

ON DOMAINS THAT ADMIT WELL-BEHAVED STRATEGY-PROOF SOCIAL CHOICE FUNCTIONS ^{*}

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Abstract

In this paper, we investigate domains which admit “well-behaved”, strategy-proof social choice functions. We show that if the number of voters is even, then every domain that satisfies a richness condition and admits an anonymous, tops-only, unanimous and strategy-proof social choice function, must be *semi-single-peaked*. Conversely every semi-single-peaked domain admits an anonymous, tops-only, unanimous and strategy-proof social choice function. Semi-single-peaked domains are generalizations of single-peaked domains on a tree introduced by Demange (1982). We provide sharper versions of the results above when tops-onlyness is replaced by tops-selectivity and the richness condition is weakened.

Keywords and Phrases: Voting-rules, Strategy-proofness, Restricted Domains, Tops-Only domains.

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1 INTRODUCTION

The celebrated Gibbard-Satterthwaite Theorem (Gibbard (1973), Satterthwaite (1975)) rules out the existence of strategy-proof, non-dictatorial social choice function over the complete domain of preferences. This is a powerful negative result and has stimulated a very large literature which has investigated the structure of strategy-proof social choice functions on

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restricted domains. One of the most salient restricted domains in this respect is the domain of single-peaked domains. It is now well-known that these domains admit a large class of strategy-proof social choice functions satisfying additional, attractive properties such as anonymity and Pareto-efficiency. These include the median voter rule (see [Moulin \(1980\)](#), [Moulin \(1984\)](#)). Of course, single-peaked domains are also very important in Arrovian aggregation theory and forms the bedrock of the modern theory of political economy. In this paper, we address a converse question. What are the domains that admit “well-behaved” strategy-proof social choice functions? In particular, do single-peaked domains emerge in a natural way from a characterization of domains that admit “well-behaved” strategy-proof social choice functions?

We consider a standard voting environment with a finite number of individuals/voters and alternatives. Preferences are assumed to be antisymmetric. We provide two partial characterizations of domains. Our first result states that if there is an even number of individuals, then any domain that satisfies a richness condition and admits an anonymous, tops-only, unanimous and strategy-proof social choice function, is *semi-single-peaked*. Moreover every semi-single-peaked domain admits an anonymous, tops-only, unanimous and strategy-proof social choice function for an arbitrary number of individuals. Our second result shows that the tops-only requirement can be dispensed with provided that the richness condition is strengthened. This additional richness requirement forces a strategy-proof, unanimous social choice function to satisfy the tops-only property. Our final result considers the case where the tops-only condition is replaced by the tops-selectivity condition. We show that if there is an even number of individuals, then any domain that satisfies a weaker richness condition than in the first result and admits an anonymous, tops-selective, unanimous and strategy-proof social choice function, is *extreme-peaked*. Moreover every extreme-peaked domain admits an anonymous, tops-selective, unanimous and strategy-proof social choice function for an arbitrary number of individuals.

As the name suggests, semi-single-peakedness is closely related to single-peakedness. In fact, it is a generalization of the notion of single-peaked preferences on a tree initially proposed by [Demange \(1982\)](#) in a different context. Semi-single-peakedness is defined in the following way. There is a tree where the nodes are alternatives. There is also a distinguished alternative on every maximal path in the tree which we refer to as the *threshold* on that path. The location of thresholds on different maximal paths are subject to strong restrictions. Semi-single peaked preferences on every path satisfy two restrictions: (i) they “decline” along the path from the peak in the direction of the threshold on that path (ii) alternatives that lie beyond the threshold are worse than the threshold. Single-peaked preferences are a special case of semi-single-peaked preferences when the underlying tree is a line and the preference restrictions are satisfied with respect to any placement of the threshold. An important respect in which semi-single-peaked preferences differ from single-peaked preferences is that, unlike the latter, restrictions are imposed in one direction. Extreme-peaked prefer-

ences are a special case of single-peaked preferences when the underlying tree is a line and the threshold is at one extremity of the line. In Section 3, we discuss semi-single peakedness at greater length and note that the domain of single-peaked preferences is the largest domain of “neutral” semi-single-peaked preferences.

