On Some Quantum Groups and Their Representations

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To

my parents

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Notations

$[n]$	Greatest integer less than or equal to n
S^1	The one dimensional torus $\{z \in \mathbb{C} : \|z\| = 1\}$
$\mathbb{C}^q$	The set $\{q^k z : k \in \mathbb{Z}, z \in S^1\} \cup \{0\}$
$C_c(X)$	Space of all continuous functions on X having compact support
$C_0(X)$	C^*-algebra of all continuous functions on X vanishing at infinity
$C_b(X)$	C^*-algebra of all bounded continuous functions on X
ℓ	The operator $e_k \mapsto e_{k-1}$ on $L_2(\mathbb{Z})$; the operator $e_k \mapsto e_{k-1}$, $k \geq 1$, $e_0 \mapsto 0$ on $\ell_2 = L_2(\mathbb{Z}_+)$
N	The operator $e_k \mapsto ke_k$ on ℓ_2 as well as on $L_2(\mathbb{Z})$
$\mathcal{H}$, $\mathcal{K}$ etc.	Hilbert spaces
$\mathcal{A}$, $\mathcal{B}$ etc.	C^*-algebras
$M(\mathcal{A})$	Multiplier algebra of $\mathcal{A}$
$a\,\eta\,\mathcal{A}$	The element a is affiliated to the C^*-algebra $\mathcal{A}$
$\mathcal{B}(\mathcal{H})$	Algebra of all bounded operators on $\mathcal{H}$
$\mathcal{B}_0(\mathcal{H})$	C^*-algebra of all compact operators on $\mathcal{H}$
S, T etc.	Operators on Hilbert spaces
$\operatorname{dom} T$	Domain of the operator T
$\sigma(T)$	Spectrum of the operator T
u, v, etc.	Elements of a Hilbert space
$\|u\rangle\langle v\|$	The operator $w \mapsto \langle v, w\rangle u$
$\langle u\|$	The functional $v \mapsto \langle u, v\rangle$

Introduction

The theory of quantum groups has become a very rapidly growing and active area of research in mathematics and mathematical physics over the last one decade. But the origin can actually be traced back in mathematical literature much earlier, in connection with the investigation for a good duality theorem for locally compact groups. In the early thirties, Pontryagin proved a duality theorem for locally compact *abelian* groups. If G is a locally compact abelian group, then the set $\hat{G}$ of characters (that is, homomorphisms into the circle group S^1) is a group in its own right. With a suitable topology, it becomes a locally compact abelian group, which we call the dual group of G. Pontryagin's theorem says that if we start with this dual group $\hat{G}$ and pass on to its dual $\hat{\hat{G}}$, what we get is nothing but the original group G that we started with. Ever since then, mathematicians had been trying to prove something similar for general locally compact groups. Success came, but very slowly. In the late thirties, Tannaka proved one duality theorem for compact groups. In 1959, Stinespring succeeded in proving a duality theorem for unimodular groups. Finally, in 1965, Tatsuuma was able to prove a duality theorem for all locally compact groups. None of these duality theorems, however, could achieve the desired symmetry, essentially due to the fact that the space of irreducible unitary representations of a nonabelian group does not have a natural group structure, so that while passing on to the dual of a nonabelian group, one gets out of the category of groups. So the search was on, this time, for a bigger category containing locally compact groups and their duals, and for which a symmetric duality result holds. To this end, Kac in the 60's introduced the notion of a *ring-group*, Takesaki in the 70's laid down the theory of *Hopf-von Neumann algebras*, and more recently, Enoch and Schwarz investigated what they called *Kac algebras*. These objects are very close in spirit to what we know as quantum groups today.

These theories all suffered from one very serious drawback, namely, that apart from locally compact groups and their duals, no good examples were known (perhaps one of the reasons why not many people were working in this area). The situation changed drastically in the early eighties. Fadeev, Sklyanin and Takhtajan were working on the quantum inverse scattering method (QISM), which is a method of constructing and studying integrable quantum systems. They encountered a certain kind of Hopf structure that naturally arise there. Drinfeld noticed the connection with Lie bialgebras, and made all the notions rigorous in his talk given in the meeting of the ICM in Berkeley. Around the same time, Jimbo published his papers on q-deformations of universal enveloping algebras, which is another way of getting lots of examples of such structures. On one hand, one had lots of examples of the kind of objects whose theory had already been developed by Kac, Takesaki et al., on the other hand, many more examples of a similar nature were now available that were not covered by these theories, as they were not sufficiently general in nature. As a result, lots of mathematicians and physicists started getting interested in this area. Slowly, connections were established with various other apparently diverse areas in mathematics and physics; for example, the theory of knot and link invariants, invariants for 3-manifolds, q-special functions, representation theory of Lie algebras in characteristic p, conformal and quantum field theories, soliton theory, solvable lattice models, to name a few. Thus quantum groups has already evolved as a very busy area of research in present day mathematics.

To understand what is a quantum group, let us start with a Lie group G. For convenience, take it to be an open connected subgroup of $GL(n)$. Let $A(G)$ be the space of coordinate functions on this group G. This is a commutative algebra, being a space of functions. The group structure of the underlying space G makes it a Hopf algebra, which, unless the underlying group is abelian, is non cocommutative. Now suppose we drop the commutativity restriction from the algebra. The resulting object then behaves like the algebra of coordinate functions on a group, but is not quite so, because, it does not consist of functions on any concrete space. So we pretend as if there is some kind of a space underneath, on which this is the space of 'coordinate functions'. Crudely speaking, this is what a quantum group is all about. It should be stressed here that it is the hidden 'space' that is the quantum group, that is the object that one is primarily interested in. $A(G)$, or rather, the noncommutative version of it is merely an associated object through which we study the underlying quantum group. In the C^*-algebra approach to the theory of quantum groups, starting with this Hopf algebra $A(G)$, one tries to build the C^*-algebra $C_0(G)$ of 'continuous functions on G vanishing at infinity', and extend the comultiplication map there. Though these involve technical and conceptual difficulties, usually they give us more information and insight about the quantum group G under consideration. It is this approach that we shall follow in this thesis.

Associated to the group G, there is another Hopf algebra, namely, the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of its Lie algebra $\mathfrak{g}$. It is noncommutative, but cocommutative, and once we know $\mathcal{U}(\mathfrak{g})$, it is possible to recover the group G from it. Again, as before, we are interested in the noncommutative version of it. This time, one gets the noncommutative version, or, what is known as a quantized universal enveloping algebra (QUEA), by dropping the cocommutativity restriction. The primary object of interest, like in the earlier case, is not this QUEA, but the underlying 'quantum group', which one studies using this associated Hopf algebra. This constitutes yet another approach to the study of quantum groups.

Of course there is a definite connection between the two approaches described above. It would become very transparent once we understand the connection in the case of a classical group. Let us again go back to our group G. It is easy to see that $\mathcal{U}(\mathfrak{g})$ is isomorphic to the space of generalised functions on G with support at the identity. It is clear intuitively that $\mathcal{U}(\mathfrak{g})^*$ contains all the coordinate functions, and possibly much much more. The Hopf algebra of coordinate functions is, therefore, a 'reduced dual' of $\mathcal{U}(\mathfrak{g})$. For quantum groups also, one observes the same phenomenon. For a deatailed account on this, see the papers of Rosso([45]) and Van Daele([61]).

Let us now come to the content of the present thesis. The origin of quantum groups, as we have mentioned, lies in the study of (classical) groups. In fact, in many ways, the two are strikingly similar. Many concepts and results from group theory admit generalization to the case of a quantum group. However, at the same time, quantum groups sometime exhibit behaviour that is very much different from a group. We take up some such concepts and results, and try to see what happens for a quantum group.

In chapters 1 and 2, we examine the notion of an induced representation, which plays a very important role in the representation theory of classical groups. Now, the class of all quantum groups is yet to be fully characterized. However, a fairly satisfactory theory for a smaller class, namely the class of all *compact quantum groups* has been laid down rigorously by Woronowicz et al. Here we develop the concept of an induced representation for this subclass.

In the remaining chapters, we take up noncompact quantum groups. Unlike in the earlier case, here the existing theory is far from satisfactory. A unified description of a locally compact quantum group is yet to be found. Various examples are being studied in order to be able to reach an appropriate definition. We deal with one specific example here, namely, the q-deformation $E_q(2)$ of the group of motions of the plane.

The first two sections of chapter 3 contain a brief description of the quantum group $E_q(2)$, and some key results of Woronowicz on the topic. We then introduce what we call the q-analogues of Bessel functions, present an explicit computation of the comultiplication map μ, and describe a feature of $E_q(2)$ that is special to quantum groups.

In chapter 4, we handle the haar measure for the group $E_q(2)$. First we prove the existence of an invariant measure, the form of which is fairly easy to guess once we know the haar measure for $SU_q(2)$, and the fact that $E_q(2)$ comes from $SU_q(2)$ via the 'contraction procedure'. The proof, however, is a bit involved. Next, we prove a few identities involving the q-Bessel functions using the invariance properties of the haar measure. Making use of these identities, we next prove the uniqueness of the haar measure. As an application of some of the identities proved in the second section, we prove the existence of a left invariant and a right invariant measure on the dual group $\widehat{E_q(2)}$.

In the last chapter, we find all the irreducible unitary representations of the quantum group $E_q(2)$. According to a theorem of Woronowicz, unitary representations are described by a pair of closed (unbounded) operators. Finding irreducible unitaries amounts to finding irreducible representations for this pair of operators. Using the identities proved in the previous chapter, orthogonality relations involving the matrix entries of the irreducible unitaries are proved. We also give a formula expressing a tensor product as a direct sum of irreducible ones. The regular representation is introduced in the second section. Section 3 contains a very general identity from which one can derive a lot of identities involving the q-Bessel functions. Finally a brief description of the quantised complex plane is given. $E_q(2)$ has an action on this noncommutative space. We describe this action in detail and show how it splits into a direct sum of irreducible representations.

We shall need the basics of the theory of C^*-algebras, as can be found, for example, in the book by Pedersen([38]). We shall also need some familiarity with the concept of a multiplier algebra, morphisms and the concept of an affiliation relation in the context of C^*-algebras. We give a brief description of each of them below and present some more relevant material in the appendix. References for these topics are [70] and [75].

Let $\mathcal{A}$ be a C^*-algebra, acting nondegenerately on a Hilbert space $\mathcal{H}$ (meaning $au = 0$ for all $a \in \mathcal{A}$ implies $u = 0$), then the *multiplier algebra* $M(\mathcal{A})$, the *left multiplier algebra* $LM(\mathcal{A})$ and the *right multiplier algebra* $RM(\mathcal{A})$ of the C^*-algebra $\mathcal{A}$ are defined as follows:

$$\begin{aligned}
M(\mathcal{A}) &= \{b \in \mathcal{B}(\mathcal{H}) : ba, ab \in \mathcal{A} \;\; \forall a \in \mathcal{A}\}, \\
LM(\mathcal{A}) &= \{b \in \mathcal{B}(\mathcal{H}) : ba \in \mathcal{A} \;\; \forall a \in \mathcal{A}\}, \\
RM(\mathcal{A}) &= \{b \in \mathcal{B}(\mathcal{H}) : ab \in \mathcal{A} \;\; \forall a \in \mathcal{A}\}.
\end{aligned}$$

Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras. A C^*-homomorphism ϕ from $\mathcal{A}$ to $M(\mathcal{B})$ is called a *morphism* if $\phi(\mathcal{A})\mathcal{B}$ is dense in $\mathcal{B}$. We denote the space of all morphisms from $\mathcal{A}$ to $\mathcal{B}$ by $\mathrm{mor}(\mathcal{A}, \mathcal{B})$.

Once again, as before, let $\mathcal{A}$ be a C^*-algebra acting nondegenerately on $\mathcal{H}$. A closed operator T on $\mathcal{H}$ is said to be *affiliated to* $\mathcal{A}$ if the following two conditions are satisfied:

$$T(I + T^*T)^{-1/2} \in M(\mathcal{A}),$$

$$(I + T^*T)^{-1/2}\mathcal{A} \quad \text{is dense in } \mathcal{A}.$$

We would always be dealing with complex separable Hilbert spaces. The inner product of a Hilbert space will be assumed to be linear in the second argument and antilinear in the first. Let us explain here a convention regarding notation that we shall follow in many places in this thesis. Let $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ etc. be Hilbert spaces. Suppose $\{e_a\}_{a\in\Lambda_1}$, $\{f_a\}_{a\in\Lambda_2}$ and $\{g_a\}_{a\in\Lambda_3}$ denote respectively orthonormal bases for the above Hilbert spaces. While dealing with tensor products of these spaces, we shall denote, unless there is any chance of ambiguity, the basis $\{e_a \otimes f_b\}_{(a,b)\in\Lambda_1\times\Lambda_2}$

for $\mathcal{H}_1 \otimes \mathcal{H}_2$ by simply $\{e_{ab}\}_{(a,b)\in\Lambda_1\times\Lambda_2}$, the basis $\{e_a \otimes f_b \otimes g_c\}_{(a,b,c)\in\Lambda_1\times\Lambda_2\times\Lambda_3}$ for $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ by $\{e_{abc}\}_{(a,b,c)\in\Lambda_1\times\Lambda_2\times\Lambda_3}$, and so on.

Throughout this thesis, q *will always denote a real number between* 0 *and* 1.

Chapter 1

Compact Quantum Groups

As has been mentioned in the introduction, we shall deal with topological quantum groups. In other words, we shall study quantum groups using the 'C^*-algebra of continuous vanishing-at-infinity functions' on them. In this chapter and the next, we shall be concerned with those quantum groups whose 'underlying spaces' are compact. Of course, there is no underlying space really. So what one means is the class of quantum groups for which the corresponding C^*-algebra of continuous vanishing-at-infinity functions that one typically constructs out of the algebra of coordinate functions, has an identity. It turns out that it is possible to characterize the C^*-algebras associated with such quantum groups among the category of unital C^*-algebras, and also to describe the comultiplication map without any reference whatsoever to the Hopf algebra $A(G)$. The theory of such quantum groups has been laid down rigorously by Woronowicz (see [67, 68, 72]). In this chapter, we mainly review some of the basic features of the theory.

1.1 Representation Theory

We start with the definition of a compact quantum group and a few examples.

Definition 1.1.1 ([72]) *Let $\mathcal{A}$ be a separable unital C^*-algebra, and $\mu : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ be a unital *-homomorphism. We call $G = (\mathcal{A}, \mu)$ a compact quantum group if the following two conditions are satisfied:*
i. $(id \otimes \mu)\mu = (\mu \otimes id)\mu$,
ii. *Linear spans of both $\{(a \otimes I)\mu(b) : a, b \in \mathcal{A}\}$ and $\{(I \otimes a)\mu(b) : a, b \in \mathcal{A}\}$ are dense in $\mathcal{A} \otimes \mathcal{A}$.*

μ is called the *comultiplication map* associated with G. We shall very often denote the underlying C^*-algebra $\mathcal{A}$ by $C(G)$ and if situation demands, the map μ by μ_G.

Example 1.1.2 Any compact group is an example of a compact quantum group. To see this, notice first that for a compact group G, there is an obvious identification between $C(G)\otimes C(G)$ and $C(G\times G)$. Define a map μ from $C(G)$ to $C(G)\otimes C(G)$ by the prescription: $\mu f(x,y) = f(xy)$, $f \in C(G)$, $x, y \in G$. Then it is trivial to verify that $(C(G), \mu)$ is a compact quantum group. Conversely, if for a compact quantum group G, $C(G)$ is abelian, then it has to be of the above form. Thus classical compact groups are precisely the ones for which $C(G)$ is abelian.

Example 1.1.3 ([24]) Take $\mathcal{A}$ to be the C^*-algebra $\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}\oplus M_2(\mathbb{C})$. Let

$$e_k = \delta_{1k}\oplus\delta_{2k}\oplus\delta_{3k}\oplus\delta_{4k}\oplus\begin{pmatrix}\delta_{5k} & \delta_{8k}\\ \delta_{7k} & \delta_{6k}\end{pmatrix},\quad k = 1,2,\ldots,8,$$

where δ denotes the Kronecker delta. Then $\{e_1, \ldots, e_8\}$ form a basis for $\mathcal{A}$. Define a map $\mu : \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ as follows:

$$\begin{aligned}
\mu(e_1) &= e_1\otimes e_1 + e_2\otimes e_2 + e_3\otimes e_3 + e_4\otimes e_4\\
&\qquad +\tfrac{1}{2}(e_5\otimes e_5 + e_6\otimes e_6 + e_7\otimes e_7 + e_8\otimes e_8),\\
\mu(e_2) &= e_1\otimes e_2 + e_2\otimes e_1 + e_3\otimes e_4 + e_4\otimes e_3\\
&\qquad +\tfrac{1}{2}(e_5\otimes e_6 + e_6\otimes e_5 + ie_7\otimes e_8 - ie_8\otimes e_7),\\
\mu(e_3) &= e_1\otimes e_3 + e_3\otimes e_1 + e_2\otimes e_4 + e_4\otimes e_2\\
&\qquad +\tfrac{1}{2}(e_5\otimes e_6 + e_6\otimes e_5 - ie_7\otimes e_8 + ie_8\otimes e_7),\\
\mu(e_4) &= e_1\otimes e_4 + e_4\otimes e_1 + e_2\otimes e_3 + e_3\otimes e_2\\
&\qquad +\tfrac{1}{2}(e_5\otimes e_5 + e_6\otimes e_6 - e_7\otimes e_7 - e_8\otimes e_8),\\
\mu(e_5) &= e_1\otimes e_5 + e_5\otimes e_1 + e_2\otimes e_6 + e_6\otimes e_2\\
&\qquad +e_3\otimes e_6 + e_6\otimes e_3 + e_4\otimes e_5 + e_5\otimes e_4,\\
\mu(e_6) &= e_1\otimes e_6 + e_6\otimes e_1 + e_2\otimes e_5 + e_5\otimes e_2\\
&\qquad +e_3\otimes e_5 + e_5\otimes e_3 + e_4\otimes e_6 + e_6\otimes e_4,\\
\mu(e_7) &= e_1\otimes e_7 + e_7\otimes e_1 - ie_2\otimes e_8 + ie_8\otimes e_2\\
&\qquad +ie_3\otimes e_8 - ie_8\otimes e_3 - e_4\otimes e_7 - e_7\otimes e_4,\\
\mu(e_8) &= e_1\otimes e_8 + e_8\otimes e_1 + ie_2\otimes e_7 - ie_7\otimes e_2\\
&\qquad -ie_3\otimes e_7 + ie_7\otimes e_3 - e_4\otimes e_8 - e_8\otimes e_4.
\end{aligned}$$

It is a matter of straightforward verification that μ is a unital $*$-homomorphism and $G = (\mathcal{A}, \mu)$ is a compact quantum group.

Example 1.1.4 ([67]) $\boldsymbol{SU_q(2)}$. Take $\mathcal{A}$ to be the canonical C^*-algebra generated by two elements α and β satisfying the following relations:

$$\alpha^*\alpha + \beta^*\beta = I, \qquad \alpha\alpha^* + q^2\beta\beta^* = I,$$
$$\alpha\beta - q\beta\alpha = 0, \qquad \alpha\beta^* - q\beta^*\alpha = 0,$$
$$\beta^*\beta \;=\; \beta\beta^*.$$

Define a map μ on the linear span of α and β as follows:

$$\begin{aligned}\mu(\alpha) &= \alpha\otimes\alpha - q\beta^*\otimes\beta,\\ \mu(\beta) &= \beta\otimes\alpha + \alpha^*\otimes\beta.\end{aligned}$$

It extends to a unital $*$-homomorphism from $\mathcal{A}$ to $\mathcal{A}\otimes\mathcal{A}$, and one can now verify that $(\mathcal{A},\mu)$ is a compact quantum group. This is known as the q-deformation of the $SU(2)$ group. For $q=1$, we get the classical $SU(2)$ group.

The C^*-algebra $\mathcal{A}$ in the above example can be described more concretely as follows. Let $\{e_i\}_{i\geq 0}$ and $\{e_i\}_{i\in\mathbb{Z}}$ be the canonical orthonormal bases for ℓ_2 and $L_2(\mathbb{Z})$ respectively. We denote by the same symbol N the operator $e_k\mapsto ke_k$, $k\geq 0$, on ℓ_2 and $e_k\mapsto ke_k$, $k\in\mathbb{Z}$, on $L_2(\mathbb{Z})$. Similarly, denote by the same symbol ℓ the operator $e_k\mapsto e_{k-1}$, $k\geq 1$, $e_0\mapsto 0$ on ℓ_2 and the operator $e_k\mapsto e_{k-1}$, $k\in\mathbb{Z}$ on $L_2(\mathbb{Z})$. Now take $\mathcal{H}$ to be the Hilbert space $\ell_2\otimes L_2(\mathbb{Z})$, and let α and β be the following operators on $\mathcal{H}$:

$$\alpha = \ell\sqrt{I-q^{2N}}\otimes I, \quad \beta = q^N\otimes\ell.$$

Then $\mathcal{A}$ is the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by α and β.

Example 1.1.5 ([69]) $\boldsymbol{SU_q(n)}$. Let $(i_1,i_2,\ldots,i_n)$ be an n-tuple of distinct natural numbers. Let $I(i_1,i_2,\ldots,i_n)$ denote the number of inversions in $(i_1,i_2,\ldots,i_n)$, which is the cardinality of the set $\{(i_r,i_s): r<s, i_r>i_s\}$. Let

$$E_{i_1,\ldots,i_n} = \begin{cases} 0 & \text{if } i_r = i_s \text{ for some } r\neq s,\\ (-q)^{I(i_1,\ldots,i_n)} & \text{otherwise.}\end{cases}$$

Let $\mathcal{A}$ be the C^*-algebra generated by the n^2 generators $\{u_{ij}: 1\leq i,j\leq n\}$ satisfying the following relations:

$$\sum_i u_{ij}^*u_{ik} \;=\; \delta_{jk}I,$$

$$\sum_j u_{ij}u_{kj}^* = \delta_{ik}I,$$
$$\sum_{j_1,\ldots,j_n} u_{i_1j_1}\ldots u_{i_nj_n}E_{j_1,\ldots,j_n} = E_{i_1,\ldots,i_n}I.$$

The map μ defined by: $\mu(u_{ij}) = \sum_k u_{ik}\otimes u_{kj}$ extends to a unital $*$-homomorphism to the whole of $\mathcal{A}$, and $(\mathcal{A},\mu)$ is a compact quantum group. As in the preceeding example, for $q = 1$, one gets back the classical group $SU(n)$.

Other examples of compact quantum groups can be found in the papers of Andruskiewitsch ([3]), Andruskiewitsch & Enriquez ([4]), and Tiraboschi ([58]). Recently, Van Daele and Wang have constructed a class of compact quantum groups, which are universal, in the sense that any compact matrix quantum group can be shown to be a subgroup of one belonging to this class. See Van Daele & Wang([62]) and Wang([63]).

Let G be a compact group, and $U : g \mapsto U_g$, a strongly continuous representation acting on a Hilbert space $\mathcal{H}$. One can then view U as an element of the multiplier algebra $M(\mathcal{B}_0(\mathcal{H})\otimes C(G))$, and the property $U_gU_h = U_{gh}$ then reads, in our present language, $(id\otimes\mu_G)U = \phi_{12}(U)\phi_{13}(U)$, where ϕ_{12} and ϕ_{13} are C^*-homomorphisms from $M(\mathcal{B}_0(\mathcal{H})\otimes C(G))$ to $M(\mathcal{B}_0(\mathcal{H})\otimes C(G)\otimes C(G))$, given on the product elements by:

$$\phi_{12}(a\otimes b) = a\otimes b\otimes I, \quad \phi_{13}(a\otimes b) = a\otimes I\otimes b.$$

Moreover, the representation U is unitary if and only if, viewed as an element of the C^*-algebra $M(\mathcal{B}_0(\mathcal{H})\otimes C(G))$, it is unitary. This description of U does not use elements of the underlying group G explicitly, and makes sense even when $(C(G),\mu_G)$ is a compact *quantum* group.

Definition 1.1.6 A *representation* of a compact quantum group $G = (\mathcal{A},\mu)$ acting on a Hilbert space $\mathcal{H}$ is an element π of the multiplier algebra $M(\mathcal{B}_0(\mathcal{H})\otimes\mathcal{A})$ that obeys $(id\otimes\mu)\pi = \pi_{12}\pi_{13}$, where $\pi_{ij} = \phi_{ij}(\pi)$, and ϕ_{12}, ϕ_{13} are as described above. It is called a *unitary representation* if, in addition, we have $\pi^*\pi = I = \pi\pi^*$.

Unless stated on the contrary, we shall always deal with unitary representations, and will often omit the adjective 'unitary'.

Let π_1 and π_2 be two representations of a compact quantum group acting on two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. A bounded operator T from $\mathcal{H}_1$ to $\mathcal{H}_2$ is said to intertwine π_1 and π_2 if $(T \otimes I)\pi_1 = \pi_2(T \otimes I)$. Denote by $\mathcal{I}(\pi_1, \pi_2)$ the set of intertwiners between π_1 and π_2. Two representations π_1 and π_2 are *equivalent* if there is an invertible intertwiner between the two. A representation π is said to be *irreducible* if $\mathcal{I}(\pi, \pi)$ is one dimensional. One also has the notions of direct sum and tensor product of representations. As before, let π_1 and π_2 be representations acting on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Let i_1 and i_2 denote the following inclusion maps:

$$i_1 : \mathcal{B}_0(\mathcal{H}_1) \hookrightarrow \mathcal{B}_0(\mathcal{H}_1 \oplus \mathcal{H}_2), \quad i_2 : \mathcal{B}_0(\mathcal{H}_2) \hookrightarrow \mathcal{B}_0(\mathcal{H}_1 \oplus \mathcal{H}_2).$$

$i_1 \otimes id$ (respectively $i_2 \otimes id$) extends to a morphism from $M(\mathcal{B}_0(\mathcal{H}_1) \otimes C(G))$ (respectively $M(\mathcal{B}_0(\mathcal{H}_2) \otimes C(G))$) to $M(\mathcal{B}_0(\mathcal{H}_1 \oplus \mathcal{H}_2) \otimes C(G))$. The *direct sum* $\pi_1 \oplus \pi_2$ is defined to be $(i_1 \otimes id)\pi_1 + (i_2 \otimes id)\pi_2$. For defining the tensor product, we use the natural identification between $\mathcal{B}_0(\mathcal{H}_1) \otimes \mathcal{B}_0(\mathcal{H}_2)$ and $\mathcal{B}_0(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Define $\phi_{13} : \mathcal{B}_0(\mathcal{H}_1) \otimes C(G) \to \mathcal{B}_0(\mathcal{H}_1) \otimes \mathcal{B}_0(\mathcal{H}_2) \otimes C(G)$, and $\phi_{23} : \mathcal{B}_0(\mathcal{H}_2) \otimes C(G) \to \mathcal{B}_0(\mathcal{H}_1) \otimes \mathcal{B}_0(\mathcal{H}_2) \otimes C(G)$ as follows:

$$\phi_{13}(a \otimes b) = a \otimes I \otimes b, \quad \phi_{23}(c \otimes b) = I \otimes c \otimes b.$$

Then ϕ_{13} (respectively ϕ_{23}) extends to a morphism from $M(\mathcal{B}_0(\mathcal{H}_1) \otimes C(G))$ (respectively $M(\mathcal{B}_0(\mathcal{H}_2) \otimes C(G))$) into $M(\mathcal{B}_0(\mathcal{H}_1) \otimes \mathcal{B}_0(\mathcal{H}_2) \otimes C(G))$. The *tensor product* $\pi_1 \bigoplus\!\!\!\!\!\top\ \pi_2$ of π_1 and π_2 is defined to be the representation $\phi_{13}(\pi_1)\phi_{23}(\pi_2)$, which acts on the space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

As in the case of a compact group, one can prove that *any unitary representation decomposes as a direct sum of finite dimensional irreducible unitary representations.*

Let $G = (C(G), \mu_G)$ be a compact quantum group. Let $A(G)$ be the unital $*$-subalgebra of $C(G)$ generated by the matrix entries of the finite dimensional representations of G. Then one has the following result (see [72]).

Theorem 1.1.7 ([72]) *Suppose G is a compact quantum group. Let $A(G)$ be as above. Then we have the following*:
(a) *$A(G)$ is a dense unital *-subalgebra of $C(G)$ and $\mu(A(G)) \subseteq A(G) \otimes_{alg} A(G)$.*
(b) *There is a complex homomorphism $\epsilon : A(G) \to \mathbb{C}$ such that*

$$(\epsilon \otimes id)\mu = id = (id \otimes \epsilon)\mu.$$

(c) *There exists a linear antimultiplicative map $\kappa : A(G) \to A(G)$ obeying*

$$m(id \otimes \kappa)\mu(a) = \epsilon(a)I = m(\kappa \otimes id)\mu(a), \quad \text{and} \quad \kappa(\kappa(a^*)^*) = a$$

for all $a \in A(G)$, where m is the operator that sends $a \otimes b$ to ab.

The maps ϵ and κ in the above theorem are called the *counit* and *coinverse* respectively of the quantum group G.

It is easy to see that, for a finite dimensional unitary representation π,

$$(id \otimes \epsilon)\pi = I, \quad (id \otimes \kappa)\pi = \pi^*. \tag{1.1.1}$$

One can check that $(A(G), \mu, \kappa, \epsilon)$ is a Hopf algebra. It is an analogue of the Hopf algebra of 'representative functions' for a compact group. For a compact lie group, it can also be called the algebra of coordinate functions. In examples 1.1.4 and 1.1.5, it is really this Hopf algebra that has been described. Constructing the C^*-algebra $C(G)$ from this is a fairly straightforward matter. Later (chapter 3) when we start dealing with noncompact quantum groups, we will see that this is not so simple any more — it involves difficulties, both of technical as well as conceptual nature.

1.2 The Haar Measure

Let G be a compact quantum group. A linear functional on the C^*-algebra $C(G)$ plays the role of a measure on G. Using the comultiplication μ, one can define a convolution product between two linear functionals ρ_1 and ρ_2:

$$\rho_1 * \rho_2(a) = (\rho_1 \otimes \rho_2)\mu(a), \quad a \in C(G).$$

One also has the notion of a convolution product between an element of $C(G)$ and a linear functional on it:

$$a * \rho = (id \otimes \rho)\mu(a), \quad \rho * a = (\rho \otimes id)\mu(a).$$

It is easy to check that if G is a group, these notions reduce to the usual convolution product of two measures and that of a measure and a function respectively.

A bounded functional λ is said to be *right invariant* if for any continuous functional ρ on $C(G)$, we have $\lambda * \rho = \rho(I)\lambda$. Similarly, λ is *left invariant* if $\rho * \lambda = \rho(I)\lambda$ for all ρ. As before, one can easily check that these coincide with the usual notions if G is a group.

One of the main achievements of the C^*-algebraic approach is that, starting from a simple set of axioms (cf. definition 1.1.1) one can prove the existence and uniqueness of a state that is both left- and right-invariant. We call this the *haar measure* on G. Here we present only a brief sketch of the proof. For further details, the reader should refer to [72].

Theorem 1.2.1 ([72]) *Let $G = (\mathcal{A}, \mu)$ be a compact quantum group. There exists a unique state h on $\mathcal{A}$ such that*

$$h * \rho = \rho * h = \rho(I)h \tag{1.2.1}$$

for any continuous linear functional ρ on $\mathcal{A}$.

Sketch of Proof: One first shows that if (1.2.1) holds for a faithful state ρ, then it holds for any continuous linear functional ρ. One has to make use of axiom (ii) of definition 1.1.1 here. Now, since the C^*-algebra $\mathcal{A}$ is separable, there exists a faithful state ρ. Write $h_n = (1/n)\sum_{k=1}^{n} \rho^{*k}$, where ρ^{*k} is the k-fold convolution product of ρ with itself. The set of states on $\mathcal{A}$ is compact with respect to the weak topology. Therefore $\{h_n : n \geq 1\}$ has a weak limit point. Let h be one such. Then it is easy to see that $h * \rho = \rho * h = \rho(I)h$. The theorem now follows from what we have observed in the beginning. □

Example 1.2.2 Let $(\mathcal{A}, \mu)$ be as in example 1.1.3. Let ρ_k denote the functional $\sum_{i=1}^{8} \alpha_i e_i \mapsto \alpha_k$. Let h be the state $\frac{1}{8}(\rho_1 + \rho_2 + \rho_3 + \rho_4) + \frac{1}{4}(\rho_5 + \rho_6)$. Direct computations show that $h * \rho_i = \rho_i * h = \rho_i(I)h$ for all i. Since the ρ_i's span the space of all functionals on $\mathcal{A}$, we have $h * \rho = \rho * h = \rho(I)h$ for any functional ρ on $\mathcal{A}$. Thus h is the haar measure for the quantum group $(\mathcal{A}, \mu)$.

Example 1.2.3 Consider the quantum group $SU_q(2)$. We have seen that $C(SU_q(2))$ is a C^*-subalgebra of $\mathcal{B}(\ell_2 \otimes L_2(\mathbb{Z}))$. Let h be the state $a \mapsto (1-q^2)\sum_{i=0}^{\infty} q^{2i}\langle e_{i0}, ae_{i0}\rangle$ on $C(SU_q(2))$. Then h is the haar measure measure for $SU_q(2)$. For a proof of this fact, see [68].

We have seen that compact quantum groups resemble their classical counterparts in almost all the aspects that have been described so far. We now mention one feature that is unique to them. For a compact group (even for locally compact groups, for that matter), haar measure is always faithful. But the same thing can not be said for a compact quantum group. It is faithful when restricted to the dense $*$-subalgebra $A(G)$ (cf. theorem 1.1.7), but is not, in general, faithful on the entire $C(G)$. Another thing to notice in this context is that the haar measure, which is actually a state on $C(G)$, is not tracial. The following theorem of Woronowicz gives the modular properties of the haar state.

Theorem 1.2.4 ([68]) *Let G be a compact quantum group. There exists a unique one (complex-) parameter group $\{f_z : z \in \mathbb{C}\}$ of linear multiplicative functionals on $A(G)$ such that for any $a \in A(G)$,*

1. *$z \mapsto f_z(a)$ is an entire function of exponential growth on the right half plane,*
2. *$f_z(\kappa(a)) = f_{-z}(a)$, $f_z(a^*) = \overline{f_{-\overline{z}}(a)}$,*
3. *$\kappa^2(a) = f_{-1} * a * f_1$,*
4. *$h(ab) = h(b(f_1 * a * f_1))$ for all $b \in C(G)$.*

1.3 Subgroups and Homomorphisms

In this section, we generalize some of the concepts for groups to our present context. Notice that if G and H are two compact groups, then a group homomorphism from H to G induces a C^*-homomorphism from $C(G)$ to $C(H)$. With this in mind, we give the following definition.

Definition 1.3.1 Let $G = (C(G), \mu_G)$ and $H = (C(H), \mu_H)$ be two compact quantum groups. A C^*-homomorphism ϕ from $C(G)$ to $C(H)$ is called a *quantum group homomorphism* from G to H if it obeys $(\phi \otimes \phi)\mu_G = \mu_H \phi$.

H is said to be a (*quantum*) *subgroup* of G if there is a quantum group homomorphism ϕ from G to H that maps $C(G)$ onto $C(H)$.

