e_j\rangle J_q(q^{m+1+i-s}, i-i'-2s). \end{aligned}$$

Similarly, for $m\in\mathbb{Z}+\frac{1}{2}$, one has

$$\langle e_{k'l'}, \xi_{rs}^{(m)} * \rho\, e_{kl}\rangle = \langle e_{k'l'}, \xi^{(m)}_{r,i-i'-s-1}e_{kl}\rangle\langle e_{j'}, U_0^{i-i'-2s-1}e_j\rangle J_q(q^{m+\frac{1}{2}+i-s}, i-i'-2s-1).$$

Therefore $\Re_\rho = R_\rho$. Extending by linearity, the same conclusion holds for any ρ of the form $\langle u_1, \pi_{U_0}(\cdot)u_2\rangle$, where u_1 and u_2 are in the linear span of the e_{ij}'s. Combining this with our earlier observation, we find that $\Re_\rho = R_\rho$ for any ρ of the form (4.5), with u_1 and u_2 coming from a dense subspace of $L_2(\mathbb{Z}) \otimes \mathcal{H}$. The set $\mathcal{D}$ of all such functionals is dense in norm topology in the space of all continuous functionals on $C_0(E_q(2))$. Hence for any continuous linear functional ρ, we have $\Re_\rho = R_\rho$, and, in particular,

$$\Re_\rho(a) = a * \rho \quad \forall\, a \in C_0(E_q(2)) \cap L_2(h).$$

We call $\Re$ the *right regular representation*. From (4.3) it is immediate that in the direct sum decomposition of $\Re$, all the infinite dimensional irreducibles appear, and each one appears countably infinite number of times.

5 Identities Involving q-Bessel Functions

We shall now use the computations done in section 3 and the observations made in the previous section to generate a class of identities involving the q-Bessel functions.

Let us take ρ to be the functional $a \mapsto \langle e_{i+i',j+j'}, a\, e_{ij}\rangle$ on $C_0(E_q(2))$. Then

$$\begin{aligned}
\Re_\rho(\mathbf{v}^r J_q(q^s\mathbf{n}, t)) &= \Re_\rho\Big(w^{(s-1+\frac{r-t}{2})}_{[\frac{r+t}{2}],[\frac{r-t}{2}]}\Big)\\
&= q^{[\frac{r+t}{2}]-s+1-\frac{r-t}{2}}\Re_\rho\Big(\xi^{(s-1+\frac{r-t}{2})}_{[\frac{r+t}{2}],[\frac{r-t}{2}]}\Big)\\
&= q^{[\frac{r+t}{2}]-s+1-\frac{r-t}{2}}\sum_p \Big\langle e_p, w_\rho^{(s-1+\frac{r-t}{2})} e_{[\frac{r-t}{2}]}\Big\rangle \xi^{(s-1+\frac{r-t}{2})}_{[\frac{r+t}{2}],p}\\
&= \sum_p \Big\langle e_{p,i+i',j+j'}, w^{(s-1+\frac{r-t}{2})} e_{[\frac{r-t}{2}],i,j}\Big\rangle w^{(s-1+\frac{r-t}{2})}_{[\frac{r+t}{2}],p}.
\end{aligned}$$

After simplification, this yields

$$\Re_\rho(\mathbf{v}^r J_q(q^s\mathbf{n}, t)) = \delta_{t-r,i'+j'} J_q(q^{i+s}, j')\mathbf{v}^{t-i'} J_q(q^{s-j'}\mathbf{n}, t-j'). \tag{5.1}$$

Hence for any $u \in L_2(h)$,

$$\begin{aligned}
\Re_\rho u &= \Re_\rho\Big(\sum_{r,s,t} q^{2(s-t-1)}\, \langle \mathbf{v}^r J_q(q^s\mathbf{n}, t), u\rangle\, \mathbf{v}^r J_q(q^s\mathbf{n}, t)\Big)\\
&= \sum_{\substack{r,s,t\\ t-r=i'+j'}} q^{2(s-t-1)}\, \langle \mathbf{v}^r J_q(q^s\mathbf{n}, t), u\rangle\, J_q(q^{i+s}, j')\mathbf{v}^{t-i'} J_q(q^{s-j'}\mathbf{n}, t-j')\\
&= \sum_{s,t} q^{2(s-t-1)} \Big\langle \mathbf{v}^{t-i'-j'} J_q(q^s\mathbf{n}, t), u\Big\rangle\, J_q(q^{i+s}, j')\mathbf{v}^{t-i'} J_q(q^{s-j'}\mathbf{n}, t-j'),
\end{aligned}$$

so that

$$\langle e_{k'l'}, \Re_\rho u\, e_{kl}\rangle$$
$$= \sum_{s,t} q^{2(s-t-1)} J_q(q^{i+s}, j') J_q(q^{s-j'+k}, t-j') \left\langle \mathbf{v}^{t-i'-j'} J_q(q^s\mathbf{n}, t), u\right\rangle \delta_{k',k-t+i'}\delta_{l',l+t-j'}. \quad (5.2)$$