One feature of our results deserves special mention. The notions of semi-single-peakedness and extreme-peakedness are based on an underlying structure on alternatives (a tree in the case of semi-single-peakedness and a line in the case of extreme-peakedness). We *do not* start with the assumption that there is an underlying structure of a tree or a line on alternatives; instead we uncover this structure as a consequence of our assumption that the domain admits well-behaved, strategy-proof social choice functions. The only structure that we impose on alternatives is via a richness condition on domains that we have used in our earlier papers (Aswal et al. (2003) and Chatterji and Sen (2010)). For the semi-single-peakedness result, we assume that the domain is strongly path-connected. Two alternatives a_i and a_j are strongly path-connected if there exists an ordering in the domain where a_i and a_j are ranked first and second respectively and another one where the reverse is true; moreover the rankings of other alternatives are the same in the two orderings. We require that the graph of strong connections be path connected; i.e. that we can find a path between any pair of alternatives in terms of strong connections. Importantly, we do not make any other assumptions on this graph. One of the major steps in our proof is to show that if such a domain admits a well-behaved strategy-proof social choice function, then this graph must be a tree. We go on to show that there must be appropriate thresholds on every path of the tree and that preferences must satisfy appropriate restrictions with the respect to the tree and the specification of the thresholds. For the extreme peakedness result we require an even weaker notion of path-connectedness. Two alternatives a_i and a_j are weakly connected if there exists an ordering in the domain where a_i and a_j are ranked first and second respectively and another one where the reverse is true. We say that the domain is rich if there exists a path between any pair of alternatives in terms of weak path-connectedness. We then show that if such a domain admits an anonymous, tops-selective and strategy-proof social choice function, then the graph of weak connections must be a line. In addition, the threshold must be an extreme point of the line and that preferences must be extreme-peaked.

In a series of papers (Nehring and Puppe (2007b), Nehring and Puppe (2007a)) investigate the structure of strategy-proof social choice functions in an abstract algebraic setting. Formally our results are independent of theirs; however some of their results are motivated by similar concerns. We discuss the relationship between our results in Section 3.1. There are several other papers (Danilov (1994), Schummer and Vohra (2002)) that investigate the structure of strategy-proof social choice functions that choose locations on trees where preferences are single-peaked-like (such as quadratic). It is clear that our focus is different from these papers; however our results confirm that such domains are salient from the point of view of admitting well-behaved, strategy-proof social choice functions.

Recently [Ballester and Haeringer \(2010\)](#) have provided a characterization of single-peaked preferences using axioms directly on voter preferences and profiles. In contrast, our approach focuses on domains that admit well-behaved strategy-proof social choice functions.

The paper is organized as follows. In Section 2, we set out the model and the basic definitions. Sections 3 and 4 are concerned with semi-single-peaked and extreme-peaked domains respectively. Section 5 concludes.

2 PRELIMINARIES

We let $A = \{a_1, \dots, a_m\}$ denote the set of alternatives where $\infty > m \geq 3$. There is a finite set of voters or individuals $N = \{1, \dots, n\}$ with $n \geq 2$. Each voter i is assumed to have a linear order P_i over the elements of the set A which we shall refer to as her *preference ordering*. For all $a_j, a_k \in A$, $a_j P_i a_k$ will signify the statement “ a_j is strictly preferred to a_k according to P_i ”. We let \mathcal{P} be the set of all linear orders over the elements of the set A . The set of all *admissible* orderings is a set $\mathbb{D} \subset \mathcal{P}$. A preference profile $P = (P_1, \dots, P_n) \in \mathbb{D}^n$ is a list of admissible preference orderings, one for each voter.

For all $s = 1, \dots, m$, $P_i \in \mathbb{D}$, and $a_j \in A$, we shall say that a_j is s^{th} ranked in P_i if $|\{a_k \in A | a_j P_i a_k\}| = m - s$. We will write $a_j = r_s(P_i)$ if a_j is s^{th} ranked in P_i .

The object of study of the paper is a *social choice function (SCF)*. An SCF is a mapping $f : \mathbb{D}^n \rightarrow A$. We restrict attention to SCF’s that satisfy *unanimity*, that is, $f(P) = a_j$ whenever $P \in \mathbb{D}^n$ is such that $a_j = r_1(P_i), i = 1, \dots, n$. We will also assume that \mathbb{D} satisfies *minimal richness*, which requires that for each $a_j \in A$, there exists $P_i \in \mathbb{D}$ such that $r_1(P_i) = a_j, j = 1, \dots, m$.