Proposition 1.3.2 *Suppose ϕ is a homomorphism from G to H. Then it maps $A(G)$ into $A(H)$, and the following diagrams commute:*

$$\begin{tikzcd} A(G) \arrow[r, "\phi"] \arrow[d, "\kappa_G"] & A(H) \arrow[d, "\kappa_H"] \\ A(G) \arrow[r, "\phi"] & A(H) \end{tikzcd} \qquad \begin{tikzcd} A(G) \arrow[r, "\phi"] \arrow[dr, "\epsilon_G"] & A(H) \arrow[d, "\epsilon_H"] \\ & \mathbb{C} \end{tikzcd}$$

Proof: Observe that $A(G)$ is the $*$-subalgebra of $C(G)$ generated by elements of the form $(\rho \otimes id)\pi$, where π is a unitary representation acting on some finite dimensional space $\mathcal{H}_\pi$ and ρ is a linear functional on $\mathcal{B}(\mathcal{H}_\pi)$. Take an element a of $A(G)$ of the form $(\rho \otimes id)\pi$. Then $\phi(a) = \phi((\rho \otimes id)\pi) = (\rho \otimes id)((id \otimes \phi)\pi)$. Now it is easy to see that if π is a unitary representation of G, $(id \otimes \phi)\pi$ is a unitary representation of H acting on the same space. Therefore the right hand side of the above equation belongs to $A(H)$, which implies that $\phi(A(G)) \subseteq A(H)$. Also, using (1.1.1), we get

$$\begin{aligned}
\phi(\kappa_G(a)) &= \phi\kappa_G((\rho \otimes id)\pi) \\
&= \phi(\rho \otimes id)((id \otimes \kappa_G)\pi) \\
&= \phi(\rho \otimes id)\pi^* \\
&= (\rho \otimes id)(id \otimes \phi)\pi^* \\
&= (\rho \otimes id)((id \otimes \phi)\pi)^* \\
&= (\rho \otimes id)(id \otimes \kappa_H)(id \otimes \phi)\pi \\
&= \kappa_H\phi(a),
\end{aligned}$$

and

$$\begin{aligned}
\epsilon_H\phi(a) &= \epsilon_H\phi((\rho \otimes id)\pi) \\
&= (\rho \otimes id)(id \otimes \epsilon_H\phi)\pi \\
&= (\rho \otimes id)I \\
&= (\rho \otimes id)((id \otimes \epsilon_G)\pi) \\
&= \epsilon_G(a).
\end{aligned}$$

Thus both the diagrams commute. □

Corollary 1.3.3 *If $\mathcal{I} = ker\ \phi$, then $\mathcal{I} \cap A(G) \subseteq ker\ \epsilon_G$ and $\kappa_G(\mathcal{I} \cap A(G)) \subseteq \mathcal{I}$.*

Example 1.3.4 Suppose $\mathcal{I}$ is a closed ideal in $C(G)$ such that $\mu_G(\mathcal{I}) \subseteq \mathcal{I} \otimes C(G) + C(G) \otimes \mathcal{I}$. Let $\mathcal{A}_1 = C(G)/\mathcal{I}$, and let p be the canonical projection of $C(G)$ onto $\mathcal{A}_1$. The above condition then implies that $(p \otimes p)\mu_G(a) = 0$ whenever $p(a) = 0$.

Therefore if we define a map $\mu_1 : \mathcal{A}_1 \to \mathcal{A}_1 \otimes \mathcal{A}_1$ by $\mu_1(p(a)) = (p \otimes p)\mu_G(a), a \in C(G)$, it is well-defined. It is in fact a unital *-homomorphism and

$$\begin{aligned}
(\mu_1 \otimes id)\mu_1 p &= (\mu_1 \otimes id)(p \otimes p)\mu_G \\
&= (p \otimes p \otimes p)(\mu_G \otimes id)\mu_G \\
&= (p \otimes p \otimes p)(id \otimes \mu_G)\mu_G \\
&= (id \otimes \mu_1)(p \otimes p)\mu_G \\
&= (id \otimes \mu_1)p.
\end{aligned}$$

Hence $(\mu_1 \otimes id)\mu_1 = (id \otimes \mu_1)\mu_1$ on $\mathcal{A}_1$. Next

$$\begin{aligned}
\{(a \otimes I)\mu_1(b) : a, b \in \mathcal{A}_1\} &= \{(p \otimes p)(a \otimes I)\mu_G(b) : a, b \in C(G)\} \\
&= (p \otimes p)\{(a \otimes I)\mu_G(b) : a, b \in C(G)\}.
\end{aligned}$$

and similarly $\{(I \otimes a)\mu_1(b) : a, b \in \mathcal{A}_1\} = (p \otimes p)\{(I \otimes a)\mu_G(b) : a, b \in C(G)\}$. Therefore $\{(a \otimes I)\mu_1(b) : a, b \in \mathcal{A}_1\}$ and $\{(I \otimes a)\mu_1(b) : a, b \in \mathcal{A}_1\}$ both are total in $\mathcal{A}_1 \otimes \mathcal{A}_1$. $G_1 = (\mathcal{A}_1, \mu_1)$ is thus a compact quantum group. Check that this is a subgroup of G.

Example 1.3.5 Take the *group* $H = \{e, x_1, x_2, x_3\}$, the group operation being given by the following multiplication table:

	e	x_1	x_2	x_3
e	e	x_1	x_2	x_3
x_1	x_1	e	x_3	x_2
x_2	x_2	x_3	e	x_1
x_3	x_3	x_2	x_1	e

Verify that $C(H) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$, and if e_k denotes $\delta_{1k} \oplus \delta_{2k} \oplus \delta_{3k} \oplus \delta_{4k}$, $k = 1, 2, 3, 4$, then the comultiplication μ is given by

$$\begin{aligned}
\mu(e_1) &= e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 + e_4 \otimes e_4, \\
\mu(e_2) &= e_1 \otimes e_2 + e_2 \otimes e_1 + e_3 \otimes e_4 + e_4 \otimes e_3, \\
\mu(e_3) &= e_1 \otimes e_3 + e_3 \otimes e_1 + e_2 \otimes e_4 + e_4 \otimes e_2, \\
\mu(e_4) &= e_1 \otimes e_4 + e_4 \otimes e_1 + e_2 \otimes e_3 + e_3 \otimes e_2.
\end{aligned}$$

It is now trivial to see that this is a subgroup of the compact quantum group appearing in example 1.1.3.

Example 1.3.6 The circle group S^1 is a subgroup of $SU_q(2)$. It is easy to see that $C(S^1)$ is a C^*-algebra generated by a single unitary element u with spectrum $\sigma(u) = S^1$. Define a map ϕ from $C(SU_q(2))$ into $C(S^1)$ by $\phi(\alpha) = u$, $\phi(\beta) = 0$. One can check that this is a quantum group homomorphism from $C(SU_q(2))$ *onto* $C(S^1)$.

Example 1.3.7 $SU_q(n)$ is a subgroup of $SU_q(n+1)$. Denote the generators of $C(SU_q(n))$ by $u_{ij}^{(n)}$, and those of $C(SU_q(n+1))$ by $u_{ij}^{(n+1)}$. Define a map ϕ on the generators as follows:

$$\phi(u_{ij}^{(n+1)}) = \begin{cases} u_{ij}^{(n)} & \text{if } 1 \leq i, j \leq n, \\ I & \text{if } i = j = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Verify that ϕ is the required homomorphism.

As in the case of a classical group, one also has the notions of quotient spaces, normal subgroups and quotient groups. Let H be a compact quantum subgroup of G, and let P be the homomorphism mapping $C(G)$ onto $C(H)$. The *right coset space* of H in G is given via the C^*-algebra of continuous functions on it which is defined to be

$$C(G/H) = \{a \in C(G) : (P \otimes id)\mu_G(a) = I \otimes a\}. \tag{1.3.1}$$

Let h_H denote the haar measure for H. It follows from the above that

$$\begin{aligned} C(G/H) &= \{a \in C(G) : h_H P * a = a\} \\ &= \{h_H P * a : a \in C(G)\} \end{aligned} \tag{1.3.2}$$

In a similar manner, the C^*-algebra of continuous functions on the *left coset space* of H in G is given by

$$\begin{aligned} C(G\backslash H) &= \{a \in C(G) : (id \otimes P)\mu_G(a) = a \otimes I\} \\ &= \{a \in C(G) : a * h_H P = a\} \\ &= \{a * h_H P : a \in C(G)\}. \end{aligned} \tag{1.3.3}$$

From the definition, it is clear that both $C(G/H)$ and $C(G\backslash H)$ are *unital* C^*-algebras. In the language of noncommutative topology, G/H and $G\backslash H$ are compact noncommutative spaces.

It follows from (1.3.2) that $\mu_G(C(G/H)) \subseteq C(G/H) \otimes C(G)$. Let us denote the restriction of μ_G to $C(G/H)$ by ν. Then ν is a unital $*$-homomorphism from $C(G/H)$ into $C(G/H) \otimes C(G)$, and satisfies $(\nu \otimes id)\nu = (id \otimes \mu_G)\nu$. We say that ν is a *right action* of G on G/H. More generally, we have the following definition.

Definition 1.3.8 Let $\mathcal{B}$ be a unital C^*-algebra. A unital $*$-homomorphism ν from $\mathcal{B}$ to $\mathcal{B} \otimes C(G)$ is called a *right action* of G on $\mathcal{B}$ if $(\nu \otimes id)\nu = (id \otimes \mu_G)\nu$. A *left action* of G on $\mathcal{B}$ is a unital $*$-homomorphism ν from $\mathcal{B}$ to $C(G) \otimes \mathcal{B}$ satisfying $(id \otimes \nu)\nu = (id \otimes \mu_G)\nu$.

If ν is a right action of G on $\mathcal{B}$, then the subalgebra $\{a \in \mathcal{B} : \nu(a) = a \otimes I\}$ of $\mathcal{B}$ is called the fixed point subalgebra for this action ν. For a left action ν, the fixed point subalgebra is $\{a \in \mathcal{B} : \nu(a) = I \otimes a\}$. An action ν is called homogeneous if its fixed point subalgebra is $\mathbb{C}$.

We have seen that G has a right action on G/H. Similarly one can verify that G has a left action on $G\backslash H$. Both these actions are homogeneous.

If for any $a \in C(G)$, we have $C(G/H) = C(G\backslash H)$, then we call H a *normal subgroup* of G. In this case, μ_G maps $C(G/H)$ into $C(G/H)\otimes C(G/H)$, and we have the following proposition.

Proposition 1.3.9 *If H is a normal subgroup of G, then $(C(G/H), \mu_G|_{C(G/H)})$ is a compact quantum group.*

Proof: All we need to show is that condition (ii) of definition 1.1.1 holds. Notice that $B := \{h_H P * a : a \in A(G)\}$ is dense in $C(G/H)$. It follows from the equality $C(G/H) = C(G\backslash H)$ that $B = \{a * h_H P : a \in A(G)\}$, and hence κ maps B onto itself. Using this, it is easy to show that the linear span of both $\{(a \otimes I)\mu(b) : a, b \in B\}$ and $\{(I \otimes a)\mu(b) : a, b \in B\}$ are $B \otimes_{alg} B$, which is dense in $C(G/H) \otimes C(G/H)$. The proof is thus complete. □

We denote the quantum group $(C(G/H), \mu_G|_{C(G/H)})$ by G/H and call it the *quotient group* of G over H.

Remark. All the notions introduced above coincide with the already existing ones in the case when $C(G)$ is commutative.

Example 1.3.10 Take G to be the quantum group in example 1.1.3, and H to be the group appearing in example 1.3.5. Verify that

$$C(G/H) = C(G\backslash H) = \{a(e_1 + e_2 + e_3 + e_4) + b(e_5 + e_6) : a, b \in \mathbb{C}\}.$$

Thus H is a normal subgroup of G, and from the above, one can check that G/H is actually the group $\mathbb{Z}_2$. One remarkable thing to notice here is that *both H and G/H are groups, although G is not.*

Example 1.3.11 We have observed in example 1.3.6 that S^1 is a subgroup of $SU_q(2)$. The homomorphism ϕ mapping $C(SU_q(2))$ onto $C(S^1)$ was also described there. It follows from there that $C(SU_q(2)/S^1)$ is the unital C^*-subalgebra of $C(SU_q(2))$ generated by the elements $\alpha\beta$ and $\beta^*\beta$. Similarly, $C(SU_q(2)\backslash S^1)$ is the unital C^*-subalgebra generated by $\alpha\beta^*$ and $\beta^*\beta$. Thus S^1 is *not* a normal subgroup of $SU_q(2)$.

Let c be a real number, and $\mathcal{A}$ be the canonical unital C^*-algebra generated by two elements ξ_1 and ξ_2 obeying the following relations:

$$\xi_1^*\xi_1 = \xi_2 - \xi_2^2 + cI, \ \ \xi_1\xi_1^* = q^2\xi_2 - q^4\xi_2^2 + cI, \ \ \xi_1\xi_2 = q^2\xi_2\xi_1, \ \ \xi_2^* = \xi_2.$$

The underlying noncommutative space is known as the Podleś sphere S^2_{qc} (see [40]). The C^*-algebra $C(SU_q(2)/S^1)$ in the above example can be shown to be isomorphic to S^2_{q0}. The C^*-algebra $C(S^2_{q0})$ can alternatively be described as the unital C^*-subalgebra of $\mathcal{B}(\ell_2)$ generated by the two operators $\xi_1 = \ell q^N \sqrt{1-q^{2N}}$ and $\xi_2 = q^{2N}$. Let

$$\begin{aligned}
\tilde{\xi_1} &= \xi_1 \otimes \alpha^2 - q\xi_1^* \otimes \beta^2 + (1-(1+q^2)\xi_2) \otimes \alpha\beta, \\
\tilde{\xi_2} &= \xi_1 \otimes \beta^*\alpha + \xi_1^* \otimes \alpha^*\beta + \xi_2 \otimes I + (1-(1+q^2)\xi_2) \otimes \beta^*\beta.
\end{aligned}$$

Then $\nu : \xi_1 \mapsto \tilde{\xi_1}$, $\xi_2 \mapsto \tilde{\xi_2}$ extends to a unital *-homomorphism from $C(S^2_{q0})$ to $C(S^2_{q0}) \otimes C(SU_q(2))$ satisfying $(\nu \otimes id)\nu = (id \otimes \mu)\nu$. That is, ν is a right action of $SU_q(2)$ on S^2_{q0}. Check that this action is homogeneous. Let λ denote the state $a \mapsto (1-q^2)\sum_{i=0}^{\infty} q^{2i}\langle e_i, ae_i\rangle$ on $C(S^2_{q0})$. Then one can show that $(\lambda \otimes id)\nu(a) = \lambda(a)I$ for any $a \in C(S^2_{q0})$. Thus λ is invariant under this action.

1.4 A Counterexample

In section 1.2 we have come across some properties of classical groups that lose their validity in the quantum situation. Here we describe yet another. If G is a locally compact group, and ν is an idempotent measure on G, that is, if ν satisfies the equation $\nu * \nu = \nu$, then the support H of ν is a compact subgroup of G, and moreover, ν is the haar measure of H. This no longer remains valid if G is a compact quantum group. To see this, take G to be the quantum group of example 1.1.3. Let us first find all the idempotent measures on G. Let $\rho_1, \rho_2, \ldots, \rho_8$ be as in example 1.2.2. Then it is not too difficult to show that $\rho_1, \rho_2, \ldots, \rho_6$, along with $\psi_1 = \frac{1}{2}(\rho_5+\rho_6+\rho_7+\rho_8)$ and $\psi_2 = \frac{1}{2}(\rho_5+\rho_6+i\rho_7-i\rho_8)$ are all states, and they span the space of all functionals on $\mathcal{A}$. Therefore any state ρ will be of the form

$$\begin{aligned}
\rho &= \sum_{i=1}^{6} c_i\rho_i + c_7\psi_1 + c_8\psi_2 \\
&= \sum_{i=1}^{4} c_i\rho_i + (c_5 + \frac{1}{2}(c_7+c_8))\rho_5 + (c_6 + \frac{1}{2}(c_7+c_8))\rho_6 \\
&\qquad + \frac{1}{2}(c_7+ic_8)\rho_7 + \frac{1}{2}(c_7-ic_8)\rho_8,
\end{aligned}$$

where $c_i \geq 0$ for all i, and $\sum_{i=1}^{8} c_i = 1$. Evaluating the two functionals $\rho * \rho$ and ρ at the basis elements and equating them, we get a system of equations which lead to the following possibilities:

a) $\rho = \rho_1$,
b) $\rho = \frac{1}{2}(\rho_1 + \rho_2)$,
c) $\rho = \frac{1}{2}(\rho_1 + \rho_3)$,
d) $\rho = \frac{1}{2}(\rho_1 + \rho_4)$,
e) $\rho = \frac{1}{4}(\rho_1 + \rho_2 + \rho_3 + \rho_4)$,
f) $\rho = \frac{1}{8}(\rho_1 + \rho_2 + \rho_3 + \rho_4) + \frac{1}{4}(\rho_5 + \rho_6)$,
g) $\rho = \frac{1}{4}(\rho_1 + \rho_4) + \frac{1}{2}\rho_5$,
h) $\rho = \frac{1}{4}(\rho_1 + \rho_4) + \frac{1}{2}\rho_6$.
These are all the idempotent states on $\mathcal{A}$. We now prove the following.

Proposition 1.4.1 *The state* $\rho = \frac{1}{4}(\rho_1 + \rho_4) + \frac{1}{2}\rho_6$ *is not the haar state of any subgroup of* G.

Proof: Suppose, if possible, $H = (C(H), \mu_H)$ is a subgroup of G and ρ is the haar state of H. This means that there is a unital $*$-homomorphism ϕ from $C(G)$ *onto* $C(H)$ obeying $(\phi \otimes \phi)\mu = \mu_H \phi$, and $\rho = h\phi$, where h is the haar state of $C(H)$. Let $\mathcal{I} = \{a \in C(G) : \rho(a^* a) = 0\}$. Using the modular properties of the haar state (part 4 of theorem 1.2.4), we find that $\mathcal{I}$ is a closed two-sided ideal. Now, observe that $e_7 \in \mathcal{I}$, but $e_7^* \notin \mathcal{I}$, which contradicts the fact that $\mathcal{I}$ is an ideal. $\square$

Chapter 2

Induced Representations

The concept of an induced representation plays an extremely important role in the representation theory of classical groups. For a large class of locally compact groups, for example, one can obtain families of irreducible representations as induced representations of one dimensional representations of appropriate subgroups. Quantum groups, just like their classical counterparts, have a very rich representation theory. Therefore it is natural to try and see how far can this concept be developed and exploited in the case of a quantum group. In this chapter, we attack this problem for compact quantum groups. The way we proceed is as follows. In section 2.2, we give an alternative description of a unitary representation as an isometric comodule, hoping to make things more transparent this way. Using this comodule description, we then introduce the concept of an induced representation in section 2.3, and prove an analogue of the Frobenius reciprocity theorem followed by an application in the last section. We shall need a little bit of the theory of Hilbert C^*-modules, which is presented in the first section.

2.1 Hilbert C^*-modules

The notion of a Hilbert C^*-module was introduced by Paschke ([33]) and was later developed by Kasparov ([25]) in the context of his KK-theory. For details, the reader should refer to Jensen & Thomsen ([19]).

Definition 2.1.1 Let $\mathcal{A}$ be a unital C^*-algebra. A vector space X having a right $\mathcal{A}$-module structure is called a *Hilbert $\mathcal{A}$-module* if it is equipped with an $\mathcal{A}$-valued inner product that satisfies

i. $\langle x,y\rangle^* = \langle y,x\rangle$,
ii. $\langle x,x\rangle \geq 0$,
iii. $\langle x,x\rangle = 0 \Rightarrow x = 0$,
iv. $\langle x,yb\rangle = \langle x,y\rangle b$ for $x,y \in X$, $b \in \mathcal{A}$,
and if $\|x\| := \|\langle x,x\rangle\|^{1/2}$ makes X a Banach Space.

A few examples are in order.

Example 2.1.2 (a) Any Hilbert space $\mathcal{H}$ with its usual inner product is a Hilbert $\mathbb{C}$-module.
(b) Any unital C^*-algebra $\mathcal{A}$, with $\langle a,b\rangle := a^*b$, is a Hilbert $\mathcal{A}$-module.

Example 2.1.3 Take X to be the Banach space $\mathcal{B}(\mathcal{H},\mathcal{K})$ of bounded operators from $\mathcal{H}$ to $\mathcal{K}$ and $\mathcal{A}$ to be the C^*-algebra $\mathcal{B}(\mathcal{H})$. Then X has a natural right $\mathcal{A}$-module structure. Define an inner product on X by: $\langle S,T\rangle := S^*T$. One can check that the natural norm of X coincides with the norm arising out of this inner product, and X is a Hilbert $\mathcal{A}$-module.

The next example is a rather crucial one for our purpose.

Example 2.1.4 Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A}$ be a unital C^*-algebra. On their algebraic tensor product $X_0 = \mathcal{H} \otimes_{alg} \mathcal{A}$, define a right $\mathcal{A}$-module structure and an $\mathcal{A}$-valued inner product as follows:

$$\begin{aligned} (\textstyle\sum_i u_i \otimes a_i)a &:= \textstyle\sum_i u_i \otimes a_i a, \quad u_i \in \mathcal{H}, a, a_i \in \mathcal{A}, \\ \langle \textstyle\sum_i u_i \otimes a_i, \sum_j v_i \otimes b_i\rangle &:= \textstyle\sum \langle u_i, v_j\rangle {a_i}^* b_j, \quad u_i, v_j \in \mathcal{H}, a_i, b_j \in \mathcal{A}. \end{aligned} \tag{2.1.1}$$

It is easy to see that $\|\sum_i u_i \otimes a_i\| := \|\langle \sum_i u_i \otimes a_i, \sum_i u_i \otimes a_i\rangle\|^{1/2}$ defines a norm on X_0, and both the right $\mathcal{A}$-module structure and the $\mathcal{A}$-valued inner product extends to the completion X of X_0 with respect to this norm, and they make X a Hilbert $\mathcal{A}$-module. We call X the *external tensor product* of $\mathcal{H}$ and $\mathcal{A}$, and denote it by $\mathcal{H} \otimes \mathcal{A}$.

Remark. For $\mathcal{A} = C(G)$, G being a compact quantum group, $\mathcal{H} \otimes \mathcal{A}$ will play the role of the space of $\mathcal{H}$-valued continuous functions on G.

Example 2.1.5 The construction in the above example can be generalised a little further. Instead of $\mathcal{H}$ and $\mathcal{A}$, we take a Hilbert $\mathcal{A}$-module X and a unital C^*-algebra

$\mathcal{B}$, then on their algebraic tensor product $X \otimes_{alg} \mathcal{B}$, one can define a right $\mathcal{A} \otimes_{alg} \mathcal{B}$-module structure, and a $\mathcal{A} \otimes \mathcal{B}$-valued inner product by:

$$\begin{aligned} (\sum x_i \otimes b_i)(a \otimes b) &:= \sum x_i a \otimes b_i b, \\ \langle \sum x_i \otimes b_i, \sum x_i' \otimes b_i' \rangle &:= \sum \langle x_i, x_j' \rangle \otimes \langle b_i, b_j' \rangle, \end{aligned} \tag{2.1.2}$$

where $x_i, x_i' \in X$, $b, b_i, b_i' \in \mathcal{B}$ and $a \in \mathcal{A}$. As before, if one defines now $\| \sum x_i \otimes b_i \| = \| \langle \sum x_i \otimes b_i, \sum x_i \otimes b_i \rangle \|^{1/2}$, then this gives a norm on $X \otimes_{alg} \mathcal{B}$. Denote the completion with respect to this norm by $X \otimes \mathcal{B}$. The inner product extends to $X \otimes \mathcal{B}$ and the right $\mathcal{A} \otimes_{alg} \mathcal{B}$-module structure also extends to give $X \otimes \mathcal{B}$ the structure of a Hilbert $\mathcal{A} \otimes \mathcal{B}$-module. We call this the external tensor product of X and $\mathcal{B}$.

Let us now state a couple of lemmas involving the Hilbert $\mathcal{A}$-module $\mathcal{H} \otimes \mathcal{A}$ which we shall need subsequently.

Lemma 2.1.6 *Let $\{e_i\}$ be an orthonormal basis for $\mathcal{H}$, and x be an element of $\mathcal{H} \otimes \mathcal{A}$. Let $x_i = (\langle e_i | \otimes id)x$. Then $\sum x_i{}^* x_i$ converges in norm and $x = \lim_{n \to \infty} \sum_{i=1}^n e_i \otimes x_i$.*

Proof: Straightforward. □

Lemma 2.1.7 *Let $\mathcal{H}_0$ be a closed subspace of $\mathcal{H}$ and $\mathcal{B}$, a C^*-subalgebra of $\mathcal{A}$. Let $x \in \mathcal{H} \otimes \mathcal{A}$. Then we have*
i. *If $(\langle u | \otimes id)x \in \mathcal{B}\ \forall u \in \mathcal{H}$, then $x \in \mathcal{H} \otimes \mathcal{B}$; and*
ii. *If $(I \otimes \rho)x \in \mathcal{H}_0$ for all continuous linear functionals ρ on $\mathcal{A}$, then $x \in \mathcal{H}_0 \otimes \mathcal{A}$.*

Proof: Choose an orthonormal basis $\{e_i\}_{i \text{ even}}$ for $\mathcal{H}_0$. Extend it to an orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$ for $\mathcal{H}$. Now if $(\langle u | \otimes id)x \in \mathcal{B}$ for all $u \in \mathcal{H}$, then each x_i belongs to $\mathcal{B}$, so that $x \in \mathcal{H} \otimes \mathcal{B}$.

For the second part, notice that for any continuous linear functional ρ on $\mathcal{A}$, $\rho((\langle u | \otimes id)x) = \langle u, (I \otimes \rho)x \rangle = 0$ whenever $u \perp \mathcal{H}_0$. Therefore $(\langle u | \otimes id)x = 0$ for $u \perp \mathcal{H}_0$. Hence $x_i = 0$ if i is odd. This means that all the summands $\sum_1^n e_i \otimes x_i$ belong to $\mathcal{H}_0 \otimes \mathcal{A}$. Hence $x \in \mathcal{H}_0 \otimes \mathcal{A}$. □

We have seen that both $\mathcal{B}(\mathcal{K}, \mathcal{H} \otimes \mathcal{K})$ and $\mathcal{H} \otimes \mathcal{B}(\mathcal{K})$ are Hilbert $\mathcal{B}(\mathcal{K})$-modules. We now establish an isometric module map ϑ from $\mathcal{H} \otimes \mathcal{B}(\mathcal{K})$ into $\mathcal{B}(\mathcal{K}, \mathcal{H} \otimes \mathcal{K})$ through which one can embed $\mathcal{H} \otimes \mathcal{B}(\mathcal{K})$ in the latter.

Take $\sum u_i \otimes a_i \in \mathcal{H} \otimes_{alg} \mathcal{B}(\mathcal{K})$. Define an operator $\vartheta(\sum u_i \otimes a_i)$ from $\mathcal{K}$ to $\mathcal{H} \otimes \mathcal{K}$ by the prescription $\vartheta(\sum u_i \otimes a_i)(v) = \sum u_i \otimes a_i(v)$, $v \in \mathcal{K}$. A simple calculation shows that $\langle\vartheta(x),\vartheta(y)\rangle = \langle x,y\rangle$ for all $x,y \in \mathcal{H}\otimes_{alg}\mathcal{B}(\mathcal{K})$, and consequently $\|\vartheta(x)\| = \|x\|$ for all $x \in \mathcal{H}\otimes_{alg}\mathcal{B}(\mathcal{K})$. The map ϑ, therefore, extends to a bounded operator from $\mathcal{H}\otimes\mathcal{B}(\mathcal{K})$ to $\mathcal{B}(\mathcal{K},\mathcal{H}\otimes\mathcal{K})$, and we have

$$\begin{aligned} \langle\vartheta(x),\vartheta(y)\rangle &= \langle x,y\rangle \quad \forall x,y \in \mathcal{H}\otimes\mathcal{B}(\mathcal{K}), \\ \vartheta(xb) &= \vartheta(x)b \quad \forall x \in \mathcal{H}\otimes\mathcal{B}(\mathcal{K}),\ b \in \mathcal{B}(\mathcal{K}). \end{aligned} \tag{2.1.3}$$

Observe two things here: first, if $\mathcal{H} = \mathbb{C}$, ϑ is just the identity map. And, ϑ is *onto* if and only if $\mathcal{H}$ is finite dimensional. The following lemma, the proof of which is fairly straightforward, gives a very useful property of ϑ.

Lemma 2.1.8 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. Let ϑ_i be the map ϑ constructed above with $\mathcal{H}_i$ replacing $\mathcal{H}$, $i = 1,2$. Let $S \in \mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$ and $x \in \mathcal{H}_1\otimes\mathcal{B}(\mathcal{K})$. Then $\vartheta_2((S\otimes id)x) = (S\otimes I)\vartheta_1(x)$.*

We now use this map ϑ to introduce another map Ψ and study a few of its properties.

Proposition 2.1.9 *Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces, and ϑ be the isometry from $\mathcal{H}\otimes\mathcal{B}(\mathcal{K})$ to $\mathcal{B}(\mathcal{K},\mathcal{H}\otimes\mathcal{K})$ as constructed above. Then there is a unique, linear, injective contraction Ψ from the left multiplier algebra $LM(\mathcal{B}_0(\mathcal{H})\otimes\mathcal{B}(\mathcal{K}))$ of $\mathcal{B}_0(\mathcal{H})\otimes\mathcal{B}(\mathcal{K})$ to the space $\mathcal{B}(\mathcal{H},\mathcal{H}\otimes\mathcal{B}(\mathcal{K}))$ of bounded operators from $\mathcal{H}$ to $\mathcal{H}\otimes\mathcal{B}(\mathcal{K})$ satisfying the following:*

$$\vartheta(\Psi(T)(u))(v) = T(u\otimes v) \ \ \forall u \in \mathcal{H}, v \in \mathcal{K},\ \ T \in LM(\mathcal{B}_0(\mathcal{H})\otimes\mathcal{B}(\mathcal{K})). \tag{2.1.4}$$

Proof: Take a $T \in \mathcal{B}_f(\mathcal{H})\otimes_{alg}\mathcal{B}(\mathcal{K})$, $\mathcal{B}_f(\mathcal{H})$ being the space of all finite-rank operators. Such a T can be written in the form $\sum|e_i\rangle\langle e_j|\otimes T_{ij}$, where $\{e_i\}$ is an orthonormal basis for $\mathcal{H}$, T_{ij}'s$\in\mathcal{B}(\mathcal{K})$, and all but finitely many of them are zero. Consider now the map $e_k \mapsto \sum e_i\otimes T_{ik}$ from the linear span of the e_i's to $\mathcal{H}\otimes\mathcal{B}(\mathcal{K})$. The following calculation shows that it is bounded:

$$\begin{aligned} &\sup_{\sum|\alpha_k|^2\le 1}\Big\|\sum_{k,l}\overline{\alpha}_k\alpha_l\sum_i {T_{ik}}^*T_{il}\Big\| \\ &= \sup_{\sum|\alpha_k|^2\le 1}\ \sup_{\substack{\|v\|\le 1\\ \|w\|\le 1}}\Big|\langle v,\sum_{k,l,i}\overline{\alpha}_k\alpha_l{T_{ik}}^*T_{il}(w)\rangle\Big| \\ &= \sup_{\sum|\alpha_k|^2\le 1}\ \sup_{\substack{\|v\|\le 1\\ \|w\|\le 1}}\Big|\sum_i\Big\langle\sum_k\alpha_kT_{ik}(v),\sum_l\alpha_lT_{il}(w)\Big\rangle\Big| \end{aligned}$$

$$\begin{aligned}
&= \sup_{\sum|\alpha_k|^2\le 1}\ \sup_{\substack{\|v\|\le 1\\ \|w\|\le 1}} \Big|\Big\langle \sum_i e_i\otimes\sum_k \alpha_k T_{ik}(v), \sum e_i\otimes\sum_l \alpha_l T_{il}(w)\Big\rangle\Big| \\
&\le \sup\Big\{\Big|\Big\langle T(\sum\alpha_k e_k\otimes v), T(\sum\alpha_k e_k\otimes w)\Big\rangle\Big| : \|v\|^2\le 1, \|w\|^2\le 1, \sum|\alpha_k|^2\le 1\Big\} \\
&\le \|T\|^2.
\end{aligned}$$

Observe that all the summations involved here are finite, so that convergence problems do not arise.

The map therefore extends to the whole of $\mathcal{H}$. Call this map $\Psi(T)$. The above calculation shows that

$$\|\Psi(T)\| \le \|T\|. \tag{2.1.5}$$

Also, it is easy to see that for $u\in\mathcal{H}$, $v\in\mathcal{K}$,

$$\vartheta\Big(\Psi(T)(u)\Big)(v) = T(u\otimes v). \tag{2.1.6}$$

It is clear from (2.1.6) that $\Psi(T)$ does not depend on the choice of the particular basis.

From (2.1.5) it follows that the map $T\mapsto\Psi(T)$ extends to the closure $\mathcal{B}_0(\mathcal{H})\otimes\mathcal{B}(\mathcal{K})$ of $\mathcal{B}_f(\mathcal{H})\otimes_{alg}\mathcal{B}(\mathcal{K})$, and (2.1.5) and (2.1.6) still hold.

Finally, take a $T\in LM(\mathcal{B}_0(\mathcal{H})\otimes\mathcal{B}(\mathcal{K}))$. Let P_n be a sequence of finite-rank projections strongly converging to I. Then $T^{(n)} := T(P_n\otimes I)\in\mathcal{B}_0(\mathcal{H})\otimes\mathcal{B}(\mathcal{K})$, so that $\vartheta(\Psi(T^{(n)})(u))(v) = T^{(n)}(u\otimes v)\ \ \forall u\in\mathcal{H}, v\in\mathcal{K}$. For an operator S on $\mathcal{H}\otimes\mathcal{K}$, and $u\in\mathcal{H}$, let S_u denote the operator $v\mapsto S(u\otimes v)$ from $\mathcal{K}$ to $\mathcal{H}\otimes\mathcal{K}$. Then we have $\vartheta(\Psi(T^{(n)})(u)) = T_u^{(n)}$. Now

$$\begin{aligned}
\|T_u - T_u^{(n)}\| &= \sup_{\|v\|\le 1}\|T(u\otimes v) - T^{(n)}(u\otimes v)\| \\
&= \sup_{\|v\|\le 1}\|T((u-P_n u)\otimes v)\| \\
&\le \|T\|\,\|u-P_n u\|.
\end{aligned}$$

Hence T_u is the norm limit of $T_u^{(n)}$. Since $T_u^{(n)}\in\operatorname{range}\vartheta$ for all n, T_u also belongs to $\operatorname{range}\vartheta$. Define $\Psi(T)$ by the following:

$$\Psi(T)(u) = \vartheta^{-1}(T_u),\quad u\in\mathcal{H}.$$

One can verify now that Ψ is well-defined, linear, injective and satisfies (2.1.4) and (2.1.5). Uniqueness follows from (2.1.4). □

Proposition 2.1.10 *Let* $\Psi : LM(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})) \to \mathcal{B}(\mathcal{H}, \mathcal{H} \otimes \mathcal{B}(\mathcal{K}))$ *be the map constructed in proposition 2.1.9. Then we have the following:*
i. Ψ *maps isometries in* $LM(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}))$ onto *the isometries in* $\mathcal{B}(\mathcal{H}, \mathcal{H} \otimes \mathcal{B}(\mathcal{K}))$.
ii. *For any* $T \in LM(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}))$ *and* $S \in \mathcal{B}_0(\mathcal{H})$,

$$\Psi(T(S \otimes I)) = \Psi(T) \circ S, \quad \Psi((S \otimes I)T) = (S \otimes id) \circ \Psi(T).$$

iii. *If* $\mathcal{A}$ *is any* C^**-subalgebra of* $\mathcal{B}(\mathcal{K})$ *containing its identity, then* $T \in LM(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A})$ *if and only if* range $\Psi(T) \subseteq \mathcal{H} \otimes \mathcal{A}$.