Let us now compute the quantity $\langle e_{k'l'}, \Re_\rho u\, e_{kl}\rangle$ in another way, using equation (3.1). Take a u in $L_2(h)\cap C_0(E_q(2))$. Then

$$\begin{aligned}\langle e_{k'l'}, \Re_\rho u\, e_{kl}\rangle &= \langle e_{k'l'}, u*\rho\, e_{kl}\rangle \\ &= \langle e_{k',l',i+i',j+j'}, \mu(u) e_{k,l,i,j}\rangle \\ &= \sum_m J_q(q^{i+i'-k'+1}, m) J_q(q^{i-k+1}, m-j') \\ &\qquad \langle e_{k'+m,l'+m}, u\, e_{k-j'+m,l-j'+m}\rangle \delta_{i+i'-j-j'-k'-l',i-j-k-l}.\end{aligned} \quad (5.3)$$

Take $k' = k+c$ and $l' = l-c+i'-j'$. Then from (5.2) and (5.3), we get

$$\begin{aligned}&\sum_m J_q(q^{i-k+1+i'-c}, m) J_q(q^{i-k+1}, m-j') \langle e_{k+c+m,l-c+i'-j'+m}, u\, e_{k-j'+m,l-j'+m}\rangle \\ &= \sum_s q^{2(s-i'+c-1)} J_q(q^{s+i}, j') J_q(q^{s-j'+k}, i'-j'-c) \left\langle \mathbf{v}^{-j'-c} J_q(q^s\mathbf{n}, i'-c), u\right\rangle.\end{aligned} \quad (5.4)$$

Taking various choices for the element u and the integers i, j, i', j', k, l and c, one can generate a whole lot of identities involving the q-Bessel functions. As an illustration, we prove a few identities below.

Proposition 5.1 *For any integers i, j, r and s, we have*

$$\begin{aligned}&\sum_m J_q(q^i, m-r) J_q(q^j, m-s) J_q(q^{i-m}, j-m-1) \\ &\qquad = J_q(q^{r-1}, r-s-2) J_q(q^{i-s}, j-r+1).\end{aligned} \quad (5.5)$$

Proof: Take $u = \mathbf{v}^{-j'-c} J_q(q^s\mathbf{n}, i'-c)$ in equation (5.4), use part 5 of proposition 2.1 and make some change of variables to get the required identity. □

If we take $i = j = r = 1$ and $s = -1$ in (5.5) and use part 5 of proposition 2.1, we get the following.

$$\sum_m J_q(q, m-1) J_q(q, m+1) J_q(q^{m+1}, 0) = J_q(1,0)^2. \quad (5.6)$$

Proposition 5.2 *For any integers* a*,* b*,* i*,* j *and* k*, we have*

$$\sum_s q^{2(s-a-1+j+k)} J_q(q^{s+i}, b) J_q(q^{s+j}, a-b) J_q(q^{s+j+k}, a) = J_q(q^{i-j+a-b+1}, k) J_q(q^{i-j-b+1}, k-b). \tag{5.7}$$

Proof: Take $u = \mathbf{v}^{-j'-c} g(\mathbf{n})$, where $g(q^d z) = I_{\{k-j'+j\}}(d) z^{i'-c}$, $d \in \mathbb{Z}$, $z \in S^1$. Now use (5.4) and make some change of variables. □

The following identities can all be derived from (5.7) by taking appropriate choices of the integers a, b, i, j and k.

$$\begin{aligned}
\sum_s q^{2(s-a-1)} J_q(q^s, a) J_q(q^{s+i}, b) J_q(q^{s+j}, a-b) &= J_q(q^{i-j+a-b+1}, -j) J_q(q^{i-j-b+1}, -j-b), \\
\sum_s q^{2s} J_q(q^s, 0) J_q(q^{s+i}, 0) J_q(q^{s+j}, 0) &= q^2 J_q(q^{i-j+1}, -j)^2, \\
\sum_s q^{2(s+j+k-1)} J_q(q^{s+i}, 0) J_q(q^{s+j}, 0) J_q(q^{s+j+k}, 0) &= J_q(q^{i-j+1}, k)^2, \\
\sum_s q^{2s} J_q(q^{s+i}, 0) J_q(q^{s+1}, 0)^2 &= J_q(q^i, 0)^2, \\
\sum_s q^{2s} J_q(q^{s+1}, 0)^3 &= J_q(q, 0)^2.
\end{aligned}$$

Remark: q-analogues of Graf's identities were first proved by Koelink in [2] using quantum group theoretic, but more algebraic arguments. Later, he and Swarttouw gave an analytical proof ([3]), but this time avoiding the use of quantum groups. The proof presented here uses the quantum $E(2)$ group, which is the natural setting for q-Bessel functions, and is also analytic in nature.

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