In our framework each voters’ preference ordering is private information; they must therefore be elicited by the mechanism designer. If an SCF is strategy-proof, then no voter can benefit by misrepresenting her preferences irrespective of her beliefs about the preference announcement of other voters. Formally, an SCF is *strategy-proof* if for all $P = (P_i, P_{-i}) \in \mathbb{D}^n$, and for all $P'_i \in \mathbb{D}$, we have either $f(P_i, P_{-i}) = f(P'_i, P_{-i})$ or $f(P_i, P_{-i}) P_i f(P'_i, P_{-i})$. An SCF is *tops-only* if it is determined completely by the peaks of the voters preferences, that is, $f(P) = f(P')$ whenever $r_1(P_i) = r_1(P'_i), i = 1, \dots, n$. An SCF is *top-selective* if for each profile P of preference orderings, $f(P) \in \{a_k | a_k = r_1(P_i), i \in \{1, \dots, n\}\}$. In order to define an anonymous SCF, we let $\eta : N \rightarrow N$ denote a one to one function and define the η permutation of a profile P of preference orderings as the profile $P^\eta = (P_{\eta(1)}, \dots, P_{\eta(n)})$. An SCF is *anonymous* if for any profile P and any η permutation of P , $f(P) = f(P^\eta)$.

3 TOPS-ONLYNESS AND SEMI-SINGLE-PEAKED DOMAINS

In this section we investigate domains which admit a strategy-proof, anonymous, unanimous and tops-only SCF. We require domains under consideration to satisfy a further richness

condition which we describe below.

DEFINITION 1 *Two alternatives a_j, a_k are strongly connected in \mathbb{D} , denoted $a_j \approx a_k$, if there exists $P_i, \bar{P}_i \in \mathbb{D}$ such that*

$$(i) \quad r_1(P_i) = a_j = r_2(\bar{P}_i)$$

$$(ii) \quad r_2(P_i) = a_k = r_1(\bar{P}_i)$$

$$(iii) \quad r_j(P_i) = r_j(\bar{P}_i), \quad j = 3, \dots, m.$$

According to the definition, two alternatives a_j, a_k are strongly connected if there exists an admissible ordering where a_j and a_k are ranked first and second respectively; another ordering where a_k and a_j are ranked first and second respectively while both orderings agree in the ranking of the rest of the alternatives. This notion is a strengthening of the connectedness condition (see Definition 6 below) that was introduced in [Aswal et al. \(2003\)](#) and used subsequently in [Chatterji and Sen \(2010\)](#).

DEFINITION 2 *The domain \mathbb{D} is strongly path-connected iff for all $a_r, a_s \in A$, there exists a sequence of alternatives $a_{j(k)} \in A$, $k = 0, \dots, T$ such that*

$$(i) \quad a_{j(0)} = a_r$$

$$(ii) \quad a_{j(T)} = a_s$$

$$(iii) \quad a_{j(k)} \approx a_{j(k+1)}, \quad k = 0, \dots, T - 1.$$

It is convenient to think of strong path-connectedness in terms of graphs. Fix a domain \mathbb{D} . Consider a graph whose nodes are the elements of A . Two nodes in this graph constitute an edge if they are strongly connected. The domain \mathbb{D} is strongly path-connected if every pair of nodes in this graph can be joined by a sequence of edges, i.e if the graph is connected.

Strong path-connectedness is a richness condition on the admissible domain of preferences in that it requires that there be sufficiently many strong connections among alternatives. This condition is satisfied by many of the admissible domains that have been studied in the literature on strategy-proofness. We give some examples below.

EXAMPLE 1 The domain of all preference orderings \mathbb{P} is clearly a strongly path connected domain. The associated strong connectivity graph is the complete graph on A . Note that there are much smaller domains whose strong connectivity graph is the complete graph on A . The smallest such domain has $M(M - 1)$ orderings.