Proof: i. Suppose $T \in LM(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}))$ is an isometry. By (2.1.3), $\langle \Psi(T)u, \Psi(T)v\rangle = \langle \vartheta^{-1}(T_u), \vartheta^{-1}(T_v)\rangle = \langle T_u, T_v\rangle = \langle u, v\rangle I$ for $u, v \in \mathcal{H}$. Thus $\Psi(T)$ is an isometry.

Conversely, take an isometry $\pi : \mathcal{H} \to \mathcal{H} \otimes \mathcal{B}(\mathcal{K})$ and define an operator T on the product vectors in $\mathcal{H} \otimes \mathcal{K}$ by $T(u \otimes v) = \vartheta(\pi(u))(v)$, ϑ being the map constructed prior to lemma 2.1.8. It is clear that T is an isometry. It is enough, therefore to show that $T(|u\rangle\langle v| \otimes S) \in \mathcal{B}_0(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$ whenever $S \in \mathcal{B}(\mathcal{K})$ and u, v are unit vectors in $\mathcal{H}$ such that $\langle u, v\rangle = 0$ or 1.

Choose an orthonormal basis $\{e_i\}$ for $\mathcal{H}$ such that $e_1 = u$, $e_r = v$ where

$$r = \begin{cases} 1 & \text{if } \langle u, v\rangle = 1, \\ 2 & \text{if } \langle u, v\rangle = 0. \end{cases}$$

Let $\pi_{ij} = (\langle e_i| \otimes id)\pi(e_j)$. Then $T(|u\rangle\langle v| \otimes S) = \sum |e_i\rangle\langle e_r| \otimes \pi_{i1}S$ where the right hand side converges strongly. Since $\pi(e_1) \in \mathcal{H} \otimes \mathcal{B}(\mathcal{K})$, lemma 2.1.6 tells us that $\sum_i {\pi_{i1}}^*\pi_{i1}$ converges in norm. Consequently the right hand side above converges in norm, which means $T(|u\rangle\langle v| \otimes S) \in \mathcal{B}_0(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$.

ii. Straightforward, from (2.1.6) and proposition 2.1.9.

iii. Take $T = |u\rangle\langle v| \otimes a$, $u, v \in \mathcal{H}$, $a \in \mathcal{A}$. For any $w \in \mathcal{H}$, $\Psi(T)(w) = \langle v, w\rangle u \otimes a \in \mathcal{H} \otimes \mathcal{A}$. Since Ψ is a contraction, and the norm closure of all linear combinations of such T's is $\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}$, we have range $\Psi(T) \subseteq \mathcal{H} \otimes \mathcal{A}$ for all $T \in \mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}$.

Assume next that $T \in LM(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A})$. Then $T(|u\rangle\langle u| \otimes I) \in \mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}$ for all $u \in \mathcal{H}$. Hence $\Psi(T(|u\rangle\langle u| \otimes I))(u) \in \mathcal{H} \otimes \mathcal{A}$, which means, by part (ii), that $\Psi(T)(u) \in \mathcal{H} \otimes \mathcal{A}$ for all $u \in \mathcal{H}$. Thus range $\Psi(T) \subseteq \mathcal{H} \otimes \mathcal{A}$.

To prove the converse, it is enough to show that $T(|u\rangle\langle v| \otimes a) \in \mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}$ whenever $a \in \mathcal{A}$ and $u, v \in \mathcal{H}$ are such that $\langle u, v\rangle = 0$ or 1. Rest of the proof now goes along the same lines as the proof of the last part of (i). $\square$

Before going to the next proposition, let us make the following observation.

Let $\mathcal{K}_1$, $\mathcal{K}_2$ be two Hilbert spaces, $\mathcal{A}_i$ being a C^*-subalgebra of $\mathcal{B}(\mathcal{K}_i)$ containing its identity. Suppose ϕ is a unital *-homomorphism from $\mathcal{A}_1$ to $\mathcal{A}_2$. Then $id \otimes \phi : S \otimes a \mapsto S \otimes \phi(a)$ extends to a *-homomorphism from $\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}_1$ to $\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}_2$. Moreover the linear span of $\{ ((id \otimes \phi)(a))b : a \in \mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}_1, b \in \mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}_2\}$ is dense in $\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}_2$. Therefore $id \otimes \phi$ extends to an algebra homomorphism from $LM(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}_1)$ to $LM(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}_2)$ by the following prescription: for all $a \in LM(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}_1)$, $b \in \mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}_1$, $c \in \mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}_2$,

$$\Big((id \otimes \phi)a\Big)\Big((id \otimes \phi)b\Big)c := \Big((id \otimes \phi)(ab)\Big)c.$$

Proposition 2.1.11 *Let ϕ be as above, and Ψ_i be the map Ψ constructed earlier with $\mathcal{K}_i$ replacing $\mathcal{K}$. Then for $T \in LM(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}_1)$,*

$$(I \otimes \phi)\Psi_1(T) = \Psi_2\Big((id \otimes \phi)T\Big).$$

Proof: It is enough to prove that

$$(\langle u| \otimes id)\Big((I \otimes \phi)\Psi_1(T)(v)\Big) = (\langle u| \otimes id)\Psi_2\Big((id \otimes \phi)T\Big)(v), \quad \forall\, u, v \in \mathcal{H}.$$

Rest now is a careful application of lemma 2.1.8. $\square$

Consider the homomorphic embeddings $\phi_{12} : \mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}_1 \to \mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}_1 \otimes \mathcal{A}_2$ and $\phi_{13} : \mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}_2 \to \mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}_1 \otimes \mathcal{A}_2$ given on the product elements by

$$\phi_{12}(a \otimes b) = a \otimes b \otimes I, \quad \phi_{13}(a \otimes c) = a \otimes I \otimes c.$$

Each of their ranges contains an approximate identity for $\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}_1 \otimes \mathcal{A}_2$, so that their extensions respectively to $LM(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}_1)$ and $LM(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}_2)$ are also homomorphic embeddings.

Proposition 2.1.12 *Let Ψ_1, Ψ_2 be as in the previous proposition, and let Ψ_0 be the map Ψ with $\mathcal{A}_1 \otimes \mathcal{A}_2$ replacing $\mathcal{A}$. Let $S \in LM(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}_1)$, $T \in LM(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}_2)$. Then*

$$\Psi_0(\phi_{12}(S)\phi_{13}(T)) = (\Psi_1(S) \otimes id)\Psi_2(T).$$

Proof: Observe that for $u_1, \ldots, u_n \in \mathcal{H}$,

$$\Big(\Big(\langle \Psi_1(S)(u_i), \Psi_1(S)(u_j)\rangle\Big)\Big) \leq \|S\|^2 \,((\langle u_i, u_j\rangle I)).$$

Therefore $\Psi_1(S)\otimes id$ is a well-defined bounded operator from $\mathcal{H}\otimes\mathcal{A}_2$ to $\mathcal{H}\otimes\mathcal{A}_1\otimes\mathcal{A}_2$. Take an orthonormal basis $\{e_i\}$ for $\mathcal{H}$. Define S_{ij}'s and T_{ij}'s as follows:

$$S_{ij} : v \mapsto (\langle e_i| \otimes I)S(e_j \otimes v), \;\; T_{ij} : v \mapsto (\langle e_i| \otimes I)T(e_j \otimes v).$$

Let $P_n := \sum_{i=1}^n |e_i\rangle\langle e_i|$. Then

$$\begin{aligned}(\Psi_1(S) \otimes id)(P_n \otimes id)\Psi_2(T)(e_i) &= (\Psi_1(S) \otimes id)\Big(\sum_{j\le n} e_j \otimes T_{ij}\Big)\\ &= \sum_{j\le n}\Big(\sum_k e_k \otimes S_{kj}\Big) \otimes T_{ij}.\end{aligned}$$

Hence for $v \in \mathcal{K}_1$, $w \in \mathcal{K}_2$,

$$\begin{aligned}&\vartheta\Big((\Psi_1(S) \otimes id)(P_n \otimes id)\Psi_2(T)(e_i)\Big)(v \otimes w)\\ &= \sum_{j\le n}\sum_k e_k \otimes S_{kj}(v) \otimes T_{ji}(w)\\ &= \Big(\sum_{j\le n}\sum_{k,r} |e_k\rangle\langle e_r| \otimes S_{kj} \otimes T_{ji}\Big)(e_i \otimes v \otimes w)\\ &= \phi_{12}(S)(P_n \otimes I \otimes I)\phi_{13}(T)(e_i \otimes v \otimes w).\end{aligned}$$

This converges to $\phi_{12}(S)\phi_{13}(T)(e_i \otimes v \otimes w)$ as $n \to \infty$. On the other hand,

$$\lim_{n\to\infty}(\Psi_1(S) \otimes id)(P_n \otimes id)\Psi_2(T)(e_i) = (\Psi_1(S) \otimes id)\Psi_2(T)(e_i),$$

which implies $\lim_{n\to\infty} \vartheta((\Psi_1(S)\otimes id)(P_n\otimes id)\Psi_2(T)(e_i)) = \vartheta((\Psi_1(S)\otimes id)\Psi_2(T)(e_i))$. Therefore

$$\begin{aligned}\vartheta\Big((\Psi_1(S) \otimes id)\Psi_2(T)(e_i)\Big)(v \otimes w) &= \phi_{12}(S)\phi_{13}(T)(e_i \otimes v \otimes w)\\ &= \vartheta\Big(\Psi_0\Big(\phi_{12}(S)\phi_{13}(T)\Big)(e_i)\Big)(v \otimes w).\end{aligned}$$

Thus $(\Psi_1(S) \otimes id)\Psi_2(T) = \Psi_0(\phi_{12}(S)\phi_{13}(T))$. □

2.2 Isometric Comodules

Let $G = (\mathcal{A}, \mu)$ be a compact quantum group. Throughout this section we shall assume that $\mathcal{A}$ acts nondegenerately on a Hilbert space $\mathcal{K}$, i.e. $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{K})$ containing its identity. We call a map π from $\mathcal{H}$ to $\mathcal{H} \otimes \mathcal{A}$ an *isometry* if $\langle\pi(u), \pi(v)\rangle = \langle u, v\rangle I$ for all $u, v \in \mathcal{H}$. If $\pi : \mathcal{H} \to \mathcal{H} \otimes \mathcal{A}$ is an isometry, then

$\pi \otimes id : u \otimes a \mapsto \pi(u) \otimes a$ extends to a bounded map from $\mathcal{H} \otimes \mathcal{A}$ to $\mathcal{H} \otimes \mathcal{A} \otimes \mathcal{A}$. π is called an *isometric comodule map* if it is an isometry, and satisfies $(\pi \otimes id)\pi = (I \otimes \mu)\pi$. The pair $(\mathcal{H}, \pi)$ is called an *isometric comodule*. We shall often just say π is a comodule, omitting the $\mathcal{H}$.

A closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ is said to be *invariant* under π if $\pi(\mathcal{H}_0) \subseteq \mathcal{H}_0 \otimes \mathcal{A}$. π is called *irreducible* if it does not have any nontrivial invariant subspace. If π is an isometric comodule, so is its restriction to any invariant subspace.

Let π_1 and π_2 be two isometric comodule maps acting on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. A bounded map $T : \mathcal{H}_1 \to \mathcal{H}_2$ is said to *intertwine* π_1 and π_2 if $\pi_2 T = (T \otimes id)\pi_1$. π_1 and π_2 are *equivalent* if there is a bounded invertible intertwiner. They are called *disjoint* if there is no nonzero intertwiner. It is easy to see that any two irreducible isometric comodules are either equivalent or disjoint.

We have seen in section 1.1 that any unitary representations decompose into a direct sum of finite dimensional irreducible unitary representations. The following proposition is a similar statement about isometric comodules.

Proposition 2.2.1 *Let $(\mathcal{H}, \pi)$ be an isometric comodule. Then $\mathcal{H}$ decomposes into a direct sum of finite dimensional subspaces $\mathcal{H} = \oplus \mathcal{H}_\alpha$ such that each $\mathcal{H}_\alpha$ is π-invariant and $\pi|_{\mathcal{H}_\alpha}$ is an irreducible isometric comodule.*

Proof: By proposition 2.1.10, there is an isometry $\hat{\pi}$ in $LM(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A})$ such that $\Psi(\hat{\pi}) = \pi$. Using propositions 2.1.11 and 2.1.12, we get $\hat{\pi}_{12}\hat{\pi}_{13} = (id \otimes \mu)\hat{\pi}$ where $\hat{\pi}_{12} = \phi_{12}(\hat{\pi})$, $\hat{\pi}_{13} = \phi_{13}(\hat{\pi})$, ϕ_{12} and ϕ_{13} being as in proposition 2.1.12 with $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$.

Let $\mathcal{I} = \{a \in \mathcal{A} : h(a^*a) = 0\}$. From the properties of the haar state, $\mathcal{I}$ is an ideal in $\mathcal{A}$. For any unit vector u in $\mathcal{H}$, let $Q(u) = (id \otimes h)(\hat{\pi}(|u\rangle\langle u| \otimes I)\hat{\pi}^*)$. Then $Q(u)^* = Q(u) \in \mathcal{B}_0(\mathcal{H})$. If $Q(u) = 0$, then $|\hat{\pi}(|u\rangle\langle u| \otimes I)\hat{\pi}^*|^{1/2} \in \mathcal{B}_0(\mathcal{H}) \otimes \mathcal{I}$. Therefore $\hat{\pi}(|u\rangle\langle u| \otimes I)\hat{\pi}^* \in \mathcal{B}_0(\mathcal{H}) \otimes \mathcal{I}$. It follows then that $|u\rangle\langle u| \otimes I \in \mathcal{B}_0(\mathcal{H}) \otimes \mathcal{I}$. This forces u to be zero. Thus for a nonzero u, $Q(u) \neq 0$. Choose and fix any nonzero unit vector u. Then

$$\begin{aligned}
&\hat{\pi}(Q(u) \otimes I)\hat{\pi}^* \\
&\quad = \quad (id \otimes id \otimes h)\Big(\hat{\pi}_{12}\hat{\pi}_{13}(|u\rangle\langle u| \otimes I \otimes I)\hat{\pi}_{13}^*\hat{\pi}_{12}^*\Big) \\
&\quad = \quad (id \otimes id \otimes h)\Big(\hat{\pi}_{12}\hat{\pi}_{13}(|u\rangle\langle u| \otimes I \otimes I)\Big(\hat{\pi}_{12}\hat{\pi}_{13}(|u\rangle\langle u| \otimes I \otimes I)\Big)^*\Big) \\
&\quad = \quad (id \otimes id \otimes h)\Big((id \otimes \mu)(\hat{\pi})(id \otimes \mu)(|u\rangle\langle u| \otimes I)
\end{aligned}$$

$$
\begin{aligned}
&\times\Big((id\otimes\mu)(\hat{\pi})(id\otimes\mu)(|u\rangle\langle u|\otimes I)\Big)^*\Big)\\
=\;& (id\otimes id\otimes h)\Big((id\otimes\mu)\Big(\hat{\pi}(|u\rangle\langle u|\otimes I)\Big)\Big((id\otimes\mu)\Big(\hat{\pi}(|u\rangle\langle u|\otimes I)\Big)\Big)^*\Big)\\
=\;& (id\otimes id\otimes h)\Big((id\otimes\mu)\Big(\hat{\pi}(|u\rangle\langle u|\otimes I)\Big)(id\otimes\mu)\Big((|u\rangle\langle u|\otimes I)\hat{\pi}^*\Big)\Big)\\
=\;& (id\otimes id\otimes h)\Big((id\otimes\mu)\Big(\hat{\pi}(|u\rangle\langle u|\otimes I)\hat{\pi}^*\Big)\Big)\\
=\;& \Big(id\otimes(id\otimes h)\mu\Big)\Big(\hat{\pi}(|u\rangle\langle u|\otimes I)\hat{\pi}^*\Big)\\
=\;& Q(u)\otimes I.
\end{aligned}
$$

Thus $\hat{\pi}(Q(u)\otimes I)=(Q(u)\otimes I)\hat{\pi}$. If P is any finite dimensional spectral projection of $Q(u)$, then $\hat{\pi}(P\otimes I)=(P\otimes I)\hat{\pi}$, which means, by an application of part (ii) of proposition 2.1.10, that $\pi P=(P\otimes id)\pi$. Standard arguments now tell us that π can be decomposed into a direct sum of finite dimensional isometric comodules. Finite dimensional comodules, in turn, can easily be shown to decompose into a direct sum of irreducible isometric comodules. The proof is thus complete. □

Using the above proposition, we can now establish the equivalence between unitary representations and isometric comodules.

Theorem 2.2.2 *Let π be an isometric comodule map acting on $\mathcal{H}$. Then $\Psi^{-1}(\pi)$ is a unitary representation acting on $\mathcal{H}$. Conversely, if $\hat{\pi}$ is a unitary representation of G on $\mathcal{H}$, then $(\mathcal{H},\Psi(\hat{\pi}))$ is an isometric comodule.*

Proof: Let $\hat{\pi}$ be a unitary representation. By proposition 2.1.10, $\Psi(\hat{\pi})$ is an isometry from $\mathcal{H}$ to $\mathcal{H}\otimes C(G)$. Using Propositions 2.1.11 and 2.1.12, we conclude that $\Psi(\hat{\pi})$ is an isometric comodule.

For the converse, take an isometric comodule π. If π is finite dimensional, it is easy to see that $\Psi^{-1}(\pi)$ is a unitary representation. So assume that π is infinite dimensional. By the lemma above, there is a family $\{P_\alpha\}$ of finite dimensional projections in $\mathcal{B}(\mathcal{H})$ satisfying

$$P_\alpha P_\beta=\delta_{\alpha\beta}P_\alpha,\;\; \sum P_\alpha=I,\;\; \pi P_\alpha=(P_\alpha\otimes id)\pi\;\;\forall\alpha \tag{2.2.1}$$

such that $\pi|_{P_\alpha\mathcal{H}}=\pi P_\alpha$ is an irreducible isometric comodule. $\pi|_{P_\alpha\mathcal{H}}$ is finite dimensional, therefore $\Psi^{-1}(\pi|_{P_\alpha\mathcal{H}})$ is a unitary element of $LM(\mathcal{B}_0(P_\alpha\mathcal{H})\otimes\mathcal{A})=\mathcal{B}(P_\alpha\mathcal{H})\otimes\mathcal{A}$. Let us denote $\Psi^{-1}(\pi)$ by $\hat{\pi}$. Then the above implies that in the bigger space $\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$,

$$(\hat{\pi}(P_\alpha\otimes I))^*(\hat{\pi}(P_\alpha\otimes I)) = P_\alpha\otimes I = (\hat{\pi}(P_\alpha\otimes I))(\hat{\pi}(P_\alpha\otimes I))^*.$$

The second equality implies that $\hat{\pi}(P_\alpha\otimes I)\hat{\pi}^* = P_\alpha\otimes I$ for all α, so that $\hat{\pi}\hat{\pi}^* = I$. We already know from proposition 2.1.10 that $\hat{\pi}^*\hat{\pi} = I$ and from propositions 2.1.11 and 2.1.12 that $\hat{\pi}_{12}\hat{\pi}_{13} = (id\otimes\mu)\hat{\pi}$. Thus it remains only to show that $\hat{\pi}\in M(\mathcal{B}_0(\mathcal{H})\otimes\mathcal{A})$. It is enough to show that for any $S\in\mathcal{B}_0(\mathcal{H})$ and $a\in\mathcal{A}$, $(S\otimes a)\hat{\pi}\in\mathcal{B}_0(\mathcal{H})\otimes\mathcal{A}$. Now from (2.2.1) and proposition 2.1.10, $\hat{\pi}(P_\alpha\otimes I) = (P_\alpha\otimes I)\hat{\pi}$ for all α. Therefore $(S\otimes a)(P_\alpha\otimes I)\hat{\pi} = (S\otimes a)\hat{\pi}(P_\alpha\otimes I)\in\mathcal{B}_0(\mathcal{H})\otimes\mathcal{A}$. Since $(S\otimes a)\hat{\pi}$ is the norm limit of finite sums of such terms, $(S\otimes a)\hat{\pi}\in\mathcal{B}_0(\mathcal{H})\otimes\mathcal{A}$. Thus $\hat{\pi}$ is a unitary representation acting on $\mathcal{H}$. □

Next we introduce the right regular comodule. Denote by $L_2(G)$ the GNS space associated with the haar state h on G. Then $\mathcal{A}$ is a dense subspace of $L_2(G)$. One can also see that $\mathcal{A}\otimes\mathcal{A}$ can be regarded as a subspace of $L_2(G)\otimes\mathcal{A}$. Consider the map $\mu:\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$.

$$\langle\mu(a),\mu(b)\rangle = (h\otimes id)(\mu(a)^*\mu(b)) = (h\otimes id)\mu(a^*b) = h(a^*b)I = \langle a,b\rangle I$$

for all $a,b\in\mathcal{A}$. Therefore μ extends to an isometry from $L_2(G)$ into $L_2(G)\otimes\mathcal{A}$. Denote it by $\Re$. The maps $(I\otimes\mu)\Re$ and $(\Re\otimes id)\Re$ both are isometries from $L_2(G)$ to $L_2(G)\otimes\mathcal{A}\otimes\mathcal{A}$ and they coincide on $\mathcal{A}$. Hence $(I\otimes\mu)\Re = (\Re\otimes id)\Re$. Thus $\Re$ is an isometric comodule map. We call it the *right regular comodule* of G. By theorem 2.2.2, $\Psi^{-1}(\Re)$ is a unitary representation acting on $L_2(G)$. This is the right regular representation introduced by Woronowicz ([72]).

Following is the Peter-Weyl theorem for compact quantum groups.

Theorem 2.2.3 *There is a family $\{P_\gamma\}$ of finite dimensional projections on $L_2(G)$ obeying*

$$\Re P_\gamma = (P_\gamma\otimes id)\Re\ \ \forall\gamma,\quad P_\gamma P_{\gamma'} = \delta_{\gamma\gamma'}P_\gamma,\quad \sum P_\gamma = I.$$

If $\mathcal{H}_\gamma$ denotes $P_\gamma(L_2(G))$, then $\Re$ restricted to $\mathcal{H}_\gamma$ decomposes into irreducible comodules of only one type. Moreover, if $\mathcal{H}$ is any closed $\Re$-invariant subspace of $L_2(G)$ then $\mathcal{H} = \oplus_\gamma(\mathcal{H}\cap\mathcal{H}_\gamma)$.

Sketch of proof: Let f_z be the one parameter family of multiplicative functionals introduced in theorem 1.2.4. Let π^γ_{ij} denote the ij th entry, in some fixed basis, of the irreducible representation of type γ. Let $\chi^\gamma := \sum_i \pi^\gamma_{ii}$. Note that χ^γ does not depend on the choice of the basis. Denote by ρ_γ the functional $a \mapsto f_{-1}(\chi^\gamma)h(a\kappa^{-1}(\chi^\gamma * f_{-1}))$ defined on $C(G)$. Then $P_\gamma := \Re_{\rho_\gamma}$ are the required projections. $\square$

As an application of the above theorem, we give here a small lemma that will be needed in the next section.

Lemma 2.2.4 $\{ u \in L_2(G) \;:\; \Re(u) \in L_2(G) \otimes_{alg} C(G) \} = A(G)$.

Proof: $\Re$ coincides with μ on $A(G)$. Therefore by theorem 1.1.7, $A(G)$ is contained in the left hand side of the above equation.

To prove the reverse inclusion, take an $u \in L_2(G)$ such that $\Re(u) \in L_2(G) \otimes_{alg} C(G)$. $\Re(u)$ is then a finite sum of the form $\sum u_i \otimes a_i$, where $u_i \in L_2(G)$, $a_i \in C(G)$. Let $\mathcal{H}$ be the subspace of $L_2(G)$ spanned by the u_i's. Then $\dim \mathcal{H} < \infty$. For any continuous linear functional ρ on $C(G)$, $\Re_\rho : v \mapsto (id \otimes \rho)\Re(v)$ is bounded, and $\Re_\rho(u) = \sum \rho(a_i)u_i \in \mathcal{H}$. Therefore the smallest $\Re$-invariant subspace $\mathcal{H}_0$ of $L_2(G)$ containing u is contained in $\mathcal{H}$ and hence is finite dimensional. By theorem 2.2.3, $\mathcal{H}_0 = \oplus_\gamma(\mathcal{H}_0 \cap \mathcal{H}_\gamma)$. Since $\mathcal{H}_0$ is finite dimensional, there are only a finite number of summands, so that the elements of $\mathcal{H}_0$ are finite linear combinations of the π^γ_{ij}'s. In particular, u is a finite combination of the π^γ_{ij}'s. Therefore $u \in A(G)$, which completes the proof. $\square$

2.3 Induced Representations

In this section, we shall introduce the concept of an induced representation for a compact quantum group. We shall see that Frobenius reciprocity theorem remains valid in the noncommutative setup also. Throughout this section, $H = (C(H), \mu_H)$ will denote a subgroup of the compact quantum group $G = (C(G), \mu_G)$.

We start with the following lemma concerning the boundedness of the left convolution operator.

Lemma 2.3.1 *Let $G = (\mathcal{A}, \mu)$ be a compact quantum group. Then the map $L_\rho : \mathcal{A} \to \mathcal{A}$ given by $L_\rho(a) = (\rho \otimes id)\mu(a)$ extends to a bounded operator from $L_2(G)$ into itself.*

Proof: Let us first prove the following inequality:

$$\rho_1(((\rho_2 \otimes id)c)^*(\rho_2 \otimes id)c) \leq (\rho_2 \otimes \rho_1)(c^*c), \quad c \in \mathcal{A} \otimes \mathcal{A}. \tag{2.3.1}$$

Take $c = \sum a_i \otimes b_i \in \mathcal{A} \otimes_{alg} \mathcal{A}$. The matrix $(\!(\rho_1({b_i}^* b_j))\!)$ is positive. Hence for any real t, $\sum(a_i - t\rho_2(a_i)I)^* \rho_1({b_i}^* b_j)(a_j - t\rho_2(a_j)I) \geq 0$. Applying ρ_2, we get

$$\sum \rho_1({b_i}^* b_j)\rho_2({a_i}^* a_j) + t^2 \sum \overline{\rho_2(a_i)}\rho_2(a_j)\rho_1({b_i}^* b_j) - 2t \sum \overline{\rho_2(a_i)}\rho_2(a_j)\rho_1({b_i}^* b_j) \geq 0$$

for all real t. Therefore $\sum \overline{\rho_2(a_i)}\rho_2(a_j)\rho_1({b_i}^* b_j) \leq \sum \rho_2({a_i}^* a_j)\rho_1({b_i}^* b_j)$ which means (2.3.1) holds for $c \in \mathcal{A} \otimes_{alg} \mathcal{A}$. By continuity, the same thing holds for all $c \in \mathcal{A} \otimes \mathcal{A}$.

Putting $c = \mu(a)$ in (2.3.1), we get the following:

$$\rho_1((\rho_2 * a)^*(\rho_2 * a)) \leq \rho_2 * \rho_1(a^* a) \quad \forall a \in \mathcal{A}.$$

The proof now follows by writing $\rho_1 = h$, $\rho_2 = \rho$. □

Let $\hat{\pi}$ be a unitary representation of H acting on the space $\mathcal{H}_0$. $\pi := \Psi(\hat{\pi})$ is then an isometric comodule map from $\mathcal{H}_0$ to $\mathcal{H}_0 \otimes C(H)$. Consider the following map from $\mathcal{H}_0 \otimes L_2(G)$ to $\mathcal{H}_0 \otimes L_2(G) \otimes C(G)$:

$$I \otimes \Re^G : u \otimes v \mapsto u \otimes \Re^G(v)$$

where $\Re^G$ is the right regular comodule of G. It is easy to see that this is an isometric comodule map acting on $\mathcal{H}_0 \otimes L_2(G)$.

Let ϕ be the homomorphism from G to H (cf. definition 1.3.1). Let $\mathcal{H}$ denote the following Hilbert space:

$$\{u \in \mathcal{H}_0 \otimes L_2(G) : (I \otimes L_{\rho \cdot \phi})u = (\pi_\rho \otimes I)u \text{ for all continuous functionals } \rho \text{ on } C(H)\}.$$

Then $I \otimes \Re^G$ keeps $\mathcal{H}$ invariant; the restriction of $I \otimes \Re^G$ to $\mathcal{H}$ is therefore an isometric comodule, so that $\Psi^{-1}((I \otimes \Re^G)|_{\mathcal{H}})$ is a unitary representation of G acting on $\mathcal{H}$. We call this *the representation induced by* $\hat{\pi}$, and denote it by $\mathrm{ind}_H^G \hat{\pi}$ or simply by $\mathrm{ind}\,\hat{\pi}$ when there is no ambiguity about G and H.

Let $\hat{\pi}_1$ and $\hat{\pi}_2$ be two unitary representations of H. Then clearly we have

i. $\mathrm{ind}\,\hat{\pi}_1$ and $\mathrm{ind}\,\hat{\pi}_2$ are equivalent whenever $\hat{\pi}_1$ and $\hat{\pi}_2$ are equivalent, and

ii. $\mathrm{ind}\,(\hat{\pi}_1 \oplus \hat{\pi}_2)$ and $\mathrm{ind}\,\hat{\pi}_1 \oplus \mathrm{ind}\,\hat{\pi}_2$ are equivalent.

Before going to the Frobenius reciprocity theorem, let us briefly describe what we mean by the restriction of a representation to a subgroup. Let $\hat{\pi}^G$ be a unitary representation of G acting on a Hilbert space $\mathcal{H}_0$. We call $(id \otimes \phi)\hat{\pi}^G$ the restriction of $\hat{\pi}^G$ to H and denote it by $\hat{\pi}^G|^H$. To see that it is indeed a unitary representation, observe that $\Psi((id \otimes \phi)\hat{\pi}^G) = (I \otimes \phi)\Psi(\hat{\pi}^G)$ which is clearly an isometric comodule.

Therefore by theorem 2.2.3, $\hat{\pi}^G|^H$ is a unitary representation of H acting on $\mathcal{H}_0$. Denote $\Psi(\hat{\pi}^G)$ by π^G and $\Psi(\hat{\pi}^G|^H)$ by $\pi^G|^H$.

Theorem 2.3.2 (Frobenius reciprocity theorem) *Let $\hat{\pi}^G$ and $\hat{\pi}^H$ be irreducible unitary representations of G and H respectively. Then the multiplicity of $\hat{\pi}^G$ in $\mathrm{ind}_H^G \hat{\pi}^H$ is the same as that of $\hat{\pi}^H$ in $\hat{\pi}^G|^H$.*

Proof: Let $\mathcal{I}(\hat{\pi}^G|^H, \hat{\pi}^H)$ (respectively $\mathcal{I}(\hat{\pi}^G, \mathrm{ind}\,\hat{\pi}^H)$) denote the space of intertwiners between $\hat{\pi}^G|^H$ and $\hat{\pi}^H$ (respectively $\hat{\pi}^G$ and $\mathrm{ind}\,\hat{\pi}^H$). Assume that $\hat{\pi}^G$ and $\hat{\pi}^H$ act on $\mathcal{K}_0$ and $\mathcal{H}_0$ respectively. $\mathcal{K}_0 \otimes C(G)$ can be regarded as a subspace of $\mathcal{K}_0 \otimes L_2(G)$ and hence π^G, as a map from $\mathcal{K}_0$ into $\mathcal{K}_0 \otimes L_2(G)$. Since $\pi^G = \Psi(\hat{\pi}^G)$ is unitary, we have for $u, v \in \mathcal{K}_0$,

$$\langle \pi^G(u), \pi^G(v) \rangle_{\mathcal{K}_0 \otimes L_2(G)} = h(\langle \pi^G(u), \pi^G(v) \rangle_{\mathcal{K}_0 \otimes C(G)}) = h(\langle u, v \rangle I) = \langle u, v \rangle.$$

Thus $\pi^G : \mathcal{K}_0 \to \mathcal{K}_0 \otimes L_2(G)$ is an isometry. Let $S : \mathcal{K}_0 \to \mathcal{H}_0$ be an element of $\mathcal{I}(\hat{\pi}^G|^H, \hat{\pi}^H)$. $(S \otimes I)\pi^G$ is then a bounded map from $\mathcal{K}_0$ into $\mathcal{H}_0 \otimes L_2(G)$. Denote it by $f(S)$. It is not too dificult to see that $f(S)$ actually maps $\mathcal{K}_0$ into $\mathcal{H}$, and intertwines $\hat{\pi}^G$ and $\mathrm{ind}\,\hat{\pi}^H$. $f : S \mapsto f(S)$ is thus a linear map from $\mathcal{I}(\hat{\pi}^G|^H, \hat{\pi}^H)$ to $\mathcal{I}(\hat{\pi}^G, \mathrm{ind}\,\hat{\pi}^H)$.

We shall now show that f is invertible by exhibiting the inverse of f. Take a T: $\mathcal{K}_0 \to \mathcal{H}$ that intertwines $\hat{\pi}^G$ and $\mathrm{ind}\,\hat{\pi}^H$. For any $u \in \mathcal{H}_0$, $T^u := (\langle u| \otimes I)T$ is a map from $\mathcal{K}_0$ to $L_2(G)$ intertwining $\hat{\pi}^G$ and the right regular representation $\Re^G$ of G, i.e. $\Re^G T^u = (T^u \otimes id)\pi^G$. Now, π^G is finite dimensional, so that $\pi^G(\mathcal{K}_0) \subseteq \mathcal{K}_0 \otimes_{alg} A(G)$. Hence $\Re^G T^u(\mathcal{K}_0) \subseteq L_2(G) \otimes_{alg} A(G)$. By lemma 2.2.4, $T^u(\mathcal{K}_0) \subseteq A(G)$. Since this is true for all $u \in \mathcal{H}_0$, $T(\mathcal{K}_0) \subseteq \mathcal{H}_0 \otimes_{alg} A(G)$. Therefore $(I \otimes \epsilon_G)T$ is a bounded operator from $\mathcal{K}_0$ to $\mathcal{H}_0$. Denote it by $g(T)$.