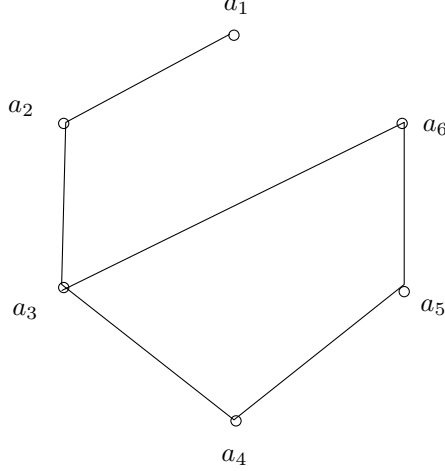


Figure 1: The graph \overline{G}

EXAMPLE 2 *Single-Peaked Domains* These domains were introduced in [Black \(1948\)](#) and have been extensively studied in the context of strategy-proofness. (see, for example [Moulin \(1980\)](#)).

Let $>$ be a linear ordering over A . A preference ordering P_i is *single-peaked* (with respect to $>$) if for all $a, b \in A$, $[r_1(P_i) > a > b \text{ or } b > a > r_1(P_i)] \implies aP_ib$.

Alternatives are ordered, say on the real line. An ordering is single-peaked if alternative a which lies “between” the peak of the ordering and another alternative b , is strictly preferred to b . We will let \mathbb{D}^{SP} denote the set of all single-peaked preferences with respect to some fixed order $>$.

We claim that \mathbb{D}^{SP} is a strongly connected domain. To see this assume without loss of generality that $a_1 > a_2 > \dots > a_m$. Note that for any ordering in \mathbb{D}^{SP} , if a_j is first-ranked, then either a_{j+1} or a_{j-1} must be ranked-second, for any $j = 2, \dots, m$. If a_1 is first, then a_2 must be second and if a_m is first, then a_{m-1} must be second. A critical observation is that if an ordering is single-peaked, then the ordering obtained by switching the first and second alternatives while leaving all other alternatives unchanged, is also single-peaked. Thus $a_1 \approx a_2 \approx \dots \approx a_m$. The strong connectivity graph for this case is shown in [Figure 2](#) below.

EXAMPLE 3 We can start with an arbitrary connected graph on A and construct a domain which induces the graph as its strong connectivity graph. For instance let $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$. Consider the graph \overline{G} in [Figure 1](#).

The domain $\overline{\mathbb{D}}$ induces \overline{G} as its associated strong connectivity graph.

Observe that $\overline{\mathbb{D}}$ is not the unique domain which induces \overline{G} . In fact, it is a minimal in the set of domains which induce \overline{G} , i.e. there does not exist a domain of smaller size which induces \overline{G} . If one assumes that domains under consideration satisfy the symmetry requirement that if there exists an ordering with a_j first and a_k second ranked, then there exists another

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
a_1	a_2	a_2	a_3	a_3	a_3	a_4	a_4	a_5	a_5	a_6	a_6
a_2	a_1	a_3	a_2	a_6	a_4	a_3	a_5	a_6	a_4	a_3	a_5
a_3	a_3	a_4	a_4	a_5	a_2	a_2	a_3	a_3	a_3	a_5	a_3
a_6	a_6	a_6	a_6	a_4	a_1	a_1	a_2	a_4	a_2	a_4	a_4
a_5	a_5	a_5	a_5	a_2	a_5	a_5	a_1	a_2	a_1	a_2	a_2
a_4	a_4	a_1	a_1	a_1	a_6	a_6	a_6	a_1	a_6	a_1	a_1

Table 1: The domain $\overline{\mathbb{D}}$

ordering with a_k first and a_j second ranked, then the maximal domain inducing \overline{G} has 288 orderings. In general, suppose G is an arbitrary strong connectivity graph. Suppose G has e edges. Let \mathbb{D} be a domain satisfying the symmetry property described above whose associated connectivity graph is G and let $|\mathbb{D}|$ denote its cardinality. Then, it is easy to verify that $2e \leq |\mathbb{D}| \leq 2e(m-2)!$. Conversely, let t be an even integer with $2e \leq t \leq 2e(m-2)!$. Then there exists a domain \mathbb{D} with $|\mathbb{D}| = t$ such that \mathbb{D} induces G .

We give an example below of a class of well-known domains that violate the property of strong path-connectedness.

EXAMPLE 4 *Separable Preferences over Product Domains (LeBreton and Sen (1999), Barberá et al. (1991), Barberà et al. (1993))*. In this setting the set A is the product of M component sets, i.e $A \equiv A_1 \times \dots \times A_M$ where $|A_j| \geq 2$ for all $j = 1, \dots, M$. We shall write a typical element $a \in A$ as $a \equiv (a_1, \dots, a_M)$ or (a_Q, a_{-Q}) where $Q \subset \{1, \dots, M\}$.