For a comodule π and a linear functional ρ, denote $(id \otimes \rho)\pi$ by π_ρ. Let ρ be a linear functional on $C(H)$. Then $\pi_\rho^H g(T) = \pi_\rho^H (I \otimes \epsilon_G)T = (I \otimes \epsilon_G)(\pi_\rho^H \otimes id)T = (I \otimes \epsilon_G)(I \otimes L_{\rho\cdot\phi})T = (I \otimes \rho \circ \phi)T$. On the other hand, since T intertwines $\hat{\pi}^G$ and $\mathrm{ind}\,\hat{\pi}^H$, we have $g(T)(\pi^G|^H)_\rho = g(T)(I \otimes \rho)\pi^G|^H = g(T)(I \otimes \rho)(I \otimes \phi)\pi^G = (I \otimes \epsilon_G)T\pi^G_{\rho\cdot\phi} = (I \otimes \epsilon_G)(I \otimes \Re^G_{\rho\cdot\phi})T = (I \otimes \rho \circ \phi)T$. Thus $\pi_\rho^H g(T) = g(T)(\pi^G|^H)_\rho$ for all continuous linear functionals ρ on $C(H)$, which implies $g(T) \in \mathcal{I}(\hat{\pi}^G|^H, \hat{\pi}^H)$. The map $T \mapsto g(T)$ is the inverse of f. Therefore $\mathcal{I}(\hat{\pi}^G|^H, \hat{\pi}^H) \cong \mathcal{I}(\hat{\pi}^G, \mathrm{ind}\,\hat{\pi}^H)$, which proves the theorem. $\square$

Corollary 2.3.3 *For any unitary representation $\hat{\pi}^G$ of G and $\hat{\pi}^H$ of H, the spaces $\mathcal{I}(\hat{\pi}^G|^H, \hat{\pi}^H)$ and $\mathcal{I}(\hat{\pi}^G, ind\,\hat{\pi}^H)$ are isomorphic.*

Corollary 2.3.4 *Let H be a subgroup of G and K be a subgroup of H. Suppose $\hat{\pi}$ is a unitary representation of K. Then $\mathrm{ind}_K^G\hat{\pi}$ and $\mathrm{ind}_H^G(\mathrm{ind}_K^H\hat{\pi})$ are equivalent.*

2.4 An Application

We have seen in section 1.3 that $SU_q(2)$ has a homogeneous action on the noncommutative sphere S^2_{q0}. This action has been decomposed by Podleś (see [40]). Here we give an alternative way of doing it using the Frobenius reciprocity theorem.

For any $n \in \{0, 1/2, 1, 3/2, \ldots\}$, if we restrict the right regular comodule $\Re$ of $SU_q(2)$ to the subspace $\mathcal{H}_n$ of $L_2(SU_q(2))$ spanned by

$$\{\alpha^{*i}\beta^{2n-i} \,:\, i = 0, 1, \ldots, 2n\}, \tag{2.4.1}$$

then we get an irreducible isometric comodule. Denote it by $u^{(n)}$. It is a well-known fact ([67]) that these constitute *all* the irreducible comodules of $SU_q(2)$. If we take the basis of $\mathcal{H}_n$ to be (2.4.1) with proper normalization, the matrix entries of $u^{(n)}$ turn out to be

$$u^{(n)}_{ij} = \left(d^{(n)}_i / d^{(n)}_j\right)^{1/2} \sum_{r=(i-j)\vee 0}^{(2n-j)\wedge i} \binom{i}{r}_{q^{-2}} \binom{2n-i}{r+j-i}_{q^{-2}} (-1)^r q^{r(2i-r+1)+(j-i)(2n-j)} \times \alpha^{*i-r}\alpha^{2n-j-r}\beta^{r+j-i}\beta^{*r},$$

where

$$d^{(n)}_k = \sum_{r=0}^{k} \binom{k}{r}_{q^{-2}} (-1)^r q^{r(2k-r+1)} \frac{1-q^2}{1-q^{4n+2r-2k+2}}\,;$$

and $\binom{k}{r}_{q^{-2}}$ are the q^{-2}-binomial coefficients given by

$$\begin{aligned} \binom{k}{r}_{q^{-2}} &:= \frac{(k)_{q^{-2}}(k-1)_{q^{-2}}\ldots(1)_{q^{-2}}}{(r)_{q^{-2}}(r-1)_{q^{-2}}\ldots(1)_{q^{-2}}\,(k-r)_{q^{-2}}(k-r-1)_{q^{-2}}\ldots(1)_{q^{-2}}}\,, \\ (k)_{q^{-2}} &:= 1 + q^{-2} + q^{-4} + \ldots + q^{-2k+2}. \end{aligned}$$

We have seen in example 1.3.6 that S^1 is a subgroup of $SU_q(2)$. Now the restriction of $u^{(n)}$ to S^1 is $u^{(n)}|^{S^1} = (I \otimes \phi)u^{(n)}$. Therefore the matrix entries of $u^{(n)}|^{S^1}$ are given by

$$(u^{(n)}|^{S^1})_{ij} = \begin{cases} u^{2(n-i)} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.4.2}$$

Therefore if n is an integer then the trivial representation occurs in $u^{(n)}|^{S^1}$ with multiplicity 1, and does not occur otherwise.

Consider now the action ν of $SU_q(2)$ on S^2_{q0}. Recall (cf. example 1.3.11) that $C(S^2_{q0}) = \{\, a \in C(SU_q(2)) \,:\, (\phi \otimes id)\mu(a) = I \otimes a \,\}$ and the action is the restriction of μ to $C(S^2_{q0})$. From the above description, $C(S^2_{q0})$ can easily be shown to be equal to $\{a \in C(S^2_{q0}) : L_{\rho\cdot\phi}(a) = \rho(I)a$ for all continuous linear functionals ρ on $C(S^1)\ \}$. Therefore when we take the closure of $C(S^2_{q0})$ with respect to the invariant inner product (arising out of the ν-invariant state) that it carries and extend the action there as an isometry, what we get is the restriction of the right regular comodule $\Re$ of $SU_q(2)$ to the subspace $\mathcal{H} = \{\, u \in L_2(SU_q(2)) \,:\, L_{\rho\cdot\phi}(u) = \rho(I)u$ for all continuous linear functionals ρ on $C(S^1)\}$, which is nothing but the representation $\hat{\pi}$ of $SU_q(2)$ induced by the trivial representation of S^1 on $\mathbb{C}$. Hence the multiplicity of $u^{(n)}$ in $\hat{\pi}$ is same as that of the trivial representation of S^1 in $u^{(n)}|^{S^1}$, which is, from (2.4.2), 1 if n is an integer and 0 if n is not. Thus the action splits into a direct sum of all the integer-spin representations.

Chapter 3

Noncompact Quantum Groups

We have so far dealt with compact quantum groups only. We have seen that it is quite easy to characterize the category of C^*-algebras associated with such objects among the class of all C^*-algebras. Unfortunately the same thing cannot be said in the noncompact situation. C^*-algebras associated with noncompact quantum groups often exhibit very strange behaviour and are, in general, much more difficult to handle. For instance, suppose G is a noncompact locally compact group, $C_0(G)$ being the C^*-algebra of continuous functions on G vanishing at infinity; and let μ denote the comultiplication map. Then $\mu(C_0(G)) \nsubseteq C_0(G) \otimes C_0(G)$. As a matter of fact, $\mu(C_0(G)) \cap (C_0(G) \otimes C_0(G)) = \{0\}$. While dealing with noncompact quantum groups, this is a major source of discomfort. One has to introduce the comultiplication μ very carefully — which in this case is a homomorphism from $C_0(G)$ to the multiplier algebra $M(C_0(G) \otimes C_0(G))$ satisfying certain properties. But unlike in the case of a classical group, one can actually have $\mu(C_0(G)) \subseteq C_0(G) \otimes C_0(G)$ in certain cases. Another problem that one often faces is the following. The Hopf algebra $A(G)$ of coordinate functions is usually not contained in $C_0(G)$. Not only that, it may not have any nontrivial intersection with it at all. In concrete example that one encounters, often $A(G)$ is given via a set of generators and relations. One has to construct the C^*-algebra $C_0(G)$ out of it. In the noncompact case, it is not at all clear what should be a canonical way of doing this, or whether this C^*-algebra contains all the information provided by $A(G)$. Therefore normally one has to study both the Hopf algebra $A(G)$ and the C^*-algebra $C_0(G)$. For a detailed discussion on various problems that arise when treating noncompact quantum groups, we refer the reader to the papers of Woronowicz (see [41], [70], [74]).

For the rest of this thesis, we shall restrict ourselves to a couple of well-known examples of noncompact quantum groups — $E_q(2)$, the q-deformation of the group of motions of the Euclidean plane, and its unitary dual $\widehat{E_q(2)}$.

3.1 Preliminaries

In the treatment of both the quantum groups $E_q(2)$ and its dual, an extremely important role is played by the set $\mathbb{C}^q = \{q^k z : z \in S^1, k \in \mathbb{Z}\} \cup \{0\}$. In the present section, we present a few results, due to Woronowicz, on normal operators having their spectrum in $\mathbb{C}^q$.

We start with a definition. But before that, some notations. For a closed operator T, denote, by V_T and $|T|$, the partial isometry and the positive self-adjoint operator appearing in its polar decomposition, i.e. V_T and $|T|$ are such that $T = V_T|T|$.

Definition 3.1.1 ([73, 76]) Let R and S be two normal operators acting on a Hilbert space $\mathcal{H}$. The pair (R, S) is called a $(q^{-2}, 1)$-*commuting pair* if the following conditions are satisfied:
1. $|R|$ and $|S|$ strongly commute,
2. $V_R V_S = V_S V_R$,
3. $V_R |S| V_R^* = q^{-1}|S|$ on $(\ker R)^\perp$,
4. $V_S |R| V_S^* = q|R|$ on $(\ker S)^\perp$.

Let T be a normal operator on $\mathcal{H}$, $P_T(\cdot)$ being the corresponding spectral measure. A vector u in $\mathcal{H}$ is said to have *compact T-support* if the support of the measure $\langle u, dP_T(\cdot)u\rangle$ is compact. The following theorem gives an easier way of checking in many cases whether a pair (R, S) is a $(q^{-2}, 1)$-commuting pair or not, besides giving a justification for calling such a pair $(q^{-2}, 1)$-commuting.

Theorem 3.1.2 ([76]) *Let R and S be normal operators acting on $\mathcal{H}$. Then (R, S) is a $(q^{-2}, 1)$-commuting pair if and only if there exists a dense domain $\mathcal{D}$ consisting of vectors with compact R- and S-support such that $\mathcal{D}$ is invariant under the actions of R, S, R^* and S^*, and for any $u \in \mathcal{D}$ one has*

$$\begin{aligned} RSu &= q^{-2} SRu, \\ RS^*u &= S^*Ru. \end{aligned} \tag{3.1.1}$$

Proof: For the proof, we refer the reader to [76]. □

Let us next introduce the following function on $\mathbb{C}^q$:

$$F_q(z) = \begin{cases} \prod_{r=0}^{\infty} \frac{1+q^{2r}\overline{z}}{1+q^{2r}z} & \text{if } z \in \mathbb{C}^q - \{-1, -q^{-2}, -q^{-4}, \ldots\}, \\ -1 & \text{if } z \in \{-1, -q^{-2}, -q^{-4}, \ldots\}. \end{cases} \tag{3.1.2}$$

This defines a bounded continuous function on $\mathbb{C}^q$. For a positive real t and for $q \neq 0$, let us denote by $(t)_q$ the number $(1-q^t)/(1-q)$. Let n be a nonnegative integer. Define the q-factorial $(n)_q!$ by:

$$(n)_q! = \begin{cases} \prod_{k=1}^{n} (k)_q & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

One can now define the q-exponential function as follows:

$$\exp_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k)_q!}.$$

This function can be shown to have the following infinite product expansion for $q > 1$:

$$\exp_q(x) = \prod_{k=1}^{\infty} \left(1 - q^{-k}(1-q)x\right),$$

from which it follows that

$$F_q(z) = \frac{\exp_{q^{-2}}(\frac{\overline{z}}{1-q^2})}{\exp_{q^{-2}}(\frac{z}{1-q^2})}. \tag{3.1.3}$$

For $(q^{-2}, 1)$-commuting pairs of normal operators having their spectrum in $\mathbb{C}^q$, this function F_q behaves like the usual exponential function. Before we make this statement more precise, let us state, without proof, two extremely useful results due to Woronowicz that deal with the sum of two unbounded normal operators.

Theorem 3.1.3 ([76]) *Let (R, S) be a $(q^{-2}, 1)$-commuting pair of normal operators. Assume that $\ker R = \{0\}$. Then the following are equivalent:*
1. $R + S$ admits a normal extension.
2. The closure $\overline{R+S}$ of $R + S$ is normal.
3. The spectrum $\sigma(R^{-1}S)$ is contained in $\mathbb{C}^q$.
If any of the above conditions hold, then $R^{-1}S$ is a normal operator and one has

$$\overline{R+S} = F_q(R^{-1}S)RF_q(R^{-1}S)^*.$$

Theorem 3.1.4 ([76]) *Let (R,S) be a $(q^{-2},1)$-commuting pair of normal operators. Assume that $\ker S=\{0\}$. Then the following are equivalent:*
1. $R+S$ admits a normal extension.
2. The closure $\overline{R+S}$ of $R+S$ is normal.
3. The spectrum $\sigma(RS^{-1})$ is contained in $\mathbb{C}^q$.
If any of the above conditions hold, then RS^{-1} is a normal operator and one has

$$\overline{R+S}=F_q(RS^{-1})^*SF_q(RS^{-1}).$$

Suppose now that (R,S) is a $(q^{-2},1)$-commuting pair of normal operators, with both their spectrums contained in $\mathbb{C}^q$. Then one can show that the third condition in the forgoing theorem is satisfied, so that $\overline{R+S}$ is a normal operator with $\sigma(\overline{R+S})\subseteq\mathbb{C}^q$. Therefore $F_q(\overline{R+S})$ is a unitary operator and we have $F_q(\overline{R+S})=F_q(R)F_q(S)$.

From here onwards, whenever R and S are two normal operators satisfying any of the conditions listed in theorems 3.1.3 or 3.1.4, we will denote $\overline{R+S}$ by simply $R+S$.

3.2 The Group $E_q(2)$

The quantum group $E_q(2)$ has been studied by several people (see [10], [59], [71]). Here we give a very brief description of the group and describe a few salient features of it. Let us start with the Hopf-algebra of coordinate functions on $E_q(2)$. Let A_0 be the unital $*$-algebra generated by two elements v and n satisfying the following relations:

$$v^*v=vv^*=I,\quad n^*n=nn^*,\quad vnv^*=qn. \tag{3.2.1}$$

Define a map μ from A_0 to $A_0\otimes A_0$ by prescribing

$$\mu(v)=v\otimes v,\quad \mu(n)=v\otimes n+n\otimes v^*.$$

Here $A_0\otimes A_0$ means their algebraic tensor product. The map μ extends to a unital $*$-homomorphism from A_0 to $A_0\otimes A_0$. Similarly, define a complex homomorphism $\epsilon: A_0\to\mathbb{C}$ and a linear antimultiplicative map $\kappa: A_0\to A_0$ by requiring that

$$\epsilon(v)=1,\quad \epsilon(n)=0,$$
$$\kappa(v)=v^*,\quad \kappa(v^*)=v,\quad \kappa(n)=-q^{-1}n,\quad \kappa(n^*)=-qn^*.$$

$(A_0, \mu, \epsilon, \kappa)$ is a Hopf*-algebra. It describes the group structure of $E_q(2)$ at an algebraic level.

Let us now try to build the C^*-algebra $C_0(E_q(2))$ of continuous functions on $E_q(2)$ vanishing at infinity. We shall follow the following scheme. First we represent A_0 faithfully as an algebra of operators (not necessarily bounded) on some Hilbert space $\mathcal{H}$. We next try to get $C_0(E_q(2))$ as an appropriate C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. First step, therefore, is to get irreducible representations of the pair (v, n) as operators on a Hilbert space. For technical reasons, along with the relations (3.2.1), we add the following spectral condition:

$$\sigma(n) \subseteq \mathbb{C}^q. \tag{3.2.2}$$

Roughly speaking, this condition ensures that $\mu(n)$ is a 'continuous function on the cartesian product $E_q(2) \times E_q(2)$'. For an explanation of how this condition arises, see [70], [73] and [75]. Now once we assume (3.2.1) and (3.2.2), it is easy to see that the irreducible representations are the following:

$$\left.\begin{array}{lcll} \pi_z & : & \left\{\begin{array}{l} v \mapsto \ell \\ n \mapsto zq^N \end{array}\right. & \text{on } L_2(\mathbb{Z}), \\ \epsilon_z & : & \left\{\begin{array}{l} v \mapsto z \\ n \mapsto 0 \end{array}\right. & \text{on } \mathbb{C}, \end{array}\right\} \quad z \in S^1. \tag{3.2.3}$$

Now let $\boldsymbol{v} = \ell \otimes I$, $\boldsymbol{n} = q^N \otimes \ell^*$ on $L_2(\mathbb{Z}) \otimes L_2(\mathbb{Z})$. Let A denote the $*$-algebra generated by $\boldsymbol{v}$ and $\boldsymbol{n}$ (we are being slightly sloppy here — one has to take into consideration the domains of the operators; but on an appropriate domain, one can form the algebra A). It is easy to verify that $v \mapsto \boldsymbol{v}$, $n \mapsto \boldsymbol{n}$ is a Hopf*-algebra isomorphism between A_0 and A. Moreover, we have $\sigma(\boldsymbol{n}) = \mathbb{C}^q$.

Take the norm closure of finite sums of the form $\sum_k \boldsymbol{v}^k f_k(\boldsymbol{n})$, where $f_k \in C_0(\mathbb{C}^q)$. This is a C^*-algebra without identity. Denote it by $C_0(E_q(2))$. It is easy to check that $\boldsymbol{v}$ and $\boldsymbol{n}$ are affiliated to $C_0(E_q(2))$, and moreover, it has the following very useful 'universality property'.

Theorem 3.2.1 ([71]) *If π is a representation of $C_0(E_q(2))$ on some Hilbert space $\mathcal{K}$, then $\pi(\boldsymbol{v})$ and $\pi(\boldsymbol{n})$ satisfy the conditions (3.2.1) and (3.2.2), with $\pi(\boldsymbol{v})$ replacing v and $\pi(\boldsymbol{n})$ replacing n. Conversely, if $\overline{\boldsymbol{v}}$ and $\overline{\boldsymbol{n}}$ are two closed operators on a Hilbert space $\mathcal{K}$ and satisfy (3.2.1) and (3.2.2), then there is a unique representation π of $C_0(E_q(2))$ such that $\pi(\boldsymbol{v}) = \overline{\boldsymbol{v}}$ and $\pi(\boldsymbol{n}) = \overline{\boldsymbol{n}}$.*

Moreover, in the above situation, if $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{K})$, then $\overline{\boldsymbol{v}}$ and $\overline{\boldsymbol{n}}$ are affiliated to $\mathcal{A}$ if and only if $\pi \in mor(C_0(E_q(2)), \mathcal{A})$.

Among other things, the above theorem helps in establishing the comultiplication map μ at the C^*-algebra level. Consider the operators $\boldsymbol{n}\otimes\boldsymbol{v}^*$ and $\boldsymbol{v}\otimes\boldsymbol{n}$ on $L_2(\mathbb{Z})^{\otimes 4}$. They form a $(q^{-2},1)$-commuting pair, and $\sigma((\boldsymbol{n}\otimes\boldsymbol{v}^*)^{-1}(\boldsymbol{v}\otimes\boldsymbol{n})) \subseteq \mathbb{C}^q$. Therefore by theorem 3.1.3, the closure of $\boldsymbol{v}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{v}^*$ is the unique normal extension of $\boldsymbol{v}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{v}^*$. By abuse of notation, we continue to denote it by the same symbol. $\boldsymbol{v}\otimes\boldsymbol{v}$ and $\boldsymbol{v}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{v}^*$ obey (3.2.1) and (3.2.2). Therefore by the theorem above, there is a representation μ of $C_0(E_q(2))$ on $L_2(\mathbb{Z})^{\otimes 4}$ such that $\mu(\boldsymbol{v}) = \boldsymbol{v}\otimes\boldsymbol{v}$ and $\mu(\boldsymbol{n}) = \boldsymbol{v}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{v}^*$. From the last part of the theorem, it follows that $\mu \in \mathrm{mor}\,(C_0(E_q(2)), C_0(E_q(2))\otimes C_0(E_q(2)))$. The coassociativity of μ also follows from the above theorem. The following lemma gives an explicit formula for μ that will be useful for computational purposes.

Lemma 3.2.2 *Let V be the unitary operator on $L_2(\mathbb{Z})^{\otimes 4}$ given on the basis elements by $e_{i,j,k,l} \mapsto e_{i,j,i+j+k,l}$. Let $W = F_q(\boldsymbol{n}^{-1}\boldsymbol{v}\otimes\boldsymbol{v}\boldsymbol{n})V$. Then $\mu(a) = W(a\otimes I)W^*$ for all $a \in C_0(E_q(2))$.*

Proof: For $a \in C_0(E_q(2))$, write $\nu(a) = W(a\otimes I)W^*$. Then both μ and ν are representations of $C_0(E_q(2))$ acting on the same space $L_2(\mathbb{Z})^{\otimes 4}$. Using theorem 3.1.3, it is easy to see that $\mu(\boldsymbol{v}) = \nu(\boldsymbol{v})$ and $\mu(\boldsymbol{n}) = \nu(\boldsymbol{n})$. Hence by theorem 3.2.1, $\mu = \nu$. $\square$

Let P denote the projection onto $L_2(\mathbb{Z}_+)\otimes L_2(\mathbb{Z})$. $C(SU_q(2))$ can be thaught of as a subalgebra of $\mathcal{B}(L_2(\mathbb{Z})\otimes L_2(\mathbb{Z}))$ via the identification $a \leftrightarrow PaP$. We shall very often write I_{SU} for this projection P. Let f_α and f_β be the following functions on $\mathbb{C}^q$:

$$f_\alpha : x \mapsto \sqrt{1-|x|^2}I_{\{|x|\le 1\}}, \quad f_\beta : x \mapsto \overline{x}I_{\{|x|\le 1\}}.$$

One can see that, via the above mentioned identification, $\alpha = \boldsymbol{v}f_\alpha(\boldsymbol{n})$ and $\beta = f_\beta(\boldsymbol{n})$. Therefore $C(SU_q(2))$ is a C^*-subalgebra of $C_0(E_q(2))$.

Define a C^*-homomorphism $\tau^k : C_0(E_q(2)) \to C_0(E_q(2))$, where $k\in\mathbb{Z}$, as follows:

$$\tau^k(a) = \boldsymbol{v}^k a \boldsymbol{v}^{-k}, \quad a \in C_0(E_q(2)). \tag{3.2.4}$$

τ^k is actually a C^*-automorphism, and moreover, we have $(\tau^k\otimes\tau^k)\mu = \mu\tau^k$ and $\tau^k\tau^l = \tau^{k+l}$. This means that $\{\tau^k\}_{k\in\mathbb{Z}}$ is a one parameter group of quantum group automorphisms of $E_q(2)$. It is quite easy to see that $\tau^k(C(SU_q(2)))$ is an increasing family of unital C^*-subalgebras of $C_0(E_q(2))$, and $\cup_k \tau^k(C(SU_q(2)))$ is dense in $C_0(E_q(2))$. $\cup_k \tau^k(C(SU_q(2)))$ plays the role of the algebra of compactly supported functions on $E_q(2)$. We denote it by $C_c(E_q(2))$. Denote by μ_{SU} the comultiplication map for the quantum group $SU_q(2)$. The following formula tells us how the two maps μ and μ_{SU} are connected.

$$\mu(a) = \lim_{k\to\infty} (\tau^k \otimes \tau^k)\mu_{SU}(\tau^{-k}a), \quad a \in C_c(E_q(2)). \tag{3.2.5}$$

Observe that for any $a \in C_c(E_q(2))$, $\tau^{-k}a \in C(SU_q(2))$ for all sufficiently large k. The above phenomenon is referred to as the *contraction procedure*. For a discussion on this and a proof of (3.2.5), see [74].

Let $\boldsymbol{t} = \prod_{k=1}^{\infty}(I_{SU} - q^{2k}\beta^*\beta)$, $X = \sum_{k=0}^{\infty} c_k(-q\beta^* \otimes \beta)^k(\boldsymbol{v} \otimes \boldsymbol{v})^{-k}$ where $c_k = \prod_{r=1}^{k}(1 - q^{2r})^{-1}$. For $a \in C_0(E_q(2))$, write Λ for $\mu_{SU}(\boldsymbol{t}^{-1/2}a\boldsymbol{t}^{-1/2})$. Then one can prove, using (3.2.5), that

$$\mu(a) = X^*(\boldsymbol{t} \otimes \boldsymbol{t})^{1/2}\Lambda(\boldsymbol{t} \otimes \boldsymbol{t})^{1/2}X, \;\; a \in C_c(E_q(2)).$$

By continuity argument, it follows that

$$\mu(a) = X^*(\boldsymbol{t} \otimes \boldsymbol{t})^{1/2}\Lambda(\boldsymbol{t} \otimes \boldsymbol{t})^{1/2}X, \;\; a \in C_0(E_q(2)). \tag{3.2.6}$$

From now on, we shall denote $\tau^k(C(SU_q(2)))$ by $\mathcal{A}_k$. For any $a \in C_0(E_q(2))$, define $p_r(a) := \tau^r(I_{SU}\tau^{-r}(a)I_{SU})$. Then p_r is a projection onto $\mathcal{A}_r$, i.e. it maps $C_0(E_q(2))$ onto $\mathcal{A}_r$, and satisfies the following: $p_r^2 = p_r$, and $\|p_r(a)\| \leq \|a\|$ for all a in $C_0(E_q(2))$. Also, for any positive element a of the form $f(\boldsymbol{n})$, one has $0 \leq p_r(a) \leq p_{r+1}(a) \leq a$. Notice that an element a of $C_0(E_q(2))$ is compactly supported if and only if $a = p_r(a)$ for some r. We can therefore define a functional ρ on $C_0(E_q(2))$ to be *compactly supported* if there is an $r \in \mathbb{Z}$ such that whenever $p_r(a) = 0$, $\rho(a)$ is also zero. In section 3.3, we shall give one example to illustrate that unlike for a classical group, the convolution product of a compactly supported element with a compactly supported functional may not be compactly supported.

We end this section by stating a theorem on unitary representations of the group $E_q(2)$ due to Woronowicz. A unitary representation π acting on a Hilbert space $\mathcal{K}$ is, by definition, a unitary element of $M(\mathcal{B}_0(\mathcal{K}) \otimes C_0(E_q(2)))$ satisfying $(id \otimes \mu)\pi = \pi_{12}\pi_{13}$.

Theorem 3.2.3 ([71]) *Let T and b be two closed operators on a Hilbert space $\mathcal{K}$ such that the following conditions are satisfied:*

$$\left.\begin{array}{rl} \text{i.} & T \text{ is self adjoint,} \\ \text{ii.} & b \text{ is normal,} \\ \text{iii.} & T \text{ and } |b| \text{ commute strongly,} \\ \text{iv.} & V_b^* T V_b = T + 2I \text{ on } (ker\, b)^\perp, \\ \text{v.} & \text{joint spectrum of } (T,|b|) \text{ is contained in the} \\ & \text{closure of the set } \{(r, q^{s+r/2}) : r, s \in \mathbb{Z}\}. \end{array}\right\} \tag{3.2.7}$$

Then $\pi = F_q(q^{T/2} b \otimes \boldsymbol{vn})(I \otimes \boldsymbol{v})^{T \otimes I}$ is a unitary representation of $E_q(2)$ acting on $\mathcal{K}$. Conversely, any unitary representation π of $E_q(2)$ is of the above form.

In the above situation, T and b are uniquely determined by π, and for any C^*-subalgebra $\mathcal{B}$ of $\mathcal{B}(\mathcal{K})$, $\pi \in M(\mathcal{B} \otimes C_0(E_q(2)))$ if and only if T and b are both affiliated to $\mathcal{B}$.

3.3 Some Computations

Let $\{e_i\}$ be the canonical orthonormal basis for $L_2(\mathbb{Z})$. Define an operator U on $L_2(\mathbb{Z})^{\otimes 4}$ by $Ue_{i,j,k,l} = e_{k-i,j,k,l}$. It is easy to see that U is unitary, and $\boldsymbol{n}^{-1}\boldsymbol{v} \otimes \boldsymbol{vn} = U^*(q^{N+1} \otimes \ell \otimes \ell \otimes \ell^*)U$. Combining this observation with lemma 3.2.2, we find that for any $a \in C_0(E_q(2))$,

$$\mu(a) = U^* F_q(q^{N+1} \otimes \ell \otimes \ell \otimes \ell^*) UV(a \otimes I)V^* U^* F_q(q^{N+1} \otimes \ell^* \otimes \ell^* \otimes \ell)U.$$

Hence

$$\begin{aligned} &\langle e_{i,j,k,l}, \mu(a) e_{r,s,t,u} \rangle = \\ &\langle V^* U^* F_q(q^{N+1} \otimes \ell^* \otimes \ell^* \otimes \ell) U e_{i,j,k,l}, (a \otimes I) V^* U^* F_q(q^{N+1} \otimes \ell^* \otimes \ell^* \otimes \ell) U e_{r,s,t,u} \rangle. \end{aligned} \tag{3.3.1}$$

Let us next compute $F_q(q^{N+1} \otimes \ell^* \otimes \ell^* \otimes \ell) U e_{i,j,k,l}$. Denote by f_k^n the k$^{\text{th}}$ Fourier coefficient of the function $F_q(q^n \cdot)$.

$$\begin{aligned} & F_q(q^{N+1} \otimes \ell^* \otimes \ell^* \otimes \ell) U e_{i,j,k,l} \\ &= F_q(q^{N+1} \otimes \ell^* \otimes \ell^* \otimes \ell) e_{k-i,j,k,l} \\ &= \sum_{m,n,p,r} \langle e_{m,n,p,r}, F_q(q^{N+1} \otimes \ell^* \otimes \ell^* \otimes \ell) e_{k-i,j,k,l} \rangle e_{m,n,p,r} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{m,n,p,r} (\delta_{m,k-i}(\int F_q(q^{k-i+1} z_1 z_2 \overline{z}_3) z_1^{j-n} z_2^{k-p} z_3^{l-r}\, dz_1\, dz_2\, dz_3)) e_{m,n,p,r} \\
&= \sum_{m,n,p,r} (\delta_{m,k-i}(\int (\sum_s f_s^{k-i+1} z_1^{j-n+s} z_2^{k-p+s} z_3^{l-r-s})\, dz_1\, dz_2\, dz_3) e_{m,n,p,r} \\
&= \sum_n f_{n-j}^{k-i+1} e_{k-i,n,k-j+n,l+j-n}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&V^* U^* F_q(q^{N+1} \otimes \ell^* \otimes \ell^* \otimes \ell) U e_{i,j,k,l} \\
&\quad = \sum_n f_{n-j}^{k-i+1} V^* e_{i-j+n,n,k-j-n,l+j-n} \\
&\quad = \sum_n f_{n-j}^{k-i+1} e_{i-j+n,n,k-i-n,l+j-n}.
\end{aligned}
$$

Now from (3.3.1), we get

$$
\begin{aligned}
&\langle e_{i,j,k,l}, \mu(a) e_{r,s,t,u} \rangle \\
&= \sum_{n,n'} f_{n-j}^{k-i+1} f_{n'-j}^{t-r+1} \langle e_{i-j+n,n,k-i-n,l+j-n}, (a \otimes I) e_{r-s+n',n',t-r-n',u+s-n'} \rangle \\
&= \begin{cases} \sum_n f_n^{k-i+1} f_{n+u-l}^{t-r+1} \langle e_{i+n,j+n}, a e_{r+u-l+n,s+u-l+n} \rangle \\ \qquad\qquad\qquad \text{if } \ t-r-s-u = k-i-j-l, \\ 0 \qquad\qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}
\end{aligned}
\tag{3.3.2}
$$

In particular, denoting $\mu(I_{\{q^\nu\}}(|\boldsymbol{n}|))$ by a_ν, we have

$$
\langle e_{i,j,k,l}, \mu(a_\nu) e_{r,s,t,u} \rangle = \begin{cases} f_{\nu-i}^{t-r+1} f_{\nu-r}^{t-r+1}, & \text{if } \ i-j = r-s, \\ & \qquad j+l = s+u, \\ & \qquad k-i = t-r \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.3}
$$

Therefore

$$
\mu(a_\nu) e_{r,s,t,u} = \sum_i f_{\nu-i}^{t-r+1} f_{\nu-r}^{t-r+1} e_{i,i+s-r,i+t-r,r+u-i}. \tag{3.3.4}
$$

We are ready for the following example.

Example 3.3.1 Take the element $a_0 = I_{\{1\}}(|\boldsymbol{n}|)$ of $C_0(E_q(2))$, and the functional $\rho : a \mapsto \langle e_{00}, a e_{00} \rangle$. It is obvious that both are compactly supported. However, from the above calculations, we get

$$
\begin{aligned}
a_0 * \rho\, e_{rs} &= \sum_{i,j} \langle e_{i,j,0,0}, \mu(a_0) e_{r,s,0,0} \rangle e_{ij} \\
&= |f_{-r}^{1-r}|^2 e_{rs}.
\end{aligned}
$$

All the numbers f_k^n can easily be shown to be nonzero. Therefore $a_0 * \rho$ is *not* compactly supported.

Let us next define a family of functions $J_q(\cdot,\cdot)$ on $\mathbb{C}^q \times \mathbb{Z}$ as follows:

$$J_q(z,k) = \int_{S^1} F_q(zu)u^{-k}\,du, \quad z \in \mathbb{C}^q, k \in \mathbb{Z}, \tag{3.3.5}$$

where F_q is the function we have already encountered in section 3.1. We call them q-analogs of Bessel functions. From equation (3.1.3), we find that for real values of z, and for $u \in S^1$,

$$F_q(zu) = \frac{\exp_{q^{-2}}\left(\frac{z}{1-q}(\frac{1}{2})_{q^2}u^{-1}\right)}{\exp_{q^{-2}}\left(\frac{z}{1-q}(\frac{1}{2})_{q^2}u\right)},$$

which is an analog of the function $\exp(\frac{1}{2}z(u^{-1}-u))$. Recall that the classical Bessel function $J(z,k)$ is the coefficient of t^k in the expansion $\exp(\frac{1}{2}z(t-t^{-1}))$.

Let us describe here another similarity with the classical Bessel functions. Let Δ_q denote the q-differential operator given by

$$\Delta_q f(x) = \frac{f(x)-f(qx)}{(1-q)x}.$$

Define a function $B_q(x,n)$ as follows:

$$B_q(x,n) = J_q(q^{n/2}(1-q)x, n), \quad |x| < q^{-n/2}(1-q)^{-1}.$$

One can see that this function $B_q(x,n)$ obeys the following 'q-differential equation'

$$qx^2\Delta_q^2 f(x) + x\Delta_q f(x) + \left(x^2 - (n)_q^2 q^{-n}\right) f(qx) = 0,$$

which is a q-analog of the classical Bessel differential equation.
Remark. There are several q-analogs of Bessel functions in the literature, the earliest one dating back to Jackson. The q-Bessel functions defined here are very closely related to the ones studied by Exton ($B_q(x,n)$ is, upto a constant factor, equal to Exton's q-Bessel function J(q;n,x); see p. 181, [16]), and seem to be the most natural. We have already cited two 'reasons' above. In chapter 5, we will mention another quantum group-theoretic reason.