The ordering P_i is *separable* if for all $Q \subset \{1, \dots, M\}$, and $a, b, c, d \in A$, we have $[(a_Q, c_{-Q})P_i(b_Q, c_{-Q}) \implies [(a_Q, d_{-Q})P_i(b_Q, d_{-Q})]$.

We shall let \mathbb{D}^{SEP} denote the set of all separable preferences over A . We claim that \mathbb{D}^{SEP} violates path-connectedness.

To see this pick P_i and let $r_1(P_i) = a$. Separability implies that the second ranked alternative in P_i must be of the form (b_k, a_{-k}) for some $b_k \in A_k$ and some $k \in \{1, \dots, M\}$. Assume without loss of generality that $k = 1$. Pick $b_2 \in A_2$ such that $b_2 \neq a_2$. Since P_i is separable, we must have $(a_1, b_2, a_{-\{1,2\}})P_i(b_1, b_2, a_{-\{1,2\}})$. Now consider any $P'_i \in \mathbb{D}^{SEP}$ such that $r_1(P'_i) = (b_1, a_{-1})$ and $r_2(P'_i) = a$. Separability of P'_i implies that $(b_1, b_2, a_{-\{1,2\}})P'_i(a_1, b_2, a_{-\{1,2\}})$. Hence P'_i contains at least two preference reversals relative to P_i contradicting part (iii) of Definition 1. Thus no pair of alternatives is strongly connected, i.e \mathbb{D}^{SEP} is not strong path-connected.

We address the following question: what strongly connected domains \mathbb{D} admit SCFs that are strategy-proof, anonymous, unanimous and satisfy the tops-only property? The class of domains that we will identify in this context is related to the domains initially identified in Demange (1982). In order to do so we need some additional concepts.

Let G be a connected graph, the set of whose nodes is A . Recall that a *path* in G is a sequence $\{a_{j(k)}\}$, $k = 0, \dots, T$ such that every pair $(a_{j(k)}, a_{j(k+1)})$, $k = 0, \dots, T - 1$ is an edge in G .

We say that G is a *tree* if there is a unique path linking every pair $a_j, a_k \in A$. In other words G contains no *cycles*. The graph in Figure 1 is not a tree but the graphs in Figures 2, 3, 4 are trees.



Figure 2: Connectivity Graph G_L : The Line

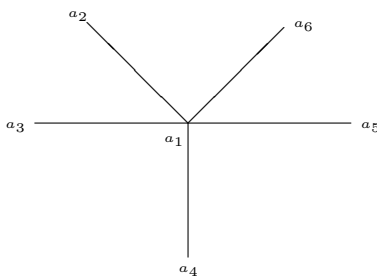


Figure 3: Connectivity Graph G_S : The Star

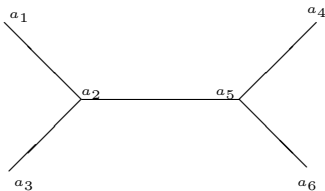


Figure 4: Connectivity Graph G_0

A path $\{a_{j(k)}\}$, $k = 0, \dots, T$ in a tree G is *maximal* if there does not exist an alternative a_r distinct from $a_{j(0)}$ and $a_{j(1)}$, or an alternative a_s distinct from $a_{j(T)}$ and $a_{j(T-1)}$ such that $(a_r, a_{j(0)})$ or $(a_s, a_{j(T)})$ are edges in G . In other words, a path is maximal if it cannot be “extended” by adding more edges at the ends. Note that every path in G can be extended to a maximal path. For any pair of distinct alternatives $a_j, a_k \in A$, we will let $\langle a_j, a_k \rangle$ denote the unique path connecting a_j and a_k . If a_r is one of the alternatives in the sequence of alternatives which comprises the path between a_j and a_k , we shall simply say that a_r belongs to the path between a_j and a_k and write it as $a_r \in \langle a_j, a_k \rangle$. We will let $\text{int}\langle a_r, a_s \rangle$ denote the alternatives in the path $\langle a_r, a_s \rangle$ excluding a_r and a_s . We will also let $\overline{\langle a_j, a_k \rangle}$ denote a maximal path containing a_j and a_k . We shall let $\mathbb{P}(G) = \{p_1, \dots, p_R\}$ denote the set of maximal paths in G .