Let us list some properties of these functions.

Proposition 3.3.2 *The functions $J_q(\cdot,\cdot)$ obey the following identities:*
1. $\overline{J_q(z,k)} = J_q(\overline{z},k)$. In particular, $J_q(z,k)$ is real whenever z is real.

2. $J_q(z,k) = (z/|z|)^k J_q(|z|,k)$. *More generally,* $J_q(z,k) = w^k J_q(z\overline{w},k)$ *for any* $w \in S^1$.
3. $J_q(-z,k) = (-1)^k J_q(z,k)$.
4. $\sum_{k\in\mathbb{Z}} \overline{J_q(z,k)} J_q(z,k+j) = \delta_{j0}$.
5. $J_q(q^{-n},k) = J_q(q^{n+2}, n+k+1)$.
6. $J_q(q^n,-k) = (-1)^k q^{-k} J_q(q^{n+k},k)$ *for* $n \geq 0$.

Proof: Proofs of 1, 2 and 3 are immediate. To prove 4, observe that for $u \in S^1$, $z \in \mathbb{C}^q$, $F_q(zu) = \sum_k J_q(z,k)u^k$, and $u^{-j}F_q(zu) = \sum_k J_q(z,k+j)u^k$. Also observe that both, as functions of u, are in $L_2(S^1)$; and $|F_q(zu)| = 1$. Now compute their inner product in $L_2(S^1)$. To prove 5, use (3.3.5) and the equality: $F_q(q^{-n}z) = z^{-n-1}F_q(q^{n+2}z)$ for all $z \in S^1$. For the last identity, we find by direct computation that

$$J_q(x,k) = \sum_{\substack{r,s\geq 0\\ s-r=k}} \frac{(-1)^s q^{r(r-1)} x^{r+s}}{(1-q^2)^{r+s}(r)_{q^2}!(s)_{q^2}!}, \qquad x \in (0,1). \tag{3.3.6}$$

From this expression, 6 is immediate. $\square$

Notice that the Fourier coefficient f_k^n introduced earlier is nothing but $J_q(q^n,k)$. Therefore equations (3.3.2) and (3.3.3) will now read as follows:

$$\begin{aligned}&\langle e_{ijkl}, \mu(a)e_{rstu}\rangle\\ &= \begin{cases} \sum_m J_q(q^{k-i+1},m)J_q(q^{t-r+1},m+u-l)\langle e_{i+m,j+m}, ae_{r+u-l+m,s+u-l+m}\rangle & \\ \qquad\qquad \text{if } t-r-s-u = k-i-j-l, & \\ 0 \qquad\qquad \text{otherwise.} & \end{cases}\end{aligned} \tag{3.3.7}$$

$$\langle e_{ijkl}, \mu(a_\nu)e_{rstu}\rangle = \begin{cases} J_q(q^{k-i+1},\nu-i)J_q(q^{k-i+1},\nu-r) & \text{if } \begin{aligned} i-j &= r-s,\\ j+l &= s+u,\\ k-i &= t-r,\end{aligned}\\ 0 & \text{otherwise.}\end{cases} \tag{3.3.8}$$

Before we end the section, let us present here two identities that can be proved using equation (3.3.7) above.

$$\sum_{i\in\mathbb{Z}} q^i J_q(q^j,i)J_q(q^{j+1},i+k) = \begin{cases} 1 & \text{if } k=0,\\ q^{j-1} & \text{if } k=1,\\ 0 & \text{otherwise.}\end{cases} \tag{3.3.9}$$

$$\sum_{i\in\mathbb{Z}} q^{2i} J_q(q^j,i)J_q(q^j,i+k) = \begin{cases} 1+q^{2j-2} & \text{if } \ k=0, \\ q^j & \text{if } \ k=-1, \\ q^{j-2} & \text{if } \ k=1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.10}$$

For the proof of (3.3.9), compute the quantity $\langle e_{r+k,s+1-k,t+1-k,u+k}, \mu(\boldsymbol{n})e_{r,s,t,u}\rangle$ in two ways — first, using the fact that $\mu(\boldsymbol{n}) = \boldsymbol{v}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{v}^*$, and then using (3.3.7) and equate the two. Proof of (3.3.10) is similar. This time compute $\langle e_{r+k,s+k,t+k,u-k}, \mu(\boldsymbol{n}^*\boldsymbol{n})e_{r,s,t,u}\rangle$ in two ways.

Chapter 4

Haar Measure on $E_q(2)$

The role of measures on a noncompact quantum group are played by weights on the corresponding C^*-algebra $C_0(G)$ of 'continuous vanishing-at-infinity functions on G'. Recall that a weight λ on a C^*-algebra $\mathcal{A}$ is a mapping from the set $\mathcal{A}_+$ of positive elements in $\mathcal{A}$ to $[0, \infty]$ such that $\lambda(\alpha x) = \alpha\lambda(x)$ for $\alpha \in \mathbb{R}_+$ and for $x \in \mathcal{A}_+$; and $\lambda(x + y) = \lambda(x) + \lambda(y)$ for $x, y \in \mathcal{A}_+$. We shall very often use the terms 'measure' and 'weight' interchangeably. Let $\mathcal{A}_+^\lambda = \{a \in \mathcal{A}_+ : \lambda(a) < \infty\}$, and let $\mathcal{A}^\lambda$ be the linear span of $\mathcal{A}_+^\lambda$. It is easy to see that λ extends to a positive linear functional on $\mathcal{A}^\lambda$.

A measure λ is called *left invariant* if whenever an element a and its left convolution product $\rho * a$ with a continuous functional ρ are both in $\mathcal{A}^\lambda$, we have $\lambda(\rho * a) = \rho(I)\lambda(a)$. Similarly, λ is said to be *right invariant* if $\lambda(a * \rho) = \rho(I)\lambda(a)$ whenever a and $a * \rho$ are in $\mathcal{A}^\lambda$. It is called both-sided invariant, or just *invariant* if it is both left and right invariant.

We shall prove in the first section that the group $E_q(2)$ admits an invariant measure. Uniqueness of this measure is established in the third section. In the rest of this chapter, we deal with some applications.

4.1 Existence

We have seen in the previous chapter that $C(SU_q(2))$ is embedded in $C_0(E_q(2))$ as a subalgebra, $\tau^k(C(SU_q(2)))$ is an increasing family of C^*-subalgebras of $C_0(E_q(2))$ and $\cup_k \tau^k(C(SU_q(2)))$ is dense in $C_0(E_q(2))$. We also know that $SU_q(2)$ has a unique invariant measure h_{SU} such that $h_{SU}(I_{SU}) = 1$. Let us now try to extend this to a weight on $C_0(E_q(2))$ with the help of the automorphism group $\{\tau^k\}$. A natural candidate would be the following:

$$h(a) = \lim_{k\to\infty} (h_{SU}(\tau^{-k} I_{SU}))^{-1} h_{SU}(I_{SU}(\tau^{-k}a)I_{SU}). \tag{4.1.1}$$

Of course, one has to show first that the right hand side makes sense, i.e. the limit exists. We show below that (4.1.1) indeed defines a weight on $C_0(E_q(2))$, and it is invariant with respect to both the right and the left convolution products. We have seen in section 1.1.2 that h_{SU} is given by the following:

$$h_{SU}(a) = (1-q^2)\sum_{i\geq 0} q^{2i}\langle e_{i0}, ae_{i0}\rangle.$$

Substituting this in (4.1.1), we get

$$\begin{aligned} h(a) &= \lim_{k\to\infty} q^{-2k}(1-q^2)^{-1} h_{SU}(I_{SU}(\tau^{-k}a)I_{SU}) \\ &= \lim_{k\to\infty} \sum_{i\geq -k} q^{2i}\langle e_{i0}, ae_{i0}\rangle \\ &= \sum_{i\in\mathbb{Z}} q^{2i}\langle e_{i0}, ae_{i0}\rangle. \end{aligned} \tag{4.1.2}$$

Obviously, for any positive a, the right hand side limit exists. It is easy to verify now that h is a faithful weight. Also, one can see that $C_c(E_q(2)) \subseteq C_0(E_q(2))^h$. Thus h is a densely defined weight on $C_0(E_q(2))$. The following theorem describes the invariance properties of this weight.

In what follows, $\mathcal{A}$ will denote the C^*-algebra $C_0(E_q(2))$.

Theorem 4.1.1 *For any $a \in \mathcal{A}^h$ and any bounded functional ρ on $\mathcal{A}$, both $a * \rho$ and $\rho * a$ are in $\mathcal{A}^h$, and the following equalities hold:*

$$h(a*\rho) = h(a)\rho(I) = h(\rho * a). \tag{4.1.3}$$

Remark: Notice that although the C^*-algebra $\mathcal{A}$ does not have identity, (4.1.3) makes sense because any continuous functional on $\mathcal{A}$ admits an extension to the multiplier algebra $M(\mathcal{A})$.

We break the proof into several propositions. Let us begin with the following proposition.

Proposition 4.1.2 *Let $a \in \mathcal{A}$ and ρ be a continuous functional on $\mathcal{A}$. If both a and ρ are compactly supported, then a and $a * \rho$ are both in $\mathcal{A}^h$, and*

$$h(a*\rho) = h(a)\rho(I).$$

Proof: Observe that $C_0(E_q(2))$ is a type I C^*-algebra, so that any representation is a direct integral of the irreducible ones. Therefore any representation of the C^*-algebra $C_0(E_q(2))$ can be written as a direct sum $\pi_U \oplus \epsilon_V$, where U and V are two unitary operators acting on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, and π_U and ϵ_V are representations acting on $\ell_2(\mathbb{Z}) \otimes \mathcal{H}$ and $\ell_2(\mathbb{Z}) \otimes \mathcal{K}$ given by:

$$\pi_U : \begin{cases} \boldsymbol{v} \mapsto \ell \otimes I \\ \boldsymbol{n} \mapsto q^N \otimes U \end{cases} \qquad \epsilon_V : \begin{cases} \boldsymbol{v} \mapsto V \\ \boldsymbol{n} \mapsto 0 \end{cases}$$

Therefore any positive functional ρ is of the form

$$a \mapsto \langle u, \pi_U(a)u\rangle + \langle v, \epsilon_V(a)v\rangle. \tag{4.1.4}$$

Denote by $\rho_{u,U}$ the first term on the right hand side above. If $\mathcal{H} = \ell_2(\mathbb{Z})$, and $U = \ell^*$, then we will simply write ρ_u instead of $\rho_{u,U}$. Let $\{f_k\}$ be an orthonormal basis for $\mathcal{H}$. Denote $e_k \otimes f_j$ by e_{kj}.

Step I : Take $a \in (\mathcal{A}_0)_+$, and $\rho = \rho_{e_{mn},U}$.

$$\begin{aligned}
&\Big|hp_{3r}(a*\rho) - h\Big((id\otimes\rho)(\tau^{3r}\otimes\tau^{3r})\mu_{SU}(\tau^{-3r}a)\Big)\Big| \\
&= \Big|\sum_{i\geq -3r} q^{2i}\langle e_{i0}, (a*\rho - (\tau^{3r}\otimes\rho\tau^{3r})\mu_{SU}(\tau^{-3r}a))e_{i0}\rangle\Big| \\
&= \Big|\sum_{i\geq -3r} q^{2i}\langle e_{i\,0\,m\,n}, (id\otimes\pi_U)\Big(\mu(a) - (\tau^{3r}\otimes\tau^{3r})\mu_{SU}(\tau^{-3r}a)\Big)\, e_{i\,0\,m\,n}\rangle\Big|.
\end{aligned} \tag{4.1.5}$$

Suppose for the time being that the right hand side above tends to zero as r goes to infinity. Now,

$$\begin{aligned}
& h((id\otimes\rho)(\tau^{3r}\otimes\tau^{3r})\mu_{SU}(\tau^{-3r}a)) \\
&= h\tau^{3r}((id\otimes\rho\tau^{3r})\mu_{SU}(\tau^{-3r}a)) \\
&= q^{-6r}h((id\otimes\rho\tau^{3r})\mu_{SU}(\tau^{-3r}a)) \\
&= (1-q^2)^{-1}q^{-6r}(h_{SU}\otimes\rho\tau^{3r})\mu_{SU}(\tau^{-3r}a) \\
&= (1-q^2)^{-1}q^{-6r}h_{SU}(\tau^{-3r}a)\rho\tau^{3r}(I_{SU}) \\
&= (1-q^2)^{-1}h_{SU}(a)\rho\tau^{3r}(I_{SU}) \\
&= h(a)\rho\tau^{3r}(I_{SU}).
\end{aligned}$$

Since $\rho\tau^{3r}(I_{SU})$ tends to $\rho(I)$ as $r\to\infty$, $\lim_{r\to\infty} hp_{3r}(a*\rho) = h(a)\rho(I)$. Therefore $a*\rho \in \mathcal{A}_+^h$ and

$$h(a*\rho) = h(a)\rho(I). \tag{4.1.6}$$

We now proceed to show that the right hand side of (4.1.5) indeed goes to zero as r tends to infinity. It follows from (3.2.6) that

$$\begin{aligned}\langle e_{i\,0\,m\,n}, (id\otimes\pi_U)(\mu(a) - (\tau^{3r}\otimes\tau^{3r})\mu_{SU}(\tau^{-3r}a))e_{i\,0\,m\,n}\rangle&\\ = \sum_{\nu=1}^{6}\langle e_{i\,0\,m\,n}, (id\otimes\pi_U)E_\nu e_{i\,0\,m\,n}\rangle,&\end{aligned}\tag{4.1.7}$$

where

$$\begin{aligned}
E_1 &= X^*(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\Lambda(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\left(X - \sum_{s=0}^{r} c_s(-q\beta^*\otimes\beta)^s(\boldsymbol{v}\otimes\boldsymbol{v})^{-s}\right),\\
E_2 &= \left(X^* - \sum_{s=0}^{r} c_s(\boldsymbol{v}\otimes\boldsymbol{v})^s(-q\beta\otimes\beta^*)^s\right)(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\Lambda(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\\
&\quad\times\sum_{s=0}^{r} c_s(-q\beta^*\otimes\beta)^s(\boldsymbol{v}\otimes\boldsymbol{v})^{-s},\\
E_3 &= \left(\sum_{s=0}^{r} c_s(\boldsymbol{v}\otimes\boldsymbol{v})^s(-q\beta\otimes\beta^*)^s\right)(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\Lambda(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\\
&\qquad\qquad\times\left(\sum_{s=0}^{r} c_s(-q\beta^*\otimes\beta)^s(\boldsymbol{v}\otimes\boldsymbol{v})^{-s}\right)\\
&\quad-\left(\sum_{s=0}^{r} d_s^r(\boldsymbol{v}\otimes\boldsymbol{v})^s(-q\beta\otimes\beta^*)^s\right)(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\Lambda(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\\
&\qquad\qquad\times\left(\sum_{s=0}^{r} d_s^r(-q\beta^*\otimes\beta)^s(\boldsymbol{v}\otimes\boldsymbol{v})^{-s}\right),\\
E_4 &= \left(\sum_{s=0}^{r} d_s^r(\boldsymbol{v}\otimes\boldsymbol{v})^s(-q\beta\otimes\beta^*)^s(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2} - (\boldsymbol{v}\otimes\boldsymbol{v})^{3r}\mu_{SU}(\alpha^{*3r})\right)\\
&\quad\times\Lambda(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\sum_{s=0}^{r} d_s^r(-q\beta^*\otimes\beta)^s(\boldsymbol{v}\otimes\boldsymbol{v})^{-s},\\
E_5 &= (\boldsymbol{v}\otimes\boldsymbol{v})^{3r}\mu_{SU}(\alpha^{*3r})\Lambda\\
&\quad\times\left((\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\sum_{s=0}^{r} d_s^r(-q\beta^*\otimes\beta)^s(\boldsymbol{v}\otimes\boldsymbol{v})^{-s} - \mu_{SU}(\alpha^{3r})(\boldsymbol{v}\otimes\boldsymbol{v})^{-3r}\right),\\
E_6 &= (\tau^{3r}\otimes\tau^{3r})\left(\mu_{SU}(\alpha^{*3r}\boldsymbol{t}^{-1/2}a\boldsymbol{t}^{-1/2}\alpha^{3r}) - \mu_{SU}(\boldsymbol{v}^{-3r}a\boldsymbol{v}^{3r})\right),\\
d_s^r &= \binom{3r}{s}_{q^2}
\end{aligned}$$

Assume, for the time being, that

$$\lim_{r\to\infty} q^{-6r}\sup_{i\geq-3r}|\langle e_{i\,0\,m\,n}, (id\otimes\pi_U)E_\nu e_{i\,0\,m\,n}\rangle| = 0 \quad\text{for } \nu=1,2,\tag{4.1.8}$$

$$\lim_{r\to\infty} |\sum_{i\geq -3r} q^{2i}\langle e_{i\,0\,m\,n}, (id\otimes\pi_U)E_\nu e_{i\,0\,m\,n}\rangle| = 0 \quad \text{for } \nu = 3,\ldots,6. \tag{4.1.9}$$

These, together with (4.1.7), will then ensure that the right hand side of (4.1.5) tends to zero as r approaches infinity. So let us now prove (4.1.8) and (4.1.9).

$\boldsymbol{\nu = 1}$. For any integer k,

$$\begin{aligned}
& q^{-kr}\sup_i \Big\| \sum_{s\geq r+1} c_s(-q\beta^*\otimes\pi_U(\beta))^s(\boldsymbol{v}\otimes\pi_U(\boldsymbol{v}))^{-s}e_{i\,0\,m\,n}\Big\| \\
&= q^{-kr}\sup_i \Big\| \sum_{\substack{s\geq r+1\\ i+s\geq 0\\ m+s\geq 0}} c_s(-1)^s q^{s(i+m+2s+1)}(I\otimes I\otimes I\otimes U^*)^s e_{i+s\,s\,m+s\,n}\Big\| \\
&\leq q^{-kr}\sup_i \Big(\sum_{\substack{s\geq r+1\\ i+s\geq 0\\ m+s\geq 0}} c_s^2 q^{2s(i+s+1)+2s(m+s)}\Big)^{1/2} \\
&\leq \Big(\sum_{s\geq r+1} c_s^2 q^{2s+s(2m+s)+s^2-2kr}\Big)^{1/2},
\end{aligned}$$

and now, clearly the right hand side tends to zero as r goes to infinity. Using this for $k = 6$, we get (4.1.8) for $\nu = 1$.

$\boldsymbol{\nu = 2}$. Similar to the previous case.

$\boldsymbol{\nu = 3}$. In this case,

$$\begin{aligned}
& \Big|\sum_{i\geq -3r} q^{2i}\langle e_{i\,0\,m\,n}, (id\otimes\pi_U)E_3 e_{i\,0\,m\,n}\rangle\Big| \\
&\leq \sum_{i\geq -r} q^{2i}\Big|\Big\langle e_{i\,0\,m\,n}, \sum_{s=0}^{r}\sum_{s'=0}^{r}(c_sc_{s'} - d_s^r d_{s'}^r)(\boldsymbol{v}\otimes\pi_U(\boldsymbol{v}))^s(-q\beta\otimes\pi_U(\beta^*))^s \\
&\times (\boldsymbol{t}\otimes\pi_U(\boldsymbol{t}))^{1/2}(id\otimes\pi_U)\Lambda(\boldsymbol{t}\otimes\pi_U(\boldsymbol{t}))^{1/2}(-q\beta^*\otimes\pi_U(\beta))^{s'}(\boldsymbol{v}\otimes\pi_U(\boldsymbol{v}))^{-s'}e_{i\,0\,m\,n}\Big\rangle\Big| \\
&\leq \sum_{i\geq -r} q^{2i}\Big| \sum_{s=0\vee(-i)\vee(-m)}^{r}\ \sum_{s'=0\vee(-i)\vee(-m)}^{r} (c_sc_{s'} - d_s^r d_{s'}^r)q^{s(i+m+2s+1)+s'(i+m+2s'+1)} \\
&\qquad \times \Big\langle (I\otimes I\otimes I\otimes U^*)^s e_{i+s\,s\,m+s\,n}, \\
&\qquad (id\otimes\pi_U)((\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\Lambda(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2})(I\otimes I\otimes I\otimes U^*)^{s'}e_{i+s'\,s'\,m+s'\,n}\Big\rangle\Big| \\
&\leq \text{const.}q^{-2r}\sup_{0\leq s\leq r}(c_s - d_s^r),
\end{aligned}$$

and the right hand side here goes to zero as r approaches infinity.

$\boldsymbol{\nu = 4}$. We shall need the following lemma.

Lemma 4.1.3 *For any integer k,*

$$\lim_{r\to\infty} q^{-kr} \sup_i \Big\| \sum_{s=r+1}^{3r} d_s^r(-q\beta^*\otimes\pi_U(\beta))^s(\alpha\otimes\pi_U(\alpha))^{3r-s}(\boldsymbol{v}\otimes\pi_U(\boldsymbol{v}))^{-3r}e_{i\,0\,m\,n}\Big\| = 0.$$

Proof: $q^{-kr}\Big\|\sum_{s=r+1}^{3r} d_s^r(-q\beta^*\otimes\pi_U(\beta))^s(\alpha\otimes\pi_U(\alpha))^{3r-s}(\boldsymbol{v}\otimes\pi_U(\boldsymbol{v}))^{-3r}e_{i\,0\,m\,n}\Big\|$

$$\begin{aligned}
&\leq \text{const.}\Big(\sum_{s=(r+1)\vee(-i)\vee(-m)}^{3r} (d_s^r)^2 q^{2s(i+m+2s+1)-2kr}\Big)^{1/2}\\
&\leq \text{const.}\Big(\sum_{s=(r+1)\vee(-i)\vee(-m)}^{3r} c_s^2 q^{2s(i+s+1)+s(2m+s)+s^2-2kr}\Big)^{1/2}\\
&\leq \text{const.}\Big(\sum_{s\geq r+1} q^{2s+s(2m+s)+s^2-2kr}\Big)^{1/2}.
\end{aligned}$$

It is clear now that the required limit is zero. □

Now, using the binomial expansion for $\mu_{SU}(\alpha^{*3r})$, we get

$$\begin{aligned}
&\Big|\sum_{i\geq -3r} q^{2i}\langle e_{i\,0\,m\,n},(id\otimes\pi_U)E_4 e_{i\,0\,m\,n}\rangle\Big|\\
&= \Big|\sum_{i\geq -r} q^{2i}\langle e_{i\,0\,m\,n},(id\otimes\pi_U)\Big(\sum_{s=0}^{r} d_s^r(\boldsymbol{v}\otimes\boldsymbol{v})^s\Big((\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}-(\boldsymbol{v}\otimes\boldsymbol{v})^{3r-s}(\alpha^*\otimes\alpha^*)^{3r-s}\Big)\\
&\qquad\times(-q\beta\otimes\beta^*)^s\Lambda(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\sum_{s'=0}^{r} d_{s'}^r(-q\beta^*\otimes\beta)^{s'}(\boldsymbol{v}\otimes\boldsymbol{v})^{-s'}\Big)e_{i\,0\,m\,n}\rangle\Big|\\
&\quad+\Big|\sum_{i\geq -3r} q^{2i}\langle e_{i\,0\,m\,n},(id\otimes\pi_U)\Big(\sum_{s=r+1}^{3r} d_s^r(\boldsymbol{v}\otimes\boldsymbol{v})^{3r}(\alpha^*\otimes\alpha^*)^{3r-s}(-q\beta\otimes\beta^*)^s\\
&\qquad\times\Lambda(\boldsymbol{t}\otimes\boldsymbol{t})^{1/2}\sum_{s'=0}^{r} d_{s'}^r(-q\beta^*\otimes\beta)^{s'}(\boldsymbol{v}\otimes\boldsymbol{v})^{-s'}\Big)e_{i\,0\,m\,n}\rangle\Big|\\
&\leq \text{const.}\ q^{-2r}\sup_{0\leq s\leq r}\Big(1-\Pi_{k\geq 3r-s+1}(1-q^{2k})^{1/2}\Big)\\
&\quad+\text{const.}\ q^{-6r}\sup_i\Big\|(id\otimes\pi_U)\Big(\sum_{s=r+1}^{3r} d_s^r(-q\beta^*\otimes\beta)^s(\alpha\otimes\alpha)^{3r-s}(\boldsymbol{v}\otimes\boldsymbol{v})^{-3r}\Big)e_{i\,0\,m\,n}\Big\|.
\end{aligned}$$

The first term obviously goes to zero as r approaches infinity. By lemma 4.1.3, the same conclusion holds for the second term also. Therefore (4.1.9) holds for $\nu = 4$.

$\boldsymbol{\nu = 5}$. Similar to the previous case.

$\boldsymbol{\nu = 6}$. Let us denote by P_r the operator $\prod_{k\geq 3r+1}(1-q^{2N+2k})^{-1/2}\otimes I$. Then

$$|\sum_{i\geq -3r} q^{2i}\langle e_{i\,0\,m\,n},(id\otimes\pi_U)E_6 e_{i\,0\,m\,n}\rangle|$$

$$
\begin{aligned}
&= \Big|\sum_{i\geq -3r} q^{2i}\Big\langle e_{i+3r,0,m+3r,n}, \\
&\qquad (id\otimes\pi_U)\left(\mu_{SU}(\alpha^{*3r})\Lambda\mu_{SU}(\alpha^{3r}) - \mu_{SU}(\boldsymbol{v}^{-3r}a\boldsymbol{v}^{3r})\right) e_{i+3r,0,m+3r,n}\Big\rangle\Big| \\
&= \Big|\sum_{i\geq -3r} q^{2i}\left\langle e_{i+3r,0,m+3r,n}, (id\otimes\pi_U)\mu_{SU}\left(\tau^{-3r}(P_r a P_r - a)\right) e_{i+3r,0,m+3r,n}\right\rangle\Big| \\
&= q^{-6r}(1-q^2)^{-1}\left|h_{SU}\left(\tau^{-3r}(P_r a P_r - a)\right)\right| \\
&= (1-q^2)^{-1}|h_{SU}(P_r a P_r - a)|.
\end{aligned}
$$

Therefore (4.1.9) holds for $\nu = 6$.

Observe that in all the estimates above, we have used crucially the fact that m is not allowed to go too near minus infinity. Essentially the same calculations can therefore be used to show that the conclusion holds even when ρ is of the form $\rho_{u,U}$, where

$$
u = \sum_{i\geq m} \lambda_{ij} e_{ij}, \quad m\in\mathbb{Z}. \tag{4.1.10}
$$

Step II : Take $a \in (\mathcal{A}_0)_+$, and ρ compactly supported.

In this case, it can be shown that ρ must be of the form $\rho_{u,U} + \langle w, \epsilon_V(\cdot)w\rangle$, where u is as in (4.1.10). For $\rho = \rho_{u,U}$, the proof is already done in step I. Let us now prove the equality for $\rho = \langle w, \epsilon_V(\cdot)w\rangle$. It is easy to see that in this case, $a*\rho\in\mathcal{A}_0$ and $a*\rho = (id\otimes\rho)\mu_{SU}(a)$. Therefore $h(a*\rho) = (1-q^2)^{-1}(h_{SU}\otimes\rho)\mu_{SU}(a) = (1-q^2)^{-1}h_{SU}(a)\rho(I) = h(a)\rho(I)$.

Step III : Take $a\in(\mathcal{A}_r)_+$, and ρ to be any compactly supported state. Observe that $\tau^{-r}a*\rho\tau^r = \tau^{-r}(a*\rho)$. Since $\tau^{-r}a\in(\mathcal{A}_0)_+$ and $\rho\tau^r$ is compactly supported, we have $h(\tau^{-r}a*\rho\tau^r) = h(\tau^{-r}a)\rho\tau^r(I) = q^{2r}h(a)\rho(I)$. On the other hand $h(\tau^{-r}(a*\rho)) = q^{2r}h(a*\rho)$. Therefore $h(a*\rho) = h(a)\rho(I)$. As $\cup_r\mathcal{A}_r$ is just the linear span of the $(\mathcal{A}_r)_+$'s, and any compactly supported contnuous functional is a linear combination of compactly supported states, the equality above holds for any compactly supported a and any compactly supported continuous functional ρ. □

As a corollary of the above proposition, we now prove a few identities involving q-Bessel functions that are going to be very useful in what follows.

Corollary 4.1.4 *The functions $J_q(\cdot,\cdot)$ satisfy the following two identities:*

$$
\sum_{i\in\mathbb{Z}} q^{2i} J_q(q^{k-i}, -i-j) J_q(q^{k-i+j}, -i) = \delta_{j0}, \tag{4.1.11}
$$

$$\sum_{i\in\mathbb{Z}} q^{2i} J_q(q^{i+k}, j) J_q(q^{i+k'}, j) = \delta_{kk'} q^{2(1-k+j)}. \tag{4.1.12}$$

Proof: Let ρ be the functional $a \mapsto \langle e_{k-1,0}, ae_{j+k-1,j}\rangle$, and let $b = \boldsymbol{v}^j g(\boldsymbol{n})$, where g is the function $q^r z \mapsto I_{\{0\}}(r) z^{-j}$, $r \in \mathbb{Z}$, $z \in S^1$. Using (3.3.7), we get

$$\begin{aligned} h(b * \rho) &= \sum_{i\in\mathbb{Z}} q^{2i} \langle e_{i,0,k-1,0}, \mu(\boldsymbol{v}^j g(\boldsymbol{n})) e_{i,0,j+k-1,j}\rangle \\ &= \sum_{i,m\in\mathbb{Z}} q^{2i} J_q(q^{k-i}, m) J_q(q^{k-i+j}, m+j) \langle e_{i+m,m}, \boldsymbol{v}^j g(\boldsymbol{n}) e_{i+j+m,j+m}\rangle \\ &= \sum_{i\in\mathbb{Z}} q^{2i} J_q(q^{k-i}, -i-j) J_q(q^{k-i+j}, -i). \end{aligned}$$

Therefore by theorem 4.1.2, we get

$$\sum_{i\in\mathbb{Z}} q^{2i} J_q(q^{k-i}, -i-j) J_q(q^{k-i+j}, -i) = h(\boldsymbol{v}^j g(\boldsymbol{n}))\rho(I) = \delta_{j0},$$

as required.

For the second identity, take g to be the function $q^r z \mapsto I_{\{1-k'+j\}}(r) z^{k-k'}$, b to be the element $\boldsymbol{v}^{k-k'} g(\boldsymbol{n})$, and ρ to be the functional $a \mapsto \langle e_{1-k,0}, ae_{1-k',k'-k}\rangle$. Now, as before, use (3.3.7) and theorem 4.1.2 to get the required identity. □

The following identities are straightforward consequences of the above corollary and proposition 3.3.2.

$$\sum_{i\in\mathbb{Z}} q^{2i} J_q(q^{k-i}, j-i)^2 = q^{2j} \quad \forall\, k \in \mathbb{Z}, \tag{4.1.13}$$

$$\sum_{i\in\mathbb{Z}} q^{2i} J_q(q^{i-k+1}, j-k)^2 = q^{2j} \quad \forall\, k \in \mathbb{Z}, \tag{4.1.14}$$

$$\sum_{i\in\mathbb{Z}} q^{2i} J_q(q^{k-i}, r-i) J_q(q^{k-i+j}, r-i+j) = q^{2r}\delta_{j0} \quad \forall\, k \in \mathbb{Z}. \tag{4.1.15}$$

Proposition 4.1.5 *For any $a \in \mathcal{A}_+^h$, and state ρ, we have $h(a * \rho) \le h(a)$.*

Proof: For any $a \in \mathcal{A}$, $\lim \|p_r(a) - a\| = 0$, and for any state ρ on $\mathcal{A}$, $\lim \rho p_r(a) = \rho(a)$ for all a. This, along with the fact that $\mu(\mathcal{A}) \subseteq \mathcal{A} \otimes \mathcal{A}$, yields that $p_r(a) * \rho p_r$ converges to $a * \rho$ in norm. Also, for a and ρ positive, all these quantities remain positive. Since $p_r(a)$ and ρp_r are compactly supported, we have $h(p_r(a) * \rho p_r) = h p_r(a) \rho p_r(I)$. Therefore, combining all these observations together, we get

$$\begin{aligned} h(a*\rho) &\leq \lim_{r\to\infty} hp_r(a)\rho p_r(I) \\ &= h(a)\lim \rho p_r(I) \\ &= h(a). \end{aligned}$$

□

Let $(L_2(h),\eta_h,\pi_h)$ and $(\mathcal{H}_\rho,\eta_\rho,\pi_\rho)$ be the GNS triples associated with the weights h and $a\mapsto h(a*\rho)$ respectively. We shall very often identify a and $\eta_h(a)$ for $a\in\{b\in\mathcal{A}: h(b^*b)<\infty\}$.

Proposition 4.1.6 *Let g_{jk} be the function on $\mathbb{C}^q$ defined by $q^r z\mapsto I_{\{j\}}(r)z^k$, $r\in\mathbb{Z}$, $z\in S^1$. Then $\{q^{-j}\boldsymbol{v}^i g_{jk}(\boldsymbol{n}): i,j,k\in\mathbb{Z}\}$ is a complete orthonormal basis in $L_2(h)$ and $\{q^{-j}\eta_\rho(\boldsymbol{v}^i g_{jk}(\boldsymbol{n})): i,j,k\in\mathbb{Z}\}$ is an orthonormal system of vectors in $\mathcal{H}_\rho$.*

Proof: The first part is easy. We prove the second part here. Write a_1 for $\boldsymbol{v}^i g_{jk}(\boldsymbol{n})$ and a_2 for $\boldsymbol{v}^r g_{st}(\boldsymbol{n})$. Then from the first part, $h(a_i^*a_i)<\infty$, $i=1,2$. Now, using proposition 4.1.5, we get

$$\begin{aligned} |\langle\eta_\rho(a_1),\eta_\rho(a_2)\rangle| &= |h(a_1^*a_2*\rho)| \\ &\leq |h(a_1^*a_1*\rho)|^{1/2}|h(a_2^*a_2*\rho)|^{1/2} \\ &\leq |h(a_1^*a_1)|^{1/2}|h(a_2^*a_2)|^{1/2} \\ &< \infty. \end{aligned}$$

Next,

$$\begin{aligned} \langle\eta_\rho(a_1),\eta_\rho(a_2)\rangle &= h(a_1^*a_2*\rho) \\ &= \lim_{m\to\infty} hp_m(a_1^*a_2*\rho) \\ &= \lim_m \sum_{l\geq -m} q^{2l}\langle e_{l0},(a_1^*a_2*\rho)e_{l0}\rangle \\ &= \lim_m \rho\Big(\sum_{l\geq -m} q^{2l}\rho_{e_{l0}}*a_1^*a_2\Big) \\ &= \lim_m \rho(T_m), \end{aligned}$$

where $T_m=\sum_{l\geq -m} q^{2l}\rho_{e_{l0}}*a_1^*a_2$. Use (3.3.2) to get

$$\rho_{e_{l0}}*a_1^*a_2=\delta_{i,r+t-k}\delta_{j,s+t-k}\boldsymbol{v}^{k-t}V_{\boldsymbol{n}}^{t-k}(f_{t-k+s-l}^{N+t-k+1-l}f_{s-l}^{N+1-l}\otimes I),$$

where $V_{\boldsymbol{n}}$ is the unitary appearing in the polar decomposition of $\boldsymbol{n}$, and f_j^{N+i} denote the operator $e_k\mapsto f_j^{k+i}e_k$. Therefore

$$T_m=\delta_{i,r+t-k}\delta_{j,s+t-k}\boldsymbol{v}^{k-t}V_{\boldsymbol{n}}^{t-k}\Big(\sum_{l\geq -m} q^{2l}f_{t-k+s-l}^{N+t-k+1-l}f_{s-l}^{N+1-l}\otimes I\Big).$$

Now using corollary 4.1.11, one can show that for any positive functional ρ of the form $\rho_{u,U}$, $\lim \rho(T_m) = \delta_{ir}\delta_{js}\delta_{kt}q^{2j}\|u\|^2$. Since $\epsilon_V(T_m) = \delta_{ir}\delta_{js}\delta_{kt}q^{2j}$ for all $m > -j$, it follows that

$$\langle \eta_\rho(\boldsymbol{v}^i g_{jk}(\boldsymbol{n})), \eta_\rho(\boldsymbol{v}^r g_{st}(\boldsymbol{n}))\rangle = \delta_{ir}\delta_{js}\delta_{kt}q^{2s},$$

which proves the assertion. $\square$

Proof of theorem 4.1.1: Take $a \in \mathcal{A}_+^h$, and ρ to be a state. Define a map S from $L_2(h)$ to $\mathcal{H}_\rho$ by the prescription $=a \mapsto \eta_\rho(a)$. Then by the above proposition, S extends as an isometry to the whole of $L_2(h)$. Therefore, for any $a \in \mathcal{A}_+^h$, $\|Sa^{1/2}\| = \|a^{1/2}\|$, which means $h(a * \rho) = h(a) = h(a)\rho(I)$. By taking linear combinations, the same conclusion holds for any $a \in \mathcal{A}^h$ and any continuous functional ρ.

Proof of the other equality, namely, $h(\rho * a) = h(a)\rho(I)$, is exactly similar. $\square$

Remark 4.1.7 For a positive, theorem 4.1.1 tells us that if $h(a)$ is finite, then so also is $h(a * \rho)$, and $h(a * \rho) = h(a)$. The equality actually holds always, i.e. if $h(a) = \infty$, then $h(a * \rho)$ also is infinity. To see this, take $a_r = a - (I - \tau^r(I_{SU}))a(I - \tau^r(I_{SU}))$. Then $h(a_r) = hp_r(a)$, so that $h(a_r)$ increases to infinity. On the other hand, since $h(a_r) < \infty$, $h(a_r) = h(a_r * \rho)$, and since $a - a_r \geq 0$, $h(a * \rho) \geq h(a_r * \rho)$. Therefore we must have $h(a * \rho) = \infty$.

Let $\mathcal{M}$ be the von Neumann subalgebra of $\mathcal{B}(L_2(\mathbb{Z}) \otimes L_2(\mathbb{Z}))$ generated by $C_0(E_q(2))$. Equation (4.1.2) actually defines a normal semifinite weight on this von Neumann algebra $\mathcal{M}$. Let $\mathcal{M}_+^h$ denote $\{a \in \mathcal{M}_+ : h(a) < \infty\}$ and let $\mathcal{M}^h$ be the linear span of $\mathcal{M}_+^h$. One can see from lemma 3.2.2 that μ extends to a homomorphism from $\mathcal{M}$ to $\mathcal{M} \otimes \mathcal{M}$. Hence for an element a of $\mathcal{M}$ and a continuous functional ρ on $\mathcal{M}$, the products $a * \rho := (id \otimes \rho)\mu(a)$ and $\rho * a := (\rho \otimes id)\mu(a)$ are well defined elements of $\mathcal{M}$. Now observe that the proof of theorem 4.1.1 actually tells us the following:

Theorem 4.1.8 *For any $a \in \mathcal{M}^h$ and any bounded functional ρ on $\mathcal{M}$, both $a * \rho$ and $\rho * a$ are in $\mathcal{M}^h$, and the following equalities hold:*

$$h(a * \rho) = h(a)\rho(I) = h(\rho * a).$$

4.2 A Basis for $L_2(h)$

We present, in the following lemma, an orthonormal basis for the L_2-space associated with the haar weight, which will be extremely useful for our later purposes. Notice that for any pair of integers s, t, the function $z \mapsto J_q(q^s z, t)$ is a bounded measurable function on $\mathbb{C}^q$. Therefore $J_q(q^s \boldsymbol{n}, t)$ is a bounded normal operator on $L_2(\mathbb{Z}) \otimes L_2(\mathbb{Z})$.

Lemma 4.2.1 *Let $L_2(h)$ denote the completion of $\{a \in \mathcal{M} : h(a^*a) < \infty\}$ with respect to the inner product $\langle a, b\rangle := h(a^*b)$. Then $\{\boldsymbol{v}^r J_q(q^s\boldsymbol{n}, t) : r, s, t \in \mathbb{Z}\}$ is a complete set of orthogonal vectors in $L_2(h)$.*

Proof: Let us first compute the inner product $\langle \boldsymbol{v}^r J_q(q^s\boldsymbol{n}, t), \boldsymbol{v}^{r'} J_q(q^{s'}\boldsymbol{n}, t')\rangle$. From the definition of J_q and the commutation relation between $\boldsymbol{v}$ and $\boldsymbol{n}$, it follows that $\boldsymbol{v}^r J_q(q^s\boldsymbol{n}, t) = J_q(q^{r+s}\boldsymbol{n}, t)\boldsymbol{v}^r$. Therefore

$$
\begin{aligned}
\left\langle \boldsymbol{v}^r J_q(q^s\boldsymbol{n}, t), \boldsymbol{v}^{r'} J_q(q^{s'}\boldsymbol{n}, t')\right\rangle &= h\left(J_q(q^s\boldsymbol{n}, t)^* \boldsymbol{v}^{r-r'} J_q(q^{s'}\boldsymbol{n}, t')\right) \\
&= h\left(J_q(q^s\boldsymbol{n}, t)^* J_q(q^{r-r'+s'}\boldsymbol{n}, t')\boldsymbol{v}^{r-r'}\right) \\
&= \delta_{rr'} h\left(J_q(q^s\boldsymbol{n}, t)^* J_q(q^{s'}\boldsymbol{n}, t')\right) \\
&= \delta_{rr'} \sum_{i\in\mathbb{Z}} q^{2i} \left\langle J_q(q^s\boldsymbol{n}, t)e_{i0}, J_q(q^{s'}\boldsymbol{n}, t')e_{i0}\right\rangle \\
&= \delta_{rr'}\delta_{tt'} \sum_{i\in\mathbb{Z}} q^{2i} J_q(q^{i+s}, t) J_q(q^{i+s'}, t).
\end{aligned}
$$

From (4.1.12), we conclude that

$$
\langle \boldsymbol{v}^r J_q(q^s\boldsymbol{n}, t), \boldsymbol{v}^{r'} J_q(q^{s'}\boldsymbol{n}, t')\rangle = \delta_{rr'}\delta_{ss'}\delta_{tt'} q^{2(1-s+t)}. \tag{4.2.1}
$$

Hence all we need to prove now is that $\{\boldsymbol{v}^r J_q(q^s\boldsymbol{n}, t) : r, s, t \in \mathbb{Z}\}^\perp = \{0\}$.

Take an *operator* $a \in L_2(h)$ such that $\langle \boldsymbol{v}^i J_q(q^j\boldsymbol{n}, k), a\rangle = 0$ for all i, j and k in $\mathbb{Z}$. This implies

$$
\sum_r q^{2r} J_q(q^{j+r}, k)\langle e_{r-i,k}, a e_{r,0}\rangle \ \ \forall i, j, k \in \mathbb{Z}. \tag{4.2.2}
$$

Write $u_{ik}(z) = \sum_r q^r \langle e_{r-i,k}, a e_{r,0}\rangle z^r$, $\xi_j^k(z) = \sum_r q^{r+j-k-1} J_q(q^{j+r}, k) z^r$. It is easy to see that u_{ik} and ξ_j^k are in $L_2(S^1)$ for all i, j and k. (4.2.2) says that $\langle u_{ik}, \xi_j^k\rangle = 0$ for all i, j and k. Therefore if we can show that for each fixed $k \in \mathbb{Z}$, $\{\xi_j^k\}_{j\in\mathbb{Z}}$ is complete in $L_2(S^1)$, then it follows that $u_{ik} = 0$ for all i and k, which, in turn, implies that $a e_{r0} = 0$ for all $r \in \mathbb{Z}$. This means $h(a^*a) = 0$, i.e. a is zero in $L_2(h)$.

So we now prove that $\{\xi_j^k\}$ is complete in $L_2(S^1)$. Fix any $k \in \mathbb{Z}$. It follows from (4.1.12) that $\{\xi_j^k\}_j$ is an orthonormal set of vectors. Observe that $z^s \xi_j^k(z) =$

$\xi^k_{j-s}(z)$. Therefore $\langle \xi^k_0, \xi^k_j \rangle = \delta_{j0}$ implies that $\xi^k_0(z) \neq 0$ almost everywhere. If P is the projection onto $\{\xi^k_j : j \in \mathbb{Z}\}^\perp$, then P commutes with all the multiplication operators, and hence is itself a multiplication by an indicator. Since $P\xi^k_0 = 0$ and $\xi^k_0 \neq 0$ almost everywhere, P must be zero. □

4.3 Uniqueness of h

We shall establish the uniqueness of the weight h in this section. Before going to the proof, let us first find the modular group associated with h. Let $\mathcal{U}$ be the $*$-subalgebra of $C_0(E_q(2))$ generated by $\{\boldsymbol{v}^r J_q(q^s \boldsymbol{n}, t) \; : r, s, t \in \mathbb{Z}\}$. It is easy to see that $\mathcal{U}$ is contained in $L_2(h)$ and it follows from lemma 4.2.1 that it is dense there.

Let P_r denote the projection $\sum_s |e_{rs}\rangle\langle e_{rs}|$. For an operator a on $L_2(\mathbb{Z}) \otimes L_2(\mathbb{Z})$, denote by a^{rs} the operator $P_r a P_s$. Then any bounded operator can be written as a strong sum of the form $\sum_r \sum_s a^{r+s,s}$. Observe that for $a \in \mathcal{U}$, the first summation is finite. Define an operator Δ' on $\mathcal{U}$ as follows:

$$\Delta' a = \sum_r q^{2r} (\sum_s a^{r+s,s}). \tag{4.3.1}$$

It is straightforward to verify that $\Delta' a = (\boldsymbol{n}^*\boldsymbol{n}) a (\boldsymbol{n}^*\boldsymbol{n})^{-1}$. One can now check that the closure Δ of this operator Δ' is a positive self-adjoint operator, and is in fact the modular operator associated with the weight h. That is, we have $h(ab) = h(b\Delta a)$ for $a, b \in \mathcal{U}$. The corresponding modular automorphism group Δ^{it} is given by

$$\Delta^{it} a = U_t a U_t^*, \tag{4.3.2}$$

where $U_t = (\boldsymbol{n}^*\boldsymbol{n})^{it} = |\boldsymbol{n}|^{2it}$. One can check that

$$\Delta^{it} a = \epsilon_{q^{-it}} * a * \epsilon_{q^{-it}}, \; t \in \mathbb{R}, \tag{4.3.3}$$

where ϵ_z is as in (3.2.3).

Theorem 4.3.1 *Let h be the weight defined in section 4.1 and let h_1 be a normal semifinite weight on $\mathcal{M}$ satisfying the invariance properties described in theorem 4.1.8. Then $h_1 = c_0 h$ for some $c_0 \in \mathbb{R}_+$.*

Proof: It is easy to see from (4.3.2) that the fixed point subalgebra of $\mathcal{M}$ for the automorphism group Δ^{it} is $\{f(\boldsymbol{n}) : \; f$ is a bounded measurable function on $\mathbb{C}^q\}$. It follows from (4.3.3) that $h_1\Delta^{it} = h_1$. Thus h_1 is a $\{\Delta^{it}\}_{t\in\mathbb{R}}$ invariant semifinite weight. Therefore by a Radon-Nikodym theorem for weights due to Pedersen and Takesaki (theorem A.4.1), there is a positive measurable function f on $\mathbb{C}^q$ such that $h_1(a) = h(f(\boldsymbol{n})a)$ whenever the right hand side makes sense.

By the invariance properties of h and h_1, it follows that $h(f(z\boldsymbol{n})a) = h(f(\boldsymbol{n})a)$ for all $z \in S^1$. Fix any $z \in S^1$. Let g be the following function on $\mathbb{C}^q$: $g(q^k w) = f(w) - f(zw)$, $w \in \mathbb{C}^q$. Then $g(\boldsymbol{n}) = f(\boldsymbol{n}) - f(z\boldsymbol{n})$, and hence $h(g(\boldsymbol{n})a) = 0$ for all a for which the left hand side expression makes sense. Let

$$g_0(q^r w) = \frac{q^{4|r|}\overline{g(q^r w)}}{(1 + |g(q^r w)|^2)}, \quad r \in \mathbb{Z},\ w \in S^1.$$

Now choose $a = g_0(\boldsymbol{n})$ and make use of the faithfulness of h to get $g(\boldsymbol{n}) = 0$, i.e. $f(\boldsymbol{n}) = f(z\boldsymbol{n})$. From this we can conclude that there are positive reals c_k, for $k \in \mathbb{Z}$, such that

$$h_1(a) = \sum_{r\in\mathbb{Z}} q^{2r} c_r \langle e_{r0}, a e_{r0}\rangle. \tag{4.3.4}$$

Our claim now is that each c_r is strictly positive. Suppose, if possible, $c_\nu = 0$ for some ν. Let ρ_0 denote the functional $a \mapsto \langle e_{00}, a e_{00}\rangle$. Then using (3.3.2), we get $h_1(\rho_0 * a_\nu) = \sum_r q^{2r} c_r J_q(q^{r+1}, \nu)^2$. Therefore from the invariance properties of h_1, it follows that $\sum_r q^{2r} c_r J_q(q^{r+1}, \nu)^2 = 0$, which forces each c_r to be zero.

Next, observe that if we use the weight h_1 instead of h in the proof of proposition 4.1.4, we get the following identities:

$$\sum_{i\in\mathbb{Z}} q^{2i} c_i J_q(q^{k-i}, -i-j) J_q(q^{k-i+j}, -i) = \delta_{j0} c_0, \tag{4.3.5}$$

$$\sum_{i\in\mathbb{Z}} q^{2i} c_i J_q(q^{i+k}, j) J_q(q^{i+k'}, j) = \delta_{kk'} q^{2(1-k+j)} c_{1-k+j}. \tag{4.3.6}$$

Denote $\boldsymbol{v}^r J_q(q^s\boldsymbol{n}, t)$ by α_{rst} for the rest of this proof. Using the above equations in place of (4.1.12) in the proof of lemma 4.2.1, we find that $\{\alpha_{rst} : r, s, t \in \mathbb{Z}\}$ form an orthogonal basis for $L_2(h_1)$ also. Simple computations using the identity (4.3.6) give

$$h_1({\alpha_{r's't'}}^* \alpha_{rst}) = \delta_{rr'}\delta_{ss'}\delta_{tt'} c_{1-s+t} q^{2(1-s+t)},$$

$$h_1(\alpha_{rst}{\alpha_{r's't'}}^*) = \delta_{rr'}\delta_{ss'}\delta_{tt'} c_{1-s+t-r} q^{2(1-s+t-r)}.$$

Denote by $\Delta_{(1)}$ the modular operator associated with h_1. Then $h_1(\alpha_{rst}{\alpha_{r's't'}}^*) = h_1({\alpha_{r's't'}}^*\Delta_{(1)}(\alpha_{rst}))$. From the above computations, we get

$$h_1\left({\alpha_{r's't'}}^*\Delta_{(1)}(\alpha_{rst})\right) = \left(\frac{c_{1-s+t-r}}{c_{1-s+t}}\right) q^{-2r} h_1({\alpha_{r's't'}}^*\alpha_{rst})$$

for any r', s' and t' in $\mathbb{Z}$. Hence $\Delta_{(1)}(\alpha_{rst}) = \left(\frac{c_{1-s+t-r}}{c_{1-s+t}}\right) q^{-2r}\alpha_{rst}$. Consequently, for any $z \in S^1$, $\Delta^z_{(1)}(\alpha_{rst}) = \left(\frac{c_{1-s+t-r}}{c_{1-s+t}}\right)^z q^{-2rz}\alpha_{rst}$. It is easy to see from (4.3.4) that $\Delta_{(1)}g(\boldsymbol{n}) = g(\boldsymbol{n})$ for any compactly supported function g on $\mathbb{C}^q$. Therefore $\Delta^z_{(1)}g(\boldsymbol{n}) = g(\boldsymbol{n})$ for all $z \in S^1$.

Take any $s', t' \in \mathbb{Z}$. Let $\{g_k\}_{k\in\mathbb{Z}}$ be a sequence of compactly supported functions on $\mathbb{C}^q$ converging uniformly to $J_q(q^{s'}\cdot, t')$. Then

$$\begin{aligned}
\Delta^z_{(1)}(\alpha_{rs't'}) &= \lim_{k\to\infty} \Delta^z_{(1)}(\boldsymbol{v}^r g_k(\boldsymbol{n}))\\
&= \lim_{k\to\infty} \Delta^z_{(1)}\Big(\alpha_{rst}\Big(\frac{g_k(\boldsymbol{n})}{J_q(q^s\boldsymbol{n}, t)}\Big)\Big)\\
&= \lim_{k\to\infty} \Delta^z_{(1)}(\alpha_{rst})\Delta^z_{(1)}\Big(\frac{g_k(\boldsymbol{n})}{J_q(q^s\boldsymbol{n}, t)}\Big)\\
&= \lim_{k\to\infty} \Big(\frac{c_{1-s+t-r}}{c_{1-s+t}}\Big)^z q^{-2rz}\alpha_{rst}\Big(\frac{g_k(\boldsymbol{n})}{J_q(q^s\boldsymbol{n}, t)}\Big)\\
&= \lim_{k\to\infty} \Big(\frac{c_{1-s+t-r}}{c_{1-s+t}}\Big)^z q^{-2rz}\boldsymbol{v}^r g_k(\boldsymbol{n})\\
&= \Big(\frac{c_{1-s+t-r}}{c_{1-s+t}}\Big)^z q^{-2rz}\alpha_{rs't'}.
\end{aligned}$$

Therefore $\frac{c_{1-s+t-r}}{c_{1-s+t}}$ is independent of s and t, which means that there is a positive real c such that $c_r = c_0 c^r$. Now it remains only to show that $c = 1$.

If we use the weight h_1 instead of h in the proof of (4.1.15), we get

$$\sum_r q^{2r}c^r J_q(q^{k-r}, k-r)J_q(q^{k'-r}, k'-r) = \delta_{kk'}q^{2k}c^k. \tag{4.3.7}$$

Let us now treat the following two cases separately.

Case I. $c < q^{-1}$.

Let $\xi_k(z) = \sum_r q^{r-k}J_q(q^{k-r}, k-r)z^r$. Then $z^s\xi_k(z) = \xi_{k+s}(z)$ and from (4.1.15), $\langle\xi_k, \xi_{k'}\rangle = \delta_{kk'}$. Therefore $\{\xi_k\}_{k\in\mathbb{Z}}$ form a complete orthonormal basis for $L_2(S^1)$. Let $u(z) = \sum_r q^r c^r J_q(q^{-r}, -r)z^r$. We claim that $u \in L_2(S^1)$. Since $qc < 1$, it is enough to show that $\sum_{r<0} q^{2r}c^{2r}J_q(q^{-r}, -r)^2 < \infty$, or which is the same thing, $\sum_{r>0}(qc)^{-2r}J_q(q^r, r)^2 < \infty$. Let m be a positive integer such that $(qc)^{-2} < q^{-m}$. Then

$$\sum_{r>0}(qc)^{-2r}J_q(q^r, r)^2 < \sum_{r>0} q^{-mr}|J_q(q^r, r)|.$$

Now using the expression (3.3.6) for $J_q(\cdot,\cdot)$, we get

$$\sum_{r>0}(qc)^{-2r}J_q(q^r, r)^2 \;\leq\; \sum_{r>0}\Big|\sum_{\substack{j,k\geq 0\\ j-k=r}} \frac{(-1)^j q^{k(k-1)+r(j+k)-mr}}{(1-q^2)^{j+k}(j)_{q^2}!(k)_{q^2}!}\Big|$$

$$
\begin{aligned}
&\leq \sum_{r>0}\sum_{k\geq 0} \frac{q^{k(k-1)+r(2k+r)-mr}}{(1-q^2)^{2k+r}}\\
&\leq \sum_{r>0}\sum_{k\geq 0} \frac{q^{k(k-1)}}{(1-q^2)^{2k}}\frac{q^{r(r-m)}}{(1-q^2)^{r}}\\
&< \infty.
\end{aligned}
$$

Thus $u \in L_2(S^1)$. From (4.3.7), $\langle u, \xi_k\rangle = 0$ if $k \neq 0$. Therefore $u \in \mathbb{C}\xi_0$, which implies $c^r = 1$ for all r. Hence $c = 1$.

Case II. $c \geq q^{-1}$.
In this case, let $\xi_k(z) = \sum q^{r-k}\sqrt{c^{r-k}}J_q(q^{k-r}, k-r)z^r$. Then, as before, $z^s\xi_k(z) = \xi_{k+s}(z)$, and from (4.3.7), $\langle \xi_k, \xi_{k'}\rangle = \delta_{kk'}$. So $\{\xi_k\}$ form an orthonormal basis for $L_2(S^1)$. Let $u(z) = \sum q^r\sqrt{c^{-r}}J_q(q^{-r}, -r)z^r$. Since $qc^{-1} < 1$, one can show just like in the earlier case that $u \in L_2(S^1)$. Now from (4.1.15), $\langle u, \xi_k\rangle = 0$ for $k \neq 0$, so that $u \in \mathbb{C}\xi_0$, which is impossible since $c > 1$.

The proof is thus complete. □

4.4 Haar Measure for the Dual Group $\widehat{E_q(2)}$

As an application of the identities proved in section 4.2, we shall establish the existence of a right- and a left-invariant measure for the dual $\widehat{E_q(2)}$ of $E_q(2)$. Before going into that, we need to have an explicit formula for computing the comultiplication $\hat{\mu}$ of this quantum group.

To fix notation, let us first recall very briefly the basic facts about $\widehat{E_q(2)}$. Denote by Σ_q the set $\{(r, q^{s+r/2}) : r, s \in \mathbb{Z}\}$, and by $\overline{\Sigma_q}$, its closure. Write $\mathcal{H} = L_2(\Sigma_q)$. Let $\{e_{i,\lambda} : (i,\lambda) \in \Sigma_q\}$ be the canonical orthonormal basis for $\mathcal{H}$. Denote by $\boldsymbol{b}$ and $\boldsymbol{T}$ the following operators:

$$
\begin{aligned}
\boldsymbol{b} &: e_{i,\lambda} \mapsto \lambda e_{i+2,\lambda},\\
\boldsymbol{T} &: e_{i,\lambda} \mapsto i e_{i,\lambda}.
\end{aligned}
$$

The pair $(\boldsymbol{b}, \boldsymbol{T})$ then satisfies the conditions listed in (3.2.7). The algebra $C_0(\widehat{E_q(2)})$ of vanishing-at-infinity functions on $\widehat{E_q(2)}$ is the norm closure of all finite sums of the form $\sum_k V_{\boldsymbol{b}}^k f_k(\boldsymbol{T}, |\boldsymbol{b}|)$, where $k \in \mathbb{Z}$, $V_{\boldsymbol{b}}$ is the unitary appearing in the polar decomposition of $\boldsymbol{b}$, and f_k's are continuous functions on $\overline{\Sigma_q}$ vanishing at infinity such that $f_k(s, 0) = 0$ whenever $k \neq 0$.

There is a unique map $\hat{\mu} \in \operatorname{mor}(C_0(\widehat{E_q(2)}), C_0(\widehat{E_q(2)}) \otimes C_0(\widehat{E_q(2)}))$ such that

$$\begin{aligned} \hat{\mu}(\boldsymbol{b}) &= q^{\frac{1}{2}\boldsymbol{T}} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes q^{-\frac{1}{2}\boldsymbol{T}}, \\ \hat{\mu}(\boldsymbol{T}) &= \boldsymbol{T} \otimes I + I \otimes \boldsymbol{T}. \end{aligned} \tag{4.4.1}$$

This is the comultiplication for the dual group $\widehat{E_q(2)}$. The following proposition is similar to lemma 3.2.2 for $E_q(2)$ and tells us that the comultiplication is unitarily implemented.

Proposition 4.4.1 *Define an operator $V : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ as follows: $Ve_{i,\lambda,k,\nu} = e_{i+k,\lambda q^{-k/2},-k,\nu}$. Then V is unitary, and we have $\hat{\mu}(a) = W(a \otimes I)W^*$ for all $a \in C_0(\widehat{E_q(2)})$, where $W = F_q(\boldsymbol{b}^{-1}q^{\frac{1}{2}\boldsymbol{T}} \otimes q^{\frac{1}{2}\boldsymbol{T}}\boldsymbol{b})V$.*

Proof: It is enough to verify the equality for $a = q^{\boldsymbol{T}}$ and for $a = \boldsymbol{b}$.

Notice that $(\boldsymbol{b} \otimes q^{-\frac{1}{2}\boldsymbol{T}})(q^{\frac{1}{2}\boldsymbol{T}} \otimes \boldsymbol{b}) = q^{-2}(q^{\frac{1}{2}\boldsymbol{T}} \otimes \boldsymbol{b})(\boldsymbol{b} \otimes q^{-\frac{1}{2}\boldsymbol{T}})$. Therefore by theorem 3.1.3 and (4.4.1), it follows that

$$\hat{\mu}(\boldsymbol{b}) = F_q(\boldsymbol{b}^{-1}q^{\frac{1}{2}\boldsymbol{T}} \otimes q^{\frac{1}{2}\boldsymbol{T}}\boldsymbol{b})(\boldsymbol{b} \otimes q^{-\frac{1}{2}\boldsymbol{T}})F_q(\boldsymbol{b}^{-1}q^{\frac{1}{2}\boldsymbol{T}} \otimes q^{\frac{1}{2}\boldsymbol{T}}\boldsymbol{b})^*.$$

Now a straightforward computation shows that $\boldsymbol{b} \otimes q^{-\frac{1}{2}\boldsymbol{T}} = V(\boldsymbol{b} \otimes I)V^*$. Therefore $\hat{\mu}(\boldsymbol{b}) = W(\boldsymbol{b} \otimes I)W^*$.

Next, observe that $(q^{\boldsymbol{T}} \otimes q^{\boldsymbol{T}})(\boldsymbol{b}^{-1}q^{\frac{1}{2}\boldsymbol{T}} \otimes q^{\frac{1}{2}\boldsymbol{T}}\boldsymbol{b}) = (\boldsymbol{b}^{-1}q^{\frac{1}{2}\boldsymbol{T}} \otimes q^{\frac{1}{2}\boldsymbol{T}}\boldsymbol{b})(q^{\boldsymbol{T}} \otimes q^{\boldsymbol{T}})$. Therefore

$$\hat{\mu}(q^{\boldsymbol{T}}) = F_q(\boldsymbol{b}^{-1}q^{\frac{1}{2}\boldsymbol{T}} \otimes q^{\frac{1}{2}\boldsymbol{T}}\boldsymbol{b})(q^{\boldsymbol{T}} \otimes q^{\boldsymbol{T}})F_q(\boldsymbol{b}^{-1}q^{\frac{1}{2}\boldsymbol{T}} \otimes q^{\frac{1}{2}\boldsymbol{T}}\boldsymbol{b})^*.$$

Since $(q^{\boldsymbol{T}} \otimes q^{\boldsymbol{T}}) = V(q^{\boldsymbol{T}} \otimes I)V^*$, we have $\hat{\mu}(q^{\boldsymbol{T}}) = W(q^{\boldsymbol{T}} \otimes I)W^*$. $\square$

We shall now use this proposition to derive an equation similar to (3.3.2).

Let U be an operator on $\mathcal{H} \otimes \mathcal{H}$ given on the basis vectors by $Ue_{i,\lambda,k,\nu} = e_{i+k,\nu/\lambda,k,\nu}$. U is then unitary and $U(\boldsymbol{b}^{-1}q^{\frac{1}{2}\boldsymbol{T}} \otimes q^{\frac{1}{2}\boldsymbol{T}}\boldsymbol{b})^*U^*e_{i,\lambda,k,\nu} = \lambda q^{1+i/2}e_{i,\lambda,k-2,\nu}$. It is easy to see from this that

$$F_q(\boldsymbol{b}^{-1}q^{\frac{1}{2}\boldsymbol{T}} \otimes q^{\frac{1}{2}\boldsymbol{T}}\boldsymbol{b})^*e_{i,\lambda,k,\nu} = \sum_r J_q\left(\frac{\nu}{\lambda}q^{1+\frac{i+k}{2}}, r\right)e_{i+2r,\lambda,k-2r,\nu},$$

and hence

$$W^*e_{i,\lambda,k,\nu} = \sum_r J_q\left(\frac{\nu}{\lambda}q^{1+\frac{i+k}{2}}, r\right)e_{i+k,\lambda q^{r-k/2},-k+2r,\nu}. \tag{4.4.2}$$

Therefore

$$\begin{aligned}
&\langle e_{i',\lambda',k',\nu'}, \hat{\mu}(a)e_{i,\lambda,k,\nu}\rangle \\
&= \sum_r J_q\left(\frac{\nu}{\lambda}q^{1+\frac{i+k}{2}}, r\right) J_q\left(\frac{\nu}{\lambda}q^{1+\frac{i'+k'}{2}}, r+\frac{k'-k}{2}\right) \\
&\qquad \times \left\langle e_{i'+k',\lambda q^{r-k/2}}, ae_{i+k,\lambda q^{r-k/2}}\right\rangle \delta_{\lambda\lambda'}\delta_{\nu\nu'}. \qquad (4.4.3)
\end{aligned}$$

Writing $\lambda = q^{j+i/2}$, $\lambda' = q^{j'+i'/2}$, $\nu = q^{l+k/2}$ and $\nu' = q^{l'+k'/2}$, we get

$$\begin{aligned}
&\left\langle e_{i',q^{j'+i'/2},k',q^{l'+k'/2}}, \hat{\mu}(a)e_{i,q^{j+i/2},k,q^{l+k/2}}\right\rangle \\
&= \begin{cases} \sum_r J_q(q^{l-j+k+1}, r)J_q(q^{l'-j'+k'+1}, r+l-l') & \\ \qquad \times \left\langle e_{i'+k',q^{j+r+(i-k)/2}}, ae_{i+k,q^{j+r+(i-k)/2}}\right\rangle & \text{if } \begin{array}{rcl} i+2j &=& i'+2j', \\ k+2l &=& k'+2l', \end{array} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \qquad (4.4.4)$$

Remark 4.4.2 If in the proof of proposition 4.4.1, we use theorem 3.1.4 instead of theorem 3.1.3, we get $\hat{\mu}(a) = W_0^*(I \otimes a)W_0$, where $W_0 = V_0F_q(\boldsymbol{b}q^{-\frac{1}{2}\boldsymbol{T}} \otimes q^{-\frac{1}{2}\boldsymbol{T}}\boldsymbol{b}^{-1})$, and V_0 is the unitary operator that maps $e_{i,\lambda,k,\nu}$ to $e_{i,\lambda,i+k,\nu q^{i/2}}$. Using the above expression for $\hat{\mu}$, we get

$$W_0e_{i,\lambda,k,\nu} = \sum_r J_q\left(\frac{\lambda}{\nu}q^{1-(i+k)/2}, r\right) e_{i+2r,\lambda,i+k,\nu q^{r+i/2}},$$

and consequently,

$$\begin{aligned}
&\langle e_{i',\lambda',k',\nu'}, \hat{\mu}(a)e_{i,\lambda,k,\nu}\rangle \\
&= \sum_r J_q\left(\frac{\lambda}{\nu}q^{1-\frac{i+k}{2}}, r\right) J_q\left(\frac{\lambda}{\nu}q^{1-\frac{i'+k'}{2}}, r+\frac{i-i'}{2}\right) \\
&\qquad \times \left\langle e_{i'+k',\nu q^{r+i/2}}, ae_{i+k,\nu q^{r+i/2}}\right\rangle \delta_{\lambda\lambda'}\delta_{\nu\nu'}. \qquad (4.4.5)
\end{aligned}$$

$$\begin{aligned}
&\langle e_{i',q^{j'+i'/2},k',q^{l'+k'/2}}, \hat{\mu}(a)e_{i,q^{j+i/2},k,q^{l+k/2}}\rangle \\
&= \begin{cases} \sum_r J_q(q^{j-k-l+1}, r)J_q(q^{j'-k'-l'+1}, r+j'-j) & \\ \qquad \times \langle e_{i'+k',q^{l+r+(i+k)/2}}, ae_{i+k,q^{l+r+(i+k)/2}}\rangle & \text{if } \begin{array}{rcl} i+2j &=& i'+2j', \\ k+2l &=& k'+2l', \end{array} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \qquad (4.4.6)$$

We are now ready to prove the existence of a left and a right invariant measure on $\widehat{E_q(2)}$.

Proposition 4.4.3 *The weight $h_\ell : a \mapsto tr(q^{-\boldsymbol{T}}\boldsymbol{b}^*\boldsymbol{b}a)$ is a left invariant weight for the quantum group $\widehat{E_q(2)}$.*

Proof: Any representation π of the C^*-algebra $C_0(\widehat{E_q(2)})$ acting on a Hilbert space $\mathcal{H}_\pi$ is uniquely determined by the pair $(\pi(\boldsymbol{b}), \pi(\boldsymbol{T}))$ which satisfy the relations (3.2.7). π is irreducible if and only if the pair $(\pi(\boldsymbol{b}), \pi(\boldsymbol{T}))$ is irreducible, in the sense that there is no proper closed subspace of $\mathcal{H}_\pi$ invariant under both $\pi(\boldsymbol{b})$ and $\pi(\boldsymbol{T})$. Simple computations now yield that the following are *all* the irreducible representations of $C_0(\widehat{E_q(2)})$:

$$\left.\begin{array}{lll} \pi^{(m)} & : & \left.\begin{array}{l} \boldsymbol{b} \mapsto q^m \ell^* \\ \boldsymbol{T} \mapsto 2N \end{array}\right\} \text{ on } L_2(\mathbb{Z}), \quad m \in \mathbb{Z}, \\ \pi^{(m)} & : & \left.\begin{array}{l} \boldsymbol{b} \mapsto q^m \ell^* \\ \boldsymbol{T} \mapsto 2N + I \end{array}\right\} \text{ on } L_2(\mathbb{Z}), \quad m \in \mathbb{Z} + \frac{1}{2}, \\ \epsilon^{(m)} & : & \left.\begin{array}{l} \boldsymbol{b} \mapsto 0 \\ \boldsymbol{T} \mapsto m \end{array}\right\} \text{ on } \mathbb{C}, \quad m \in \mathbb{Z}. \end{array}\right\} \tag{4.4.7}$$

It is easy to see that $C_0(\widehat{E_q(2)}) = \bigoplus_{m \in \frac{1}{2}\mathbb{Z}} \pi^{(m)}(C_0(\widehat{E_q(2)}))$. Therefore any positive functional ρ on $C_0(\widehat{E_q(2)})$ will be of the form

$$\rho(a) = \langle u, (a \otimes I)u \rangle + \sum_{m \in \mathbb{Z}} \alpha_m \epsilon^{(m)}(a), \tag{4.4.8}$$

where $u \in L_2(\Sigma_q) \otimes \mathcal{K}$, $\mathcal{K}$ being a Hilbert space, and α_m are all nonnegative real numbers, with $\sum_m \alpha_m < \infty$. Let $\{f_k\}_{k \in \mathbb{Z}}$ be an orthonormal basis for $\mathcal{K}$, and let $\rho_{i,\lambda,j,\nu}$ denote the functional $a \mapsto \langle e_{i,\lambda}, a e_{j,\nu} \rangle$ on $C_0(\widehat{E_q(2)})$. Then from (4.4.8) it follows that there are scalars $\beta_{i,\lambda,r}$, $(i,\lambda) \in \Sigma_q$, $r \in \mathbb{Z}$, such that $\sum_{i,\lambda,r} |\beta_{i,\lambda,r}|^2 < \infty$, and

$$\rho(a) = \sum \overline{\beta_{i,\lambda,r}} \beta_{i',\lambda',r} \rho_{i,\lambda,i',\lambda'}(a) + \sum \alpha_m \epsilon^{(m)}(a). \tag{4.4.9}$$

Therefore it is enough to show that

$$\begin{aligned} h_\ell(\rho_{i,\lambda,i',\lambda'} * a) &= h_\ell(a) \rho_{i,\lambda,i',\lambda'}(I), \\ h_\ell(\epsilon^{(m)} * a) &= h_\ell(a) \epsilon^{(m)}(I). \end{aligned}$$

Let j and j' be such that $q^j = \lambda q^{-i/2}$ and $q^{j'} = \lambda' q^{-i'/2}$. Then using (4.4.4) and the identity (4.1.12),

$$\begin{aligned}
h_\ell(\rho_{i,\lambda,i',\lambda'} * a) &= \sum_{k,l} q^{2l} \left\langle e_{i,q^{j+i/2},k,q^{l+k/2}}, \hat{\mu}(a) e_{i',q^{j'+i'/2},k,q^{l+k/2}} \right\rangle \\
&= \sum_{k,l,r} q^{2l} J_q(q^{l-j+k+1}, r) J_q(q^{l-j'+k+1}, r) \\
&\qquad\qquad \times \left\langle e_{i'+k,q^{j+r+(i-k)/2}}, a e_{i+k,q^{j+r+(i-k)/2}} \right\rangle \delta_{\lambda\lambda'} \\
&= \sum_{k,r} q^{2(r+j-k)} \left\langle e_{i+k,q^{j+r+(i-k)/2}}, a e_{i+k,q^{j+r+(i-k)/2}} \right\rangle \delta_{ii'} \delta_{\lambda\lambda'} \\
&= \sum_{k,s} q^{2s} \langle e_{k,q^{s+k/2}}, a e_{k,q^{s+k/2}} \rangle \delta_{ii'} \delta_{\lambda\lambda'} \\
&= h_\ell(a) \rho_{i,\lambda,i',\lambda'}(I).
\end{aligned}$$

For the other equality, notice that if U is the unitary operator on $L_2(\Sigma_q)$ given by $U e_{i,\lambda} = e_{i-m,\lambda q^{-m/2}}$, then $\epsilon^{(m)} * a = U a U^*$ for all a. Therefore $h_\ell(\epsilon^{(m)} * a) = h_\ell(U a U^*) = \mathrm{tr}(q^{-\boldsymbol{T}} \boldsymbol{b}^* \boldsymbol{b} U a U^*) = \mathrm{tr}(U^* q^{-\boldsymbol{T}} \boldsymbol{b}^* \boldsymbol{b} U a) = \mathrm{tr}(q^{-\boldsymbol{T}} \boldsymbol{b}^* \boldsymbol{b} a) = h_\ell(a) = h_\ell(a) \epsilon^{(m)}(I)$.

The proof is thus complete. □

A similar proof, using (4.4.6) instead of (4.4.4), shows that the weight $h_r : a \mapsto tr\,(q^{\boldsymbol{T}} \boldsymbol{b}^* \boldsymbol{b} a)$ is a right invariant weight.

Remark. Existence of invariant measures for the group $E_q(2)$ and its dual have also been treated by S. Baaj ([5]). He worked with a different realization of the C^*-algebra $C_0(E_q(2))$. The advantage of using Woronowicz's realization of $C_0(E_q(2))$ (which is what we have used here) is that the relation between the haar measure for $E_q(2)$ and $SU_q(2)$ becomes very clear (see equation (4.1.1)), and suggests a possible generalization to the class of all noncompact quantum groups that are related to some compact quantum group via a contraction procedure.

Chapter 5

Representations of $E_q(2)$

In this chapter, we continue our study of the quantum group $E_q(2)$. In section 1, we find all the irreducible unitary representations of $E_q(2)$, and give a Clebsch-Gordon decomposition formula. Orthogonality relations among the matrix entries of these irreducible representations are also computed. In the next section, the right regular representation is introduced. It is described in terms of the pair of closed operators (b, T) associated with it, which makes its direct sum decomposition very clear. In the previous two chapters, we have seen examples of how to get identities involving q-functions (in this case, q-Bessel functions) using quantum group methods. Section 3 illustrates yet another technique. The identities in chapter 4 were used to prove the uniqueness of the haar weight and also for proving the existence of a left- and a right-invariant weight for the dual group $\widehat{E_q(2)}$. The identities presented in this chapter are likely to be very useful in calculations involving $SL_q(2, \mathbb{C})$, the double group over $E_q(2)$. The last two sections are devoted to the description of the quantum complex plane as a quotient space for the group $E_q(2)$, and to the study of the action of $E_q(2)$ on it.

5.1 Unitary Representations

Tensor product of two representations has been defined in section 1.1 for compact quantum groups. The same definition works in the present situation also. That is, if w_1 and w_2 are two unitary representations of $E_q(2)$ acting on the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, then their tensor product $w_1 \textcircled{\scriptsize{T}}\, w_2$ is given by $\phi_{13}(w_1)\phi_{23}(w_2)$, where ϕ_{13} and ϕ_{23} are morphisms from $M(\mathcal{B}_0(\mathcal{H}_1) \otimes C_0(E_q(2)))$ and $M(\mathcal{B}_0(\mathcal{H}_2) \otimes C_0(E_q(2)))$

respectively to $M(\mathcal{B}_0(\mathcal{H}_1)\otimes\mathcal{B}_0(\mathcal{H}_2)\otimes C_0(E_q(2)))$ given by

$$\phi_{13}(a\otimes b)=a\otimes I\otimes b,\quad \phi_{23}(a\otimes b)=I\otimes a\otimes b. \tag{5.1.1}$$

By theorem 3.2.3, there is a pair (b_1,T_1) of closed operators on $\mathcal{H}_1$ and another pair (b_2,T_2) of closed operators on $\mathcal{H}_2$, both satisfying the requirements (3.2.7), such that $w_i=F_q(q^{T_i/2}b_i\otimes \boldsymbol{vn})(I\otimes\boldsymbol{v})^{T_i\otimes I}$, $i=1,2$. Let $b_1 \bigcirc\!\!\!\!\!\top\, b_2$ and $T_1 \bigcirc\!\!\!\!\!\top\, T_2$ be the operators associated with the tensor product $w_1 \bigcirc\!\!\!\!\!\top\, w_2$. Let us express $b_1 \bigcirc\!\!\!\!\!\top\, b_2$ and $T_1 \bigcirc\!\!\!\!\!\top\, T_2$ in terms of the b_i's and the T_i's. From the definition,

$$w_1 \bigcirc\!\!\!\!\!\top\, w_2=F_q(q^{T_1/2}b_1\otimes I\otimes\boldsymbol{vn})(I\otimes I\otimes\boldsymbol{v})^{T_1\otimes I\otimes I}F_q(I\otimes q^{T_2/2}b_2\otimes\boldsymbol{vn})(I\otimes I\otimes\boldsymbol{v})^{I\otimes T_2\otimes I}. \tag{5.1.2}$$

From the commutation relations between $\boldsymbol{v}$ and $\boldsymbol{n}$, we get

$$(I\otimes I\otimes\boldsymbol{v})^{T_1\otimes I\otimes I}(I\otimes q^{T_2/2}b_2\otimes\boldsymbol{vn})(I\otimes I\otimes\boldsymbol{v}^*)^{T_1\otimes I\otimes I}=q^{T_1}\otimes q^{T_2/2}b_2\otimes\boldsymbol{vn},$$

so that

$$(I\otimes I\otimes\boldsymbol{v})^{T_1\otimes I\otimes I}F_q(I\otimes q^{T_2/2}b_2\otimes\boldsymbol{vn})(I\otimes I\otimes\boldsymbol{v}^*)^{T_1\otimes I\otimes I}=F_q(q^{T_1}\otimes q^{T_2/2}b_2\otimes\boldsymbol{vn}).$$

Substituting these in (5.1.2), we get

$$w_1 \bigcirc\!\!\!\!\!\top\, w_2=F_q(q^{T_1/2}b_1\otimes I\otimes\boldsymbol{vn})F_q(q^{T_1}\otimes q^{T_2/2}b_2\otimes\boldsymbol{vn})(I\otimes I\otimes\boldsymbol{v})^{T_1\otimes I\otimes I+I\otimes T_2\otimes I}.$$

By the remark following theorem 3.1.4, it now follows that

$$w_1 \bigcirc\!\!\!\!\!\top\, w_2=F_q\left((q^{T_1/2}\otimes q^{T_2/2})(b_1\otimes q^{-T_2/2}+q^{T_1/2}\otimes b_2)\otimes\boldsymbol{vn}\right)(I\otimes I\otimes\boldsymbol{v})^{(T_1\otimes I+I\otimes T_2)\otimes I},$$

which means,

$$\begin{aligned} b_1 \bigcirc\!\!\!\!\!\top\, b_2 &= b_1\otimes q^{-T_2/2}+q^{T_1/2}\otimes b_2,\\ T_1 \bigcirc\!\!\!\!\!\top\, T_2 &= T_1\otimes I+I\otimes T_2. \end{aligned} \tag{5.1.3}$$

Let us next find all the *irreducible* unitary representations of $E_q(2)$. Let (b,T) be a pair of operators on $\mathcal{H}$ satisfying (3.2.7). We call (b,T) *irreducible* if $\mathcal{H}$ does not have a nonzero proper closed subspace which is kept invariant under b, b^* and T. Thanks to the following proposition, finding irreducible representations of $E_q(2)$ is equivalent to finding irreducible copies of (b,T).

Proposition 5.1.1 *Let w be a unitary representation of $E_q(2)$. Then w is irreducible if and only if the associated pair (b,T) is irreducible.*

Proof: If the associated pair (b, T) is not irreducible, then clearly w can not be irreducible. We shall now prove the converse.

From (4.4.7), $\{\pi^{(m)} : m \in \frac{1}{2}\mathbb{Z}\}$ together with $\{\epsilon^{(m)} : m \in \mathbb{Z}\}$ constitute all the irreducible representations of the pair (b, T). Now $w(\epsilon^{(m)}(b), \epsilon^{(m)}(T)) = w(0, m) = \boldsymbol{v}^m \in C_b(E_q(2))$, which means that it is a one dimensional representation and hence obviously irreducible. Denote $w(\pi^{(m)}(b), \pi^{(m)}(T))$ by $w^{(m)}$. It is enough now to show that each $w^{(m)}$ is irreducible. To this end, let us compute the quantity $\langle e_{rij}, w^{(m)} e_{skl}\rangle$. For $m \in \mathbb{Z}$, we have

$$\langle e_{rij}, w^{(m)} e_{skl}\rangle = \begin{cases} J_q(q^{m+1+k-s}, j-l) & \text{if } i = k-r-s, j = l+r-s, \\ 0 & \text{otherwise,} \end{cases} \tag{5.1.4}$$

and for $m \in \mathbb{Z} + \frac{1}{2}$,

$$\langle e_{rij}, w^{(m)} e_{skl}\rangle = \begin{cases} J_q(q^{m+\frac{1}{2}+k-s}, j-l) & \text{if } i = k-r-s-1,\ j = l+r-s, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1.5}$$

Let P be a nonzero projection on $L_2(\mathbb{Z})$ such that $w^{(m)}(P \otimes I) = (P \otimes I)w^{(m)}$. Then for any continuous functional ρ on $C_0(E_q(2))$, $(id \otimes \rho)w^{(m)}$ commutes with P. Take a nonzero vector $u = \sum u_s e_s \in P(L_2(\mathbb{Z}))$. Then $u_t \neq 0$ for some t. Take any $p \in \mathbb{Z}$.
Case I: $m \in \mathbb{Z}$. Let ρ be the functional $a \mapsto \langle e_{00}, a e_{t+p,t-p}\rangle$. Then $(id \otimes \rho)w^{(m)}(u) = u_t J_q(q^{m+p+1}, p-t)e_p$.
Case II: $m \in \mathbb{Z} + \frac{1}{2}$. Take ρ to be the functional $a \mapsto \langle e_{0,-1}, a e_{t+p+1,t-p-1}\rangle$. Then $(id \otimes \rho)w^{(m)}(u) = u_t J_q(q^{m+p+\frac{3}{2}}, p-t)e_p$.
Therefore in both the cases, $e_p \in P(L_2(\mathbb{Z}))$. The choice of P being arbitrary, we have $P = I$. Thus $w^{(m)}$ is irreducible. $\square$

Notice that the proof above supplies all the irreducible representations of $E_q(2)$. Using (5.1.3), we can now prove the following Clebsch-Gordon type decomposition formula.

Proposition 5.1.2 *Let $w^{(m)}$ be as above. Then*

$$w^{(m)} \mathbin{\textcircled{\scriptsize$\top$}} w^{(n)} = \begin{cases} \bigoplus_{k\in\mathbb{Z}} w^{(k)} & \text{if } m-n \in \mathbb{Z}, \\ \bigoplus_{k\in\mathbb{Z}+\frac{1}{2}} w^{(k)} & \text{if } m-n \in \mathbb{Z}+\frac{1}{2}. \end{cases}$$

Let $w_{rs}^{(m)}$, $r, s \in \mathbb{Z}$, denote the matrix entries of $w^{(m)}$ with respect to the basis $\{e_i\}$, i.e. $w_{rs}^{(m)} = (\rho \otimes id)w^{(m)}$, where ρ is the functional $b \mapsto \langle e_r, be_s\rangle$. It is easy to see, from (5.1.4) and (5.1.5), that

$$w_{rs}^{(m)} = \begin{cases} \boldsymbol{v}^{r+s} J_q(q^{m-s+1}\boldsymbol{n}, r-s) & \text{if } m \in \mathbb{Z}, \\ \boldsymbol{v}^{r+s+1} J_q(q^{m-s+\frac{1}{2}}\boldsymbol{n}, r-s) & \text{if } m \in \mathbb{Z} + \frac{1}{2}. \end{cases} \tag{5.1.6}$$

This gives yet another justification for calling $J_q(\cdot,\cdot)$ the q-analogues of Bessel functions.

Since $J_q(q^i., j) \in C_0(\mathbb{C}^q)$ for all i and j, we have $w_{rs}^{(m)} \in C_0(E_q(2))$ for all r and s in $\mathbb{Z}$, and for all $m \in \frac{1}{2}\mathbb{Z}$. From (5.1.6), one can see that

$$\boldsymbol{v}^r J_q(q^s\boldsymbol{n}, t) = w_{ij}^{(m)}, \tag{5.1.7}$$

where $m = s - 1 + (r-t)/2$, $i = [(r+t)/2]$ and $j = [(r-t)/2]$. We are now in a position to prove the following proposition.

Proposition 5.1.3 *The matrix entries $w_{rs}^{(m)}$ satisfy the following*:

i. $w_{rs}^{(m)} \in L_2(h) \quad \forall\, r, s \in \mathbb{Z}, \forall m \in \frac{1}{2}\mathbb{Z}$.
ii. (**orthogonality relations**) $\langle w_{rs}^{(m)}, w_{r's'}^{(m')}\rangle = \delta_{mm'}\delta_{rr'}\delta_{ss'}q^{2(r-[m])}$.
iii. $\{q^{[m]-r}w_{rs}^{(m)} : r, s \in \mathbb{Z}, m \in \frac{1}{2}\mathbb{Z}\}$ *form an orthonormal basis for* $L_2(h)$.

Proof: Follows from (4.2.1), lemma 4.2.1, (5.1.6) and (5.1.7). □

Remark 5.1.4 Though the matrix entries in the given basis are all in $L_2(h)$, this is not, in general, true; that is, there are vectors u, v such that $(\langle u| \otimes id)w^{(m)}(v \otimes \cdot) \notin L_2(h)$. One could, for example, take $u = \sum_{n\geq 1} \frac{1}{n} e_{-n}$ and $v = e_0$. Thus each $w^{(m)}$ has both square-integrable and non square-integrable matrix entries. This situation can never arise for a locally compact *group* with a two-sided invariant measure. For a proof of this, see Robert([43]).

5.2 The Regular Representation

Denote $q^{[m]-r}w_{rs}^{(m)}$ by $\xi_{rs}^{(m)}$. Define two operators $\tilde{b}$ and $\tilde{T}$ on $L_2(h)$ as follows:

$$\left.\begin{array}{rcl} \tilde{b}\xi_{rs}^{(m)} &=& q^m\xi_{r,s+1}^{(m)}, \\ \tilde{T}\xi_{rs}^{(m)} &=& \begin{cases} 2s\xi_{rs}^{(m)} & \text{if } m \in \mathbb{Z}, \\ (2s+1)\xi_{rs}^{(m)} & \text{if } m \in \mathbb{Z} + \frac{1}{2}. \end{cases} \end{array}\right\} \tag{5.2.1}$$

$\tilde{b}$ and $\tilde{T}$ are then closed operators on $L_2(h)$ and they satisfy (3.2.7). Therefore $w(\tilde{b},\tilde{T})$ is a unitary representation of $E_q(2)$ acting on $L_2(h)$. We shall denote this representation by $\Re$. Notice that the restriction of $\Re$ to the closed span of $\{\xi_{rs}^{(m)} : s \in \mathbb{Z}\}$ is equivalent to $w^{(m)}$. By a slight modification in the proof of lemma 2.3.1, one can show that for any bounded linear functional ρ on $C_0(E_q(2))$, the operator $a \mapsto a*\rho$, when restricted to the dense subspace $L_2(h)\cap C_0(E_q(2))$, defines a bounded operator on $L_2(h)$. Let us denote this operator by R_ρ. We shall now show that for any bounded functional ρ on $C_0(E_q(2))$, the expression $\Re_\rho := (id\otimes\rho)\Re$ makes sense, and is precisely the operator R_ρ on $L_2(h)$. The way we proceed is as follows. First we show that for ρ belonging to a dense set $\mathcal{D}$ of continuous linear functionals, $\Re_\rho$ defines a bounded operator on $L_2(h)$, and is equal to R_ρ. Next we observe that the association $\rho \mapsto \Re_\rho$ is continuous. Take now any continuous functional ρ on $C_0(E_q(2))$. There is a sequence ρ_n in $\mathcal{D}$ such that $\|\rho_n - \rho\|$ converges to zero. Define $\Re_\rho$ to be the limit $\lim_{n\to\infty}\Re_{\rho_n}$. This limit is independent of the particular sequence chosen, and since $\Re_{\rho_n} = R_{\rho_n}$, it follows that $\Re_\rho = R_\rho$. The key step, therefore, is to show that there is a dense set $\mathcal{D}$ of continuous linear functionals on $C_0(E_q(2))$ for which $(id\otimes\rho)\Re = R_\rho$.

We have observed in the course of the proof of theorem 4.1.1 that any continuous functional ρ on $C_0(E_q(2))$ is of the form

$$\rho(a) = \langle u_1, \pi_{U_0}(a)u_2\rangle + \langle v_1, \epsilon_{V_0}(a)v_2\rangle, \tag{5.2.2}$$

where U_0 and V_0 are two unitary operators acting on the spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, and $u_1, u_2 \in L_2(\mathbb{Z})\otimes\mathcal{H}$, $v_1, v_2 \in \mathcal{K}$. Let us first show that if $\rho = \langle v_1, \epsilon_{V_0}(\cdot)v_2\rangle$, then $(id\otimes\rho)\Re$ is same as the operator R_ρ. In this case, $(id\otimes\epsilon_{V_0})\Re = (I\otimes V_0)^{\tilde{T}\otimes I}$. Therefore

$$\begin{aligned}
(id\otimes\rho)\Re\xi_{rs}^{(m)} &= (I\otimes\langle v_1|)((id\otimes\epsilon_{V_0})\Re)(\xi_{rs}^{(m)}\otimes v_2)\\
&= \begin{cases} \langle v_1, V_0^{2s}v_2\rangle\xi_{rs}^{(m)} & \text{if } m\in\mathbb{Z},\\ \langle v_1, V_0^{2s+1}v_2\rangle\xi_{rs}^{(m)} & \text{if } m\in\mathbb{Z}+\frac{1}{2}.\end{cases}
\end{aligned}$$

On the other hand, since $(id\otimes\epsilon_{V_0})\mu(\boldsymbol{v}) = \boldsymbol{v}\otimes V_0$, and $(id\otimes\epsilon_{V_0})\mu(\boldsymbol{n}) = \boldsymbol{n}\otimes V_0^*$, we have, for $m\in\mathbb{Z}$,

$$\begin{aligned}
R_\rho(\xi_{rs}^{(m)}) &= q^{m-r}(id\otimes\rho)\mu(w_{rs}^{(m)})\\
&= q^{m-r}(I\otimes\langle v_1|)\Big((id\otimes\epsilon_{V_0})\mu\left(\boldsymbol{v}^{r+s}J_q(q^{m-s+1}\boldsymbol{n}, r-s)\right)\Big)(\cdot\otimes|v_2\rangle)
\end{aligned}$$

$$
\begin{aligned}
&= q^{m-r}(I \otimes \langle v_1|)(\boldsymbol{v} \otimes V_0)^{r+s} J_q\left(q^{m-s+1}(\boldsymbol{n} \otimes V_0^*), r-s\right)(\cdot \otimes |v_2\rangle) \\
&= q^{m-r}\boldsymbol{v}^{r+s} J_q(q^{m-s+1}\boldsymbol{n}, r-s)\langle v_1, V_0^{2s} v_2\rangle \\
&= \langle v_1, V_0^{2s} v_2\rangle \xi_{rs}^{(m)}.
\end{aligned}
$$

Similarly, for $m \in \mathbb{Z} + \frac{1}{2}$, $R_\rho(\xi_{rs}^{(m)}) = \langle v_1, V_0^{2s+1} v_2\rangle \xi_{rs}^{(m)}$. Thus $\Re_\rho = R_\rho$ in this case.

Let $\{f_i\}$ be an orthonormal basis for the space $\mathcal{H}$ on which U_0 acts. Denote, as usual, $e_i \otimes f_j$ by e_{ij} on $L_2(\mathbb{Z}) \otimes \mathcal{H}$. Take ρ to be the functional

$$
\rho(a) = \langle e_{i'j'}, \pi_{U_0}(a) e_{ij}\rangle.
$$

Now, $(id \otimes \pi_{U_0})\Re = F_q(q^{\tilde{T}/2}\tilde{b} \otimes \ell q^N \otimes U_0)(I \otimes \ell \otimes U_0)^{\tilde{T}\otimes I \otimes I}$. Therefore, denoting $\pi^{(m)}(\boldsymbol{T})$ and $\pi^{(m)}(\boldsymbol{b})$ by $\boldsymbol{T}^{(m)}$ and $\boldsymbol{b}^{(m)}$ respectively, we have

$$
\begin{aligned}
&\langle \xi_{r's'}^{(m')} \otimes e_{i'j'}, ((id \otimes \pi_{U_0})\Re)\xi_{rs}^{(m)} \otimes e_{ij}\rangle \\
&= \delta_{mm'}\delta_{rr'}\left\langle e_{s'i'j'}, F_q(q^{\frac{1}{2}\boldsymbol{T}^{(m)}}\boldsymbol{b}^{(m)} \otimes \ell q^N \otimes U_0)(I \otimes \ell \otimes U_0)^{\boldsymbol{T}^{(m)}\otimes I \otimes I} e_{sij}\right\rangle \\
&= \begin{cases}
\delta_{mm'}\delta_{rr'}\delta_{s,i-i'-s'} J_q(q^{m+1+i-s}, i-i'-2s)\langle e_{j'}, U_0^{i-i'-2s} e_j\rangle & \\
\hfill \text{if } m \in \mathbb{Z}, & \\
\delta_{mm'}\delta_{rr'}\delta_{s,i-i'-s'-1} J_q(q^{m+\frac{1}{2}+i-s}, i-i'-2s-1)\langle e_{j'}, U_0^{i-i'-2s-1} e_j\rangle & \\
\hfill \text{if } m \in \mathbb{Z}+\frac{1}{2}. &
\end{cases}
\end{aligned}
$$

That is,

$$
\Re_\rho(\xi_{rs}^{(m)}) = \begin{cases}
J_q(q^{m+1+i-s}, i-i'-2s)\langle e_{j'}, U_0^{i-i'-2s} e_j\rangle \xi_{r,i-i'-s}^{(m)} & \\
\hfill \text{if } m \in \mathbb{Z}, & \\
J_q(q^{m+\frac{1}{2}+i-s}, i-i'-2s-1)\langle e_{j'}, U_0^{i-i'-2s-1} e_j\rangle \xi_{r,i-i'-s-1}^{(m)} & \\
\hfill \text{if } m \in \mathbb{Z}+\frac{1}{2}. &
\end{cases}
$$

On the other hand, observe that $(id \otimes \mu)(w^{(m)}) = \phi_{12}(w^{(m)})\phi_{13}(w^{(m)})$, where ϕ_{12} and ϕ_{13} are as in (5.1.1). Hence, if $m \in \mathbb{Z}$, then

$$
\begin{aligned}
&\langle e_{k'l'}, \xi_{rs}^{(m)} * \rho\, e_{kl}\rangle \\
&= q^{m-r}\left\langle e_{k'l'i'j'}, (id \otimes \pi_{U_0})\mu(w_{rs}^{(m)}) e_{klij}\right\rangle \\
&= q^{m-r}\left\langle e_{rk'l'i'j'}, (id \otimes \pi_{U_0})(id \otimes \mu)(w^{(m)}) e_{sklij}\right\rangle \\
&= q^{m-r}\left\langle e_{rk'l'i'j'}, (id \otimes id \otimes \pi_{U_0})\phi_{12}(w^{(m)})\phi_{13}(w^{(m)}) e_{sklij}\right\rangle \\
&= q^{m-r}\sum_p \langle e_{rk'l'}, w^{(m)} e_{pkl}\rangle \left\langle e_{pi'j'}, (id \otimes \pi_{U_0})(w^{(m)}) e_{sij}\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= q^{m-r}\sum_p \langle e_{k'l'}, w^{(m)}_{rp} e_{kl}\rangle \left\langle e_{i'j'}, (id\otimes\pi_{U_0})(w^{(m)}_{ps})e_{ij}\right\rangle \\
&= \sum_p \langle e_{k'l'}, \xi^{(m)}_{rp} e_{kl}\rangle \left\langle e_{i'j'}, (\ell^{p+s}\otimes I)J_q\left(q^{m-s+1}(q^N\otimes U_0), p-s\right)e_{ij}\right\rangle \\
&= \sum_p \langle e_{k'l'}, \xi^{(m)}_{rp} e_{kl}\rangle \delta_{p,i-i'-s}\langle e_{j'}, U_0^{p-s}e_j\rangle J_q(q^{m+1+i-s}, p-s) \\
&= \langle e_{k'l'}, \xi^{(m)}_{r,i-i'-s} e_{kl}\rangle \langle e_{j'}, U_0^{i-i'-2s}e_j\rangle J_q(q^{m+1+i-s}, i-i'-2s).
\end{aligned}
$$

Similarly, for $m\in\mathbb{Z}+\frac{1}{2}$, one has

$$
\langle e_{k'l'}, \xi^{(m)}_{rs} * \rho\, e_{kl}\rangle = \langle e_{k'l'}, \xi^{(m)}_{r,i-i'-s-1} e_{kl}\rangle \langle e_{j'}, U_0^{i-i'-2s-1}e_j\rangle J_q(q^{m+\frac{1}{2}+i-s}, i-i'-2s-1).
$$

Therefore $\Re_\rho = R_\rho$. Extending by linearity, the same conclusion holds for any ρ of the form $\langle u_1, \pi_{U_0}(\cdot)u_2\rangle$, where u_1 and u_2 are in the linear span of the e_{ij}'s. Combining this with our earlier observation, we find that $\Re_\rho = R_\rho$ for any ρ of the form (5.2.2), with u_1 and u_2 coming from a dense subspace of $L_2(\mathbb{Z})\otimes\mathcal{H}$. The set $\mathcal{D}$ of all such functionals is dense in norm topology in the space of all continuous functionals on $C_0(E_q(2))$. Therefore, as remarked earlier, one can define $(id\otimes\rho)\Re$ for any continuous linear functional ρ, and we have

$$
(id\otimes\rho)\Re(a) = a*\rho \quad \forall\, a\in C_0(E_q(2))\cap L_2(h).
$$

We call $\Re$ the *right regular representation.* From (5.2.1) it is immediate that in the direct sum decomposition of $\Re$, all the infinite dimensional irreducibles appear, and each one appears countably infinite number of times.

5.3 Some Further Identities

We shall now use the computations done in section 3.3 and the observations made in the previous section to generate some more identities involving the q-Bessel functions.

Let us take ρ to be the functional $a\mapsto\langle e_{i+i',j+j'}, a\,e_{ij}\rangle$ on $C_0(E_q(2))$. Then

$$
\begin{aligned}
\Re_\rho(\boldsymbol{v}^r J_q(q^s\boldsymbol{n}, t)) &= \Re_\rho\Big(w^{(s-1+\frac{r-t}{2})}_{[\frac{r+t}{2}],[\frac{r-t}{2}]}\Big) \\
&= q^{[\frac{r+t}{2}]-s+1-\frac{r-t}{2}}\Re_\rho\Big(\xi^{(s-1+\frac{r-t}{2})}_{[\frac{r+t}{2}],[\frac{r-t}{2}]}\Big) \\
&= q^{[\frac{r+t}{2}]-s+1-\frac{r-t}{2}}\sum_p \Big\langle e_p, w_\rho^{(s-1+\frac{r-t}{2})} e_{[\frac{r-t}{2}]}\Big\rangle \xi^{(s-1+\frac{r-t}{2})}_{[\frac{r+t}{2}],p} \\
&= \sum_p \Big\langle e_{p,i+i',j+j'}, w^{(s-1+\frac{r-t}{2})} e_{[\frac{r-t}{2}],i,j}\Big\rangle w^{(s-1+\frac{r-t}{2})}_{[\frac{r+t}{2}],p}.
\end{aligned}
$$

After simplification, this yields

$$\Re_\rho(\boldsymbol{v}^r J_q(q^s\boldsymbol{n},t)) = \delta_{t-r,i'+j'} J_q(q^{i+s},j')\boldsymbol{v}^{t-i'} J_q(q^{s-j'}\boldsymbol{n},t-j'). \tag{5.3.1}$$

Hence for any $u \in L_2(h)$,

$$\begin{aligned}
\Re_\rho u &= \Re_\rho\Big(\sum_{r,s,t} q^{2(s-t-1)} \langle \boldsymbol{v}^r J_q(q^s\boldsymbol{n},t), u\rangle\, \boldsymbol{v}^r J_q(q^s\boldsymbol{n},t)\Big)\\
&= \sum_{\substack{r,s,t\\ t-r=i'+j'}} q^{2(s-t-1)} \langle \boldsymbol{v}^r J_q(q^s\boldsymbol{n},t), u\rangle\, J_q(q^{i+s},j')\boldsymbol{v}^{t-i'} J_q(q^{s-j'}\boldsymbol{n},t-j')\\
&= \sum_{s,t} q^{2(s-t-1)} \left\langle \boldsymbol{v}^{t-i'-j'} J_q(q^s\boldsymbol{n},t), u\right\rangle J_q(q^{i+s},j')\boldsymbol{v}^{t-i'} J_q(q^{s-j'}\boldsymbol{n},t-j'),
\end{aligned}$$

so that

$$\begin{aligned}
&\langle e_{k'l'}, \Re_\rho u\, e_{kl}\rangle\\
&= \sum_{s,t} q^{2(s-t-1)} J_q(q^{i+s},j') J_q(q^{s-j'+k},t-j') \left\langle \boldsymbol{v}^{t-i'-j'} J_q(q^s\boldsymbol{n},t), u\right\rangle \delta_{k',k-t+i'}\delta_{l',l+t-j'}.
\end{aligned} \tag{5.3.2}$$

Let us now compute the quantity $\langle e_{k'l'}, \Re_\rho u\, e_{kl}\rangle$ in another way, using equation (3.3.7). Take a u in $L_2(h)\cap C_0(E_q(2))$. Then

$$\begin{aligned}
\langle e_{k'l'}, \Re_\rho u\, e_{kl}\rangle &= \langle e_{k'l'}, u * \rho\, e_{kl}\rangle\\
&= \langle e_{k',l',i+i',j+j'}, \mu(u) e_{k,l,i,j}\rangle\\
&= \sum_m J_q(q^{i+i'-k'+1},m) J_q(q^{i-k+1},m-j')\\
&\qquad \langle e_{k'+m,l'+m}, u\, e_{k-j'+m,l-j'+m}\rangle \delta_{i+i'-j-j'-k'-l',i-j-k-l}.
\end{aligned} \tag{5.3.3}$$

Take $k' = k+c$ and $l' = l-c+i'-j'$. Then from (5.3.2) and (5.3.3), we get

$$\begin{aligned}
&\sum_m J_q(q^{i-k+1+i'-c},m) J_q(q^{i-k+1},m-j') \langle e_{k+c+m,l-c+i'-j'+m}, u\, e_{k-j'+m,l-j'+m}\rangle\\
&\quad= \sum_s q^{2(s-i'+c-1)} J_q(q^{s+i},j') J_q(q^{s-j'+k},i'-j'-c) \left\langle \boldsymbol{v}^{-j'-c} J_q(q^s\boldsymbol{n},i'-c), u\right\rangle.
\end{aligned} \tag{5.3.4}$$

Taking various choices for the element u and the integers i, j, i', j', k, l and c, one can generate a whole lot of identities involving the q-Bessel functions. As an illustration, we prove a few identities below.

Proposition 5.3.1 *For any integers i, j, r and s, we have*

$$\sum_m J_q(q^i, m-r)J_q(q^j, m-s)J_q(q^{i-m}, j-m-1) = J_q(q^{r-1}, r-s-2)J_q(q^{i-s}, j-r+1). \quad (5.3.5)$$

Proof: Take $u = \boldsymbol{v}^{-j'-c}J_q(q^s\boldsymbol{n}, i'-c)$ in equation (5.3.4), use part 5 of proposition 3.3.2 and make some change of variables to get the required identity. □

If we take $i = j = r = 1$ and $s = -1$ in (5.3.5) and use part 5 of proposition 3.3.2, we get the following.

$$\sum_m J_q(q, m-1)J_q(q, m+1)J_q(q^{m+1}, 0) = J_q(1, 0)^2. \quad (5.3.6)$$

Proposition 5.3.2 *For any integers a, b, i, j and k, we have*

$$\sum_s q^{2(s-a-1+j+k)}J_q(q^{s+i}, b)J_q(q^{s+j}, a-b)J_q(q^{s+j+k}, a) = J_q(q^{i-j+a-b+1}, k)J_q(q^{i-j-b+1}, k-b). \quad (5.3.7)$$

Proof: Take $u = \boldsymbol{v}^{-j'-c}g(\boldsymbol{n})$, where $g(q^d z) = I_{\{k-j'+j\}}(d)z^{i'-c}$, $d \in \mathbb{Z}$, $z \in S^1$. Now use (5.3.4) and make some change of variables. □

The following identities can all be derived from (5.3.7) by taking appropriate choices of the integers a, b, i, j and k.

$$\sum_s q^{2(s-a-1)}J_q(q^s, a)J_q(q^{s+i}, b)J_q(q^{s+j}, a-b) = J_q(q^{i-j+a-b+1}, -j)J_q(q^{i-j-b+1}, -j-b), \quad (5.3.8)$$

$$\sum_s q^{2s}J_q(q^s, 0)J_q(q^{s+i}, 0)J_q(q^{s+j}, 0) = q^2 J_q(q^{i-j+1}, -j)^2, \quad (5.3.9)$$

$$\sum_s q^{2(s+j+k-1)}J_q(q^{s+i}, 0)J_q(q^{s+j}, 0)J_q(q^{s+j+k}, 0) = J_q(q^{i-j+1}, k)^2, \quad (5.3.10)$$

$$\sum_s q^{2s}J_q(q^{s+i}, 0)J_q(q^{s+1}, 0)^2 = J_q(q^i, 0)^2, \quad (5.3.11)$$

$$\sum_s q^{2s}J_q(q^{s+1}, 0)^3 = J_q(q, 0)^2. \quad (5.3.12)$$

5.4 The Quantum Plane

The quantum complex plane in the C^*-algebra set up was described very briefly in [70]. Here we shall give a more detailed account, and also describe how one can look at it as a quotient space of the group $E_q(2)$.

We start the section with a lemma on affiliation relation.

Lemma 5.4.1 *Let T be a normal operator, affiliated to a C^*-algebra $\mathcal{A}$. Let $\sigma(T)$ denote the spectrum of T. For a bounded operator S, if there exists a $g_0 \in C(\sigma(T))$ such that g_0 never vanishes on $\sigma(T)$ and $S\,g_0(T) \in M(\mathcal{A})$, then $S\,f(T) \in M(\mathcal{A})$ for all $f \in C_0(\sigma(T))$.*

Moreover, if there is a map $\phi : C(\sigma(T)) \to C(\sigma(T))$ such that for all $g \in C(\sigma(T))$, $S(\phi g)(T) = g(T)S$ and $\phi(g)$ never vanishes for nonvanishing g, then $S^k f(T) \in M(\mathcal{A})$ for all $f \in C_0(\sigma(T))$ and for all $k \in \mathbb{N}$.

Proof: Take an $f \in C_c(\sigma(T))$. g_0 being nonvanishing, $(g_0^{-1}f)$ also belongs to $C_c(\sigma(T))$, so that $(g_0^{-1}f)(T) \in M(\mathcal{A})$. Therefore $S\,f(T) = S\,g_0(T)\,(g_0^{-1}f)(T) \in M(\mathcal{A})$. Using the norm density of $C_c(\sigma(T))$ in $C_0(\sigma(T))$, we find that $S\,f(T) \in M(\mathcal{A})$ for $f \in C_0(\sigma(T))$ as well.

The second part will be proved by induction on k. Assume $S^k f(T) \in M(\mathcal{A})$ for all $f \in C_0(\sigma(T))$. As before, take an $f \in C_c(\sigma(T))$. Observe that $S^{k+1}f(T) = S\,g_0(T)\,S^k((\phi^k g_0)^{-1}f)(T)$. Therefore using the conditions of the lemma, we get $S^{k+1}f(T) \in M(\mathcal{A})$ for all $f \in C_c(\sigma(T))$. Since $C_c(\sigma(T))$ is norm dense in $C_0(\sigma(T))$, the same thing therefore holds for any $f \in C_0(\sigma(T))$. $\square$

Let $\boldsymbol{u} = \ell^*$ and $\boldsymbol{p} = q^N$ be operators on $L_2(\mathbb{Z})$. Then $\boldsymbol{u}$ and $\boldsymbol{p}$ satisfy the following:

$$\left.\begin{array}{l} \boldsymbol{u} \text{ is unitary,} \\ \boldsymbol{p} \text{ is positive, with } \sigma(\boldsymbol{p}) \subseteq \mathbb{C}^q \cap \mathbb{R}_+, \\ \boldsymbol{u}\boldsymbol{p}\boldsymbol{u}^* = q^{-1}\boldsymbol{p}. \end{array}\right\} \tag{5.4.1}$$

Denote by $C_0(\mathbb{C}_q)$ the norm closure of the linear span of $\{\boldsymbol{u}^k f_k(\boldsymbol{p}) : k \in \mathbb{Z}, f_k \in C_0(\mathbb{C}^q \cap \mathbb{R}_+), f_k(0) = 0 \text{ for } k \neq 0\}$. $C_0(\mathbb{C}_q)$ is then a nonunital C^*-algebra. The multiplier algebra of $C_0(\mathbb{C}_q)$ can easily be shown to be the closed linear span of $\{\boldsymbol{u}^k f_k(\boldsymbol{p}) : k \in \mathbb{Z}, f_k \in C_b(\mathbb{C}^q \cap \mathbb{R}_+)\}$. Denote this by $C_b(\mathbb{C}_q)$. We call $C_0(\mathbb{C}_q)$ the algebra of continuous vanishing-at-infinity functions on the quantized complex plane $\mathbb{C}_q$, and $C_b(\mathbb{C}_q)$, the algebra of bounded continuous functions on $\mathbb{C}_q$.

Write $\zeta = \boldsymbol{up}$. It is a closed operator affiliated to $C_0(\mathbb{C}_q)$, and satisfies:

$$\left.\begin{array}{rcl} \mathrm{dom}(\zeta) & = & \mathrm{dom}(\zeta^*), \\ \zeta^*\zeta & = & q^2\zeta\zeta^*, \\ \sigma(|\zeta|) & \subseteq & \mathbb{C}^q \cap \mathbb{R}_+. \end{array}\right\} \tag{5.4.2}$$

Proposition 5.4.2 *If ζ_0 is a closed operator acting on some Hilbert space $\mathcal{H}$, and satisfies (5.4.2) with ζ_0 replacing ζ, then there is a unique representation π of $C_0(\mathbb{C}_q)$ on $\mathcal{H}$ such that $\pi(\zeta) = \zeta_0$.*

Proof: Define π as follows: $\pi(\sum \boldsymbol{u}^k f_k(\boldsymbol{p})) = \sum V_{\zeta_0}^k f_k(|\zeta_0|)$, where $\zeta = V_{\zeta_0}|\zeta_0|$ is the polar decomposition of ζ_0, and for $k < 0$, $V_{\zeta_0}^k$ means $V_{\zeta_0}^{*\,-k}$. □

Proposition 5.4.3 *Let ζ_0 and π be as in the previous proposition. Then, for a C^*-algebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$, $\zeta_0\,\eta\,\mathcal{A}$ if and only if $\pi \in \mathrm{mor}\,(C_0(\mathbb{C}_q), \mathcal{A})$.*

Proof: Assume that $\zeta_0\,\eta\,\mathcal{A}$. Let us first show that $V_{\zeta_0}^k f(|\zeta_0|) \in M(\mathcal{A})$ for all $k \in \mathbb{Z}$ and $f \in C_0(\mathbb{C}^q \cap \mathbb{R}_+)$. For $k \geq 0$, use lemma 5.4.1 with $S = V_{\zeta_0}$, $T = |\zeta_0|$, $g_0 : x \mapsto (1+x^2)^{-1/2}$, and ϕ to be the map $(\phi f)(q^j) = f(q^{j-1})$, $j \in \mathbb{Z}$, $(\phi f)(0) = f(0)$. For $k < 0$, use lemma 5.4.1 taking $S = V_{\zeta_0}^*$, $T = |\zeta_0|$, $g_0 : x \mapsto (1+x^2)^{-1/2}$, and ϕ to be the map $(\phi f)(q^j) = f(q^{j+1})$, $j \in \mathbb{Z}$, $(\phi f)(0) = f(0)$. Next, notice that since $\zeta_0\,\eta\,\mathcal{A}$, $(I + \zeta_0^*\zeta_0)^{-1/2}\mathcal{A}$ is dense in $\mathcal{A}$, so that $\pi \in \mathrm{mor}\,(C_0(\mathbb{C}_q), \mathcal{A})$.

The converse follows from theorem A.3.3 in the appendix. □

Define $\sigma_z(a) := \epsilon_z * a$, $a \in C_0(E_q(2))$. $\{\sigma_z\}$ is a group of quantum group automorphisms of $E_q(2)$. Let $\mathcal{A} = \{a \in C_0(E_q(2)) : \sigma_z(a) = a\ \forall z \in S^1\}$. The operator $\zeta_0 = q\boldsymbol{v}^*\boldsymbol{n}$ is then a closed operator affiliated to $\mathcal{A}$ and satisfies (5.4.2). Therefore by propositions 5.4.2 and 5.4.3, there is an $\boldsymbol{i} \in \mathrm{mor}\,(C_0(\mathbb{C}_q), \mathcal{A})$ such that $\boldsymbol{i}(\zeta) = q\boldsymbol{v}^*\boldsymbol{n}$. Clearly $\|\boldsymbol{i}(a)\| = \|a\|$ for all $a \in C_0(\mathbb{C}_q)$. So $\boldsymbol{i}$ is injective. Let us show that $\boldsymbol{i}(C_0(\mathbb{C}_q)) = \mathcal{A}$. First observe that $\sigma_z(a) = (\overline{z}^N \otimes z^N)a(z^N \otimes \overline{z}^N)$. A simple computation now shows that a finite sum of the form $\sum \boldsymbol{v}^k f_k(\boldsymbol{n})$ will belong to $\mathcal{A}$ if and only if for any $j \in \mathbb{Z}$, $w \in S^1$, $f_k(q^j w) = g_k(q^j)\overline{w}^k$, g_k being a function on $\mathbb{C}^q \cap \mathbb{R}_+$. This means $f_k(\boldsymbol{n})$ can be written as $V_n^{-k} g_k(\boldsymbol{n})$, V_n being the unitary appearing in the polar decomposition of $\boldsymbol{n}$. This, in turn, implies that $\boldsymbol{i}(\sum V_\zeta^k f_k(|\zeta|)) \in \mathcal{A}$, if $f_k \in C_0(\mathbb{C}^q \cap \mathbb{R}_+)$ and $f_k(0) = 0$ for $k \neq 0$. Therefore $\boldsymbol{i}(C_0(\mathbb{C}_q)) \subseteq \mathcal{A}$. Next, take an element $x \in \mathcal{A}$. There is a sequence $x_m = \sum_k \boldsymbol{v}^k f_k^m(\boldsymbol{n})$ converging in norm to x.

Then

$$\sup_{z\in S^1} \|\sigma_z(x_m) - x_m\| \to 0 \quad \text{as } m \to \infty. \tag{5.4.3}$$

Let g_k^m be the following function on $\mathbb{C}^q$: $g_k^m(q^j z) = (\int f_k^m(q^j w) w^k \, dw) z^{-k}$. Write $y_m = \sum_k \boldsymbol{v}^k g_k^m(\boldsymbol{n})$. Using the condition (5.4.3), one can show that $\lim_{m\to\infty} \|x_m - y_m\| = 0$, so that $\lim_{m\to\infty} y_m = x$. Since each y_m belongs to $\boldsymbol{i}(C_0(\mathbb{C}_q))$, x also belongs to $\boldsymbol{i}(C_0(\mathbb{C}_q))$. Hence $\boldsymbol{i}(C_0(\mathbb{C}_q)) = \mathcal{A}$. $\boldsymbol{i}$ is thus an isomorphism between $C_0(\mathbb{C}_q)$ and $\mathcal{A}$. From now on we shall identify these two C^*-algebras via the isomorphism $\boldsymbol{i}$.

Let P be the spectral measure associated with the multiplication operator on $L_2(S^1)$. Define, for $a \in C_0(E_q(2))$, $\phi(a) = \int \epsilon_z(a) P(dz)$. Then ϕ is a C^*-homomorphism from $C_0(E_q(2))$ *onto* $C(S^1)$. Moreover, we have $\mu_{S^1}\phi = (\phi \otimes \phi)\mu$. Thus S^1 is a subgroup of $E_q(2)$.

Proposition 5.4.4 *Let ϕ be as above. Then we have the following:*

i. $C_0(\mathbb{C}_q) = \{a \in C_0(E_q(2)) : (\phi \otimes id)\mu(a) = I \otimes a\}$,

ii. $C_b(\mathbb{C}_q) = \{a \in C_b(E_q(2)) : (\phi \otimes id)\mu(a) = I \otimes a\}$,

iii. $a \,\eta\, C_0(\mathbb{C}_q)$ *if and only if* $a \,\eta\, C_0(E_q(2))$ *and* $(\phi \otimes id)\mu(a) = I \otimes a$.

Proof: i. Notice that

$$(\phi \otimes id)\mu(a) = \int P(dz) \otimes \sigma_z(a). \tag{5.4.4}$$

Hence clearly, if $\sigma_z(a) = a$ for all $z \in S^1$, then $(\phi \otimes id)\mu(a) = I \otimes a$. This means $\mathcal{A} \subseteq \{a \in C_0(E_q(2)) : (\phi \otimes id)\mu(a) = I \otimes a\}$. Now let $\delta_z : C(S^1) \to \mathbb{C}$ be the map $f \mapsto f(z)$. Then $\delta_z(\phi) = \epsilon_z$. Therefore

$$(\delta_z \otimes id)(\phi \otimes id)\mu = \sigma_z, \tag{5.4.5}$$

which means $\{a \in C_0(E_q(2)) : (\phi \otimes id)\mu(a) = I \otimes a\} \subseteq \mathcal{A}$.

ii. It is easy to see that $\mathcal{A}$ contains an approximate unit for $C_0(E_q(2))$. From this, it follows that $M(\mathcal{A}) = \{a \in C_b(E_q(2)) : \sigma_z(a) = a \ \ \forall z \in S^1\}$. The proof now is a consequence of (5.4.4) and (5.4.5).

iii. Take an operator a affiliated to $C_0(E_q(2))$, such that $(\phi \otimes id)\mu(a) = I \otimes a$. Then $z_a := a(I + a^*a)^{-1/2} \in C_b(E_q(2))$, and $(\phi \otimes id)\mu(z_a) = I \otimes z_a$. From part (ii), $z_a \in C_b(\mathbb{C}_q)$. Hence $S = (1 - {z_a}^* z_a)^{1/2} \in C_b(\mathbb{C}_q)$. Since $a \,\eta\, C_0(E_q(2))$, $S\, C_0(E_q(2))$ is dense in $C_0(E_q(2))$. By theorem A.3.2 (see appendix), $\pi(S)\mathcal{H}_\pi$ is dense in $\mathcal{H}_\pi$ for any irreducible representation π of $C_0(E_q(2))$ on $\mathcal{H}_\pi$. Now any irreducible representation of $C_0(\mathbb{C}_q)$ is the restriction of some irreducible representation of $C_0(E_q(2))$. Therefore

$\pi(S)\mathcal{H}_\pi$ is dense in $\mathcal{H}_\pi$ for any irreducible representation π of $C_0(\mathbb{C}_q)$. Again by theorem A.3.2, $S\mathcal{A}$ is dense in $\mathcal{A}$. Therefore $a\,\eta\,\mathcal{A}$.

Conversely, take an $a\,\eta\,C_0(\mathbb{C}_q)$. Then $z_a \in C_b(\mathbb{C}_q) \subseteq C_b(E_q(2))$, $(\phi\otimes id)\mu(a) = I\otimes a$ and $SC_0(\mathbb{C}_q)$ is dense in $C_0(\mathbb{C}_q)$. We have already observed that $C_0(\mathbb{C}_q)$ contains an approximate identity for $C_0(E_q(2))$. Therefore $C_0(\mathbb{C}_q)C_0(E_q(2))$ is dense in $C_0(E_q(2))$. Since $SC_0(\mathbb{C}_q)$ is dense in $C_0(\mathbb{C}_q)$, and $SC_0(\mathbb{C}_q)C_0(E_q(2)) \subseteq SC_0(E_q(2))$, $SC_0(E_q(2))$ is dense in $C_0(E_q(2))$, so that $a\,\eta\,C_0(E_q(2))$. □

Remark 5.4.5 Recall the definition of a quotient space for quantum groups from chapter 1. If G is a compact quantum group, H, a compact quantum subgroup of G, ϕ being a quantum group homomorphism mapping $C(G)$ onto $C(H)$, then the right coset space G/H is given via the C^*-algebra $C(G/H)$ of continuous functions on G/H, which is defined to be

$$\{a\in C(G) : (\phi\otimes id)\mu_G(a) = I\otimes a\}. \tag{5.4.6}$$

If G and H are locally compact, but noncompact, then one has to deal with $C_0(G)$ and $C_0(H)$ instead of $C(G)$ and $C(H)$, which are no longer C^*-algebras. But simply replacing $C(G)$ by $C_0(G)$ in (5.4.6) leads us nowhere — it neither describes $C_0(G/H)$, nor does it give us all elements affiliated to it (indeed, (5.4.6) may not have any element other than 0!). Theorem 5.4.4 gives us a clue as to how one can define a quotient space in this noncompact situation.

Definition. *Let G and H be two noncompact locally compact quantum groups, and let ϕ be a quantum group homomorphism mapping $C_0(G)$ onto $C_0(H)$. Then the C^*-algebra $C_0(G/H)$ of continuous functions on G/H vanishing at infinity is the unique C^*-subalgebra of $C_b(G) := M(C_0(G))$ such that*

$$T\,\eta\,C_0(G/H) \qquad \textit{if and only if} \quad T\,\eta\,C_0(G) \textit{ and } (\phi\otimes id)\mu(T) = I\otimes T. \tag{5.4.7}$$

For this definition to be satisfactory, one has to ensure that $C_b(G)$ has a unique C^*-subalgebra satisfying the stated property; which is not really possible at the moment, since we do not even know the definition of a locally compact quantum group. However, one should note in this connection the following: (i) if we assume that $C_b(G)$ has a C^*-subalgebra satisfying condition (5.4.7), then it must be unique, (ii) If G and H are locally compact groups, then this definition does describe the quotient space G/H, and (iii) if we take G to be a specific locally compact quantum group, for example $E_q(2)$ or the double group built over it, this definition seems to be satisfactory.

One natural question here is, why do we use just part (iii) of theorem 5.4.4 in defining the quotient space and ignore the first two parts? The reason is, part (ii) is a consequence of part (iii), and part (i) is a property very special to this group $E_q(2)$, which cannot be expected to hold in general (it fails to hold even when G is an ordinary locally compact group).

Let τ^k be the automorphism of $C_0(E_q(2))$ introduced in (3.2.4). We saw in chapter 3 that $\tau^k(C(SU_q(2)))$ is a family of C^*-subalgebras of $C_0(E_q(2))$, and the union $\cup\tau^k(C(SU_q(2)))$ is dense in $C_0(E_q(2))$. The next proposition is a similar statement about $C_0(\mathbb{C}_q)$.

Proposition 5.4.6 $\{\tau^k(C(S^2_{q0}))\}_{k\in\mathbb{Z}}$ *is an increasing family of* C^**-subalgebras of* $C_0(E_q(2))$*, and* $C_0(\mathbb{C}_q) = \overline{\cup_k\tau^k(C(S^2_{q0}))}$.

Proof: Observe that $\tau^{-1}(\eta_1) = q^{-1}(I-\eta_2)^{1/2}(I-q^2\eta_2)^{-1/2}\eta_1$ and $\tau^{-1}(\eta_2) = q^{-2}\eta_1{}^*(I-q^2\eta_2)^{-1/2}\eta_1$. Hence $\tau^{-1}(C(S^2_{q0})) \subseteq C(S^2_{q0})$, which implies $\tau^k(C(S^2_{q0})) \subseteq \tau^{k+1}(C(S^2_{q0}))$ for all k.

Let us now show that $C_0(\mathbb{C}_q) = \overline{\cup_k\tau^k(C(S^2_{q0}))}$. Clearly, $\eta_1,\eta_2 \in C_0(\mathbb{C}_q)$, so that $C(S^2_{q0}) \subseteq C_0(\mathbb{C}_q)$. For $a \in C_0(\mathbb{C}_q)$, $\sigma_z(\tau^k(a)) = \tau^k(a)$. Hence $\tau^k(C_0(\mathbb{C}_q) \subseteq C_0(\mathbb{C}_q)$. Therefore $\tau^k(C(S^2_{q0})) \subseteq C_0(\mathbb{C}_q)$ for all k. For the reverse inclusion, it is enough to show that elements of the form $V^r_{\zeta_0}f(|\zeta_0|)$ belong to $\cup_k\tau^k(C(S^2_{q0}))$, where $f \in C_c(\mathbb{C}^q \cap \mathbb{R}_+)$.
Case I: $r=0$. f has to be of the form

$$f(q^k) = \begin{cases} g(k+n) & \text{if } -n \le k, \\ 0 & \text{otherwise,} \end{cases}$$

where $n\in\mathbb{N}$ and g is a bounded function on $\mathbb{N}$. Then we have $\tau^{-n}(f(|\zeta_0|)) = g(N)$, which is a bounded function of η_2. Therefore $f(|\zeta_0|) \in \tau^n(C(S^2_{q0}))$.
Case II: $r\neq 0$. f must be of the form

$$f(q^k) = \begin{cases} g(k+n) & \text{if } -n \le k \le m-n, \\ 0 & \text{otherwise,} \end{cases}$$

where $m,n\in\mathbb{N}$ and g is a bounded function on $\{0,1,\ldots,m\}$. We have $\tau^{-n}(V^r_{\zeta_0}f(|\zeta_0|)$ $= (\eta_1{}^*(1-q^2\eta_2)^{-1/2})^r q^{-rN-r(r+1)/2}g(N)$. Here $q^{-rN-r(r+1)/2}g(N)$ is a bounded function of η_2. Therefore, as in the earlier case, $V^r_{\zeta_0}f(|\zeta_0|) \in \tau^n(C(S^2_{q0}))$. □

5.5 $E_q(2)$-action on the Quantum Plane

In chapter 1 we have seen that $SU_q(2)$ has a homogeneous action on the Podleś sphere S^2_{q0}. We shall see in the present section that $E_q(2)$ has a similar action on the quantum plane $\mathbb{C}_q$. We start with the following proposition.

Proposition 5.5.1 *There exists a morphism ν from $C_0(\mathbb{C}_q)$ to $C_0(\mathbb{C}_q) \otimes C_0(E_q(2))$ such that $(\nu \otimes id)\nu = (id \otimes \mu)\nu$.*

Proof: Let ζ be as in the previous section. Let $\zeta_0 = I \otimes q\boldsymbol{v}^*\boldsymbol{n} + \zeta \otimes \boldsymbol{v}^{*2}$. Then ζ_0 is a closed operator acting on $L_2(\mathbb{Z})^{\otimes 3}$. It is easy to see that $(\boldsymbol{i} \otimes id)\zeta_0 = \mu(q\boldsymbol{v}^*\boldsymbol{n})$ and $\mu(q\boldsymbol{v}^*\boldsymbol{n})$ is affiliated to $\mathcal{A} \otimes C_0(E_q(2))$. Also, it is clear that ζ_0 satisfies (5.4.2). Therefore by propositions 5.4.2 and 5.4.3, there is a $\nu \in \operatorname{mor}(C_0(\mathbb{C}_q), C_0(\mathbb{C}_q) \otimes C_0(E_q(2)))$ such that $\nu(\zeta) = \zeta_0$. Consider now the two morphisms $(\nu \otimes id)\nu$ and $(id \otimes \mu)\nu$. They coincide at ζ, and the value at ζ uniquely determines a morphism. Therefore we get $(\nu \otimes id)\nu = (id \otimes \mu)\nu$. $\square$

Denote the right action of $SU_q(2)$ on S^2_{q0} described in section 1.3 by ν_c. Proceeding exactly like in the proof of (3.2.5) (see [74]), one can show that for $a \in \cup_k \tau^k(C(S^2_{q0}))$,

$$\nu(a) = \lim_{k\to\infty} (\tau^k \otimes \tau^k)\nu_c(\tau^{-k}(a)),$$

from which it follows that $\nu(C_0(\mathbb{C}_q)) \subseteq C_0(\mathbb{C}_q) \otimes C_0(E_q(2))$.

We call a weight λ on $C_0(\mathbb{C}_q)$ *invariant* under the action ν if for any continuous functional ρ on $C_0(E_q(2))$, whenever a is in the domain of λ, so is $(id \otimes \rho)\nu(a)$, and we have $\lambda((id \otimes \rho)\nu(a)) = \lambda(a)\rho(I)$.

Consider the weight $\lambda : a \mapsto \sum_i q^{2i}\langle e_i, a\, e_i\rangle$ on $C_0(\mathbb{C}_q)$. Observe that $\lambda(a) = q^2 h(\boldsymbol{i}(a))$, and $\nu(a) = (\boldsymbol{i}^{-1} \otimes id)\mu(\boldsymbol{i}(a))$, where $\boldsymbol{i}$ is as in the previous section. Therefore from the proof of the invariance property of the haar weight on $E_q(2)$, it follows that λ is invariant under the action ν.

Define $L_2(\lambda)$ in the obvious way. It is easy to show that $\{q^{k-1}\boldsymbol{u}^r J_q(q^{r+k}\boldsymbol{p}, r) : r, k \in \mathbb{Z}\}$ form an orthonormal basis for $L_2(\lambda)$. Define two operators b and T on $L_2(\lambda)$ as follows:

$$\begin{aligned} b(\boldsymbol{u}^r J_q(q^{r+k}\boldsymbol{p}, r)) &= q^k \boldsymbol{u}^{r-1} J_q(q^{r+k-1}\boldsymbol{p}, r-1), \\ T(\boldsymbol{u}^r J_q(q^{r+k}\boldsymbol{p}, r)) &= -2r\boldsymbol{u}^r J_q(q^{r+k}\boldsymbol{p}, r). \end{aligned}$$

Then (b, T) is a pair of closed operators satisfying (3.2.7). Therefore $\pi := F_q(q^{T/2}b \otimes \boldsymbol{vn})(I \otimes \boldsymbol{v})^{T\otimes I}$ is a unitary representation acting on $L_2(\lambda)$. This representation is induced by the action ν in the following sense. For any functional ρ coming from $\mathcal{D}$, where $\mathcal{D}$ is as in section 5.2, $(id \otimes \rho)\pi$ defines a bounded operator on $L_2(\lambda)$, and coincides with the operator $a \mapsto (id \otimes \rho)\nu(a)$.

Let $\mathcal{H}_k$ be the closed linear span of $\{\boldsymbol{u}^r J_q(q^{r+k}\boldsymbol{p}, r) : r \in \mathbb{Z}\}$. Then $L_2(\lambda) = \bigoplus_{k\in\mathbb{Z}} \mathcal{H}_k$. The operators b, b^* and T keep each $\mathcal{H}_k$ invariant, and $b|_{\mathcal{H}_k} \cong \boldsymbol{b}^{(k)}$, $T|_{\mathcal{H}_k} \cong \boldsymbol{T}^{(k)}$. Thus π splits into a direct sum of all integer-spin representations, each one appearing exactly once.

It is interesting to compare this situation with the classical case where the group $E(2)$ acts on the complex plane $\mathbb{C}$. For $E(2)$, the equivalence classes of infinite dimensional irreducible unitary representations are parametrized by $\mathbb{R}_+$, and the action on $\mathbb{C}$ is a direct integral of *all* such representations.

Appendix

A.1 Multiplier Algebras

Definition A.1.1 Let $\mathcal{A}$ be a C^*-algebra acting nondegenerately on a Hilbert space $\mathcal{H}$. The *multiplier algebra* $M(\mathcal{A})$ of $\mathcal{A}$ is defined to be the following C^*-subalgebra of $\mathcal{B}(\mathcal{H})$:

$$M(\mathcal{A}) = \{b \in \mathcal{B}(\mathcal{H}) : ab, ba \in \mathcal{A}\ \forall\, a \in \mathcal{A}\}.$$

Notice that $\mathcal{A}$ is an ideal in $M(\mathcal{A})$. Also, if $\mathcal{A}$ is unital, then $M(\mathcal{A}) = \mathcal{A}$. If one takes $\mathcal{A}$ to be the C^*-algebra of continuous vanishing-at-infinity functions on a locally compact space X, then it is easy to see that $M(\mathcal{A})$ is the C^*-algebra of all bounded continuous functions on X.

Remark. $M(\mathcal{A})$ is the closure of $\mathcal{A}$ with respect to the almost uniform topology; that is, with respect to the topology generated by the following family of seminorms:

$$\|a\|_b := \|ab\| + \|ba\|, \quad b \in \mathcal{A}.$$

There is yet another equivalent description of a multiplier algebra, which is given below.

Let $\mathcal{B}(\mathcal{A})$ be the algebra of all bounded linear maps on $\mathcal{A}$. Let $S \in \mathcal{B}(\mathcal{A})$. We say that S has an adjoint T if there is a $T \in \mathcal{B}(\mathcal{A})$ such that for any a and b in $\mathcal{A}$, one has

$$b^*(Sa) = (Tb)^*a.$$

Definition A.1.2 For a C^*-algebra $\mathcal{A}$, the multiplier algebra $M(\mathcal{A})$ is the subalgebra of $\mathcal{B}(\mathcal{A})$ consisting of all elements that admit an adjoint. Norm of $M(\mathcal{A})$ is the one that it inherits from $\mathcal{B}(\mathcal{A})$.

It is easy to show that the two definitions A.1.1 and A.1.2 are equivalent.

A.2 Morphisms

Definition A.2.1 Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two C^*-algebras. A C^*-homomorphism ϕ from $\mathcal{A}_1$ to $M(\mathcal{A}_2)$ is said to be a *morphism* if $\{\phi(a_1)a_2 : a_1 \in \mathcal{A}_1,\, a_2 \in \mathcal{A}_2\}$ is dense in $\mathcal{A}_2$. The set of all morphisms from $\mathcal{A}_1$ to $\mathcal{A}_2$ is denoted by $\mathrm{mor}(\mathcal{A}_1, \mathcal{A}_2)$.

Any $\phi \in \mathrm{mor}(\mathcal{A}_1, \mathcal{A}_2)$ extends uniquely to a C^*-homomorphism from $M(\mathcal{A}_1)$ to $M(\mathcal{A}_2)$, as follows. For $T \in M(\mathcal{A}_1)$, $\phi(T)$ is the unique element of $\mathcal{B}(\mathcal{A}_2)$ for which

$$\phi(T)(\phi(a_1)a_2) = \phi(Ta_1)a_2, \;\; \forall a_1 \in \mathcal{A}_1,\, a_2 \in \mathcal{A}_2.$$

A.3 The Affiliation Relation

Definition A.3.1 Let $\mathcal{A}$ be a C^*-algebra and let T be a linear map defined on a dense subspace $\mathcal{D}(T)$ of $\mathcal{A}$. We say T *affiliated to* $\mathcal{A}$, and write $T\,\eta\,\mathcal{A}$, if there is an element $z = z_T$ in $M(\mathcal{A})$ such that $\|z\| \leq 1$, and the following condition holds: $a \in \mathcal{D}(T)$ and $Ta = b$ if and only if there exists an element d in $\mathcal{A}$ for which $a = (I - z^*z)^{1/2}d$ and $b = zd$.

If $\mathcal{A}$ acts nondegenerately on a Hilbert space $\mathcal{H}$, then one can show without much difficulty that the definition given in the introduction is equivalent to the one given above.

If $\mathcal{A}$ is unital, then $M(\mathcal{A}) = \mathcal{A}$, and any element affiliated to $\mathcal{A}$ must itself be an element of $\mathcal{A}$. If we take $\mathcal{A}$ to be $C_0(X)$, where X is a locally compact space, then elements affiliated to $\mathcal{A}$ are precisely the continuous functions on X.

The element z in the above definition is uniquely determined by the map T, and is called the *z-transform* of T. It can be shown that *any element z of $M(\mathcal{A})$ is the z-transform of an element T affiliated to $\mathcal{A}$ if and only if $\|z\| \leq 1$ and $(I - z^*z)^{1/2}\mathcal{A}$ is dense in $\mathcal{A}$.* Because of this fact, one is often required to check conditions like, whether for an element d of $M(\mathcal{A})$, $d\mathcal{A}$ is dense in $\mathcal{A}$ or not. The following theorem comes in very handy in such situations.

Theorem A.3.2 *Let $\mathcal{A}$ be a C^*-algebra and let $d \in M(\mathcal{A})$. Then $d\mathcal{A}$ is dense in $\mathcal{A}$ if and only if for any irreducible representation π of $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}_\pi$, range of $\pi(d)$ is dense in $\mathcal{H}_\pi$.*

For the proof, we refer the reader to [70].

The following theorem tells us that although, in general, affiliated elements are neither elements of the C^*-algebra under consideration nor of its multiplier algebra, one can talk of their image under a morphism.

Theorem A.3.3 *Suppose $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are three C^*-algebras. Let $T\eta\mathcal{A}$, and let $\phi \in mor(\mathcal{A}, \mathcal{B})$, $\psi \in mor(\mathcal{B}, \mathcal{C})$. Then there exists a unique $T'\eta\mathcal{B}$ such that $\phi(\mathcal{D}(T))\mathcal{B}$ is a core for T', and*

$$T'(\phi(a)b) = \phi(Ta)b, \ \ \forall\, a \in \mathcal{D}(T),\ b \in \mathcal{B}.$$

Denote T' by $\phi(T)$. Then we also have

$$z_{\phi(T)} = \phi(z_T),$$

$$\psi(\phi(T)) = (\psi \circ \phi)(T).$$

We again refer the reader to [70] for the proof.

A.4 A Radon-Nikodym Theorem for Weights

Let ϕ be a faithful normal semifinite weight on a von Neumann algebra $\mathcal{M}$. Suppose $\{\Delta_t\}_{t\in\mathbb{R}}$ is the modular automorphism group associated with the weight ϕ. Denote by $\mathcal{M}^\Delta$ the fixed point subalgebra for this automorphism group, i.e. $\mathcal{M}^\Delta = \{a \in \mathcal{M} : \Delta_t a = a\, \forall\, t \in \mathbb{R}\}$. Then we have the following Radon-Nikodym theorem, due to Pedersen & Takesaki.

Theorem A.4.1 *Let ϕ and $\{\Delta_t\}$ be as above. Let ψ be another normal semifinite weight on $\mathcal{M}$ such that $\psi\Delta_t = \psi$ for all t. Then there is a unique positive self-adjoint operator d affiliated to $\mathcal{M}^\Delta$ such that $\psi(\cdot) = \phi(d\cdot)$.*

For the proof, the reader should refer to [39].

Notice that the affiliation relation mentioned here is in the context of von Neumann algebras, which is standard in the literature (see, for example, [47]). The affiliation relation discussed in the previous section differs slightly from this.

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