OBSERVATION 1 Let $p_t \in \mathbb{P}(G)$ and let $a_l \notin p_t$. We claim that there exists a unique alternative, $a_r \in p_t$ such that every path from any $a_s \in p_t$ to a_l , contains a_r . To see this, pick an arbitrary $a_s \in p_t$ and consider the unique path $\langle a_s, a_l \rangle$. Without loss of generality, represent this path by the sequence $\{a_{j(k)}\}$, $k = 0, \dots, T$ where $a_{j(0)} = a_s$ and $a_{j(T)} = a_l$. Let k^* be the minimal integer such that $a_{j(k^*)} \in p_t$ and $a_{j(k^*+1)} \notin p_t$. Such an integer clearly exists. We claim that $a_{j(k^*)}$ is the alternative a_r . To verify this pick any another $a_{s'} \in p_t$ and suppose that the path from $a_{s'}$ to a_l does not contain $a_{j(k^*)}$. Then we have another path from $a_{s'}$ to a_l : from $a_{s'}$ to $a_{j(k^*)}$ on p_t and then the path $\{a_{j(k)}\}$, $k = k^*, \dots, T$. This contradicts the assumption that G is a tree. A similar argument shows that a_r is unique.

For any maximal path p_t and $a_k \notin p_t$, let $\gamma(p_t, a_k)$ be the alternative a_r described in the previous paragraph, i.e. a_r is the unique alternative in p_t with the property that every path from an alternative in p_t to a_k contains a_r .

Consider for example the graph in Figure 4. Let $p_t = \{a_1, a_2, a_3\}$ and let $a_k = a_4$. Clearly $a_k \notin p_t$. Then $\gamma(p_t, a_k) = a_2$ because all paths connecting nodes in p_t contain a_2 .

DEFINITION 3 Let G be a tree. The map $\lambda : \mathbb{P}(G) \rightarrow A$ is a *Threshold Assignment Map (TAM)* if there exists $a_k \in A$ such that

- (i) For all $p_t \in \mathbb{P}(G)$, $[a_k \in p_t] \implies [\lambda(p_t) = a_k]$.
- (ii) For all $p_t \in \mathbb{P}(G)$, $[a_k \notin p_t] \implies [\lambda(p_t) = \gamma(p_t, a_k)]$.

The function λ specifies a threshold for every maximal path in G . In particular, there exists an alternative a_k such that the threshold for every maximal path containing a_k is a_k ; for maximal paths that do not contain a_k , the threshold is the unique alternative that lies on every path from an alternative on the path and a_k (Observation 1).

Let G be a tree and λ , a TAM (for G). We shall refer to the pair (G, λ) as an *admissible pair*.

We give some examples of admissible pairs. Observe that there exists a unique maximal path $\{a_1, a_2, \dots, a_6\}$ in G_L . Here, the threshold for the unique maximal path can be any of the alternatives a_1, \dots, a_6 . Formally, let λ^i be the function that associates the alternative a_i , $i = 1, 2, \dots, 6$ with the unique maximal path. Then (G_L, λ^i) is an admissible pair.

There are 10 maximal paths in G_S of the form $\{a_j, a_1, a_k\}$ where $j, k \in \{2, 3, 4, 5, 6\}$ with $j \neq k$. One TAM λ^1 specifies a_1 as the threshold for every maximal path. In addition, let λ^j , $j = 2, 3, \dots, 6$ be the TAMs which specify a_j as the threshold for every maximal path containing a_j and a_1 for every other maximal path. The only admissible pairs here, are (G_S, λ^j) , $j = 1, 2, \dots, 6$.

Maximal paths in G_0 are as follows: $p_1 = \{a_1, a_2, a_3\}$, $p_2 = \{a_4, a_5, a_6\}$, $p_3 = \{a_1, a_2, a_5, a_4\}$, $p_4 = \{a_1, a_2, a_5, a_6\}$, $p_5 = \{a_3, a_2, a_5, a_4\}$ and $p_6 = \{a_3, a_2, a_5, a_6\}$. Define the function λ^4 as follows: $\lambda^4(p_1) = a_2$, $\lambda^4(p_t) = a_4$ for $t = 2, 3, 5$ and $\lambda^4(t) = a_5$ for $t = 4, 6$. Then λ^4 is a TAM and (G_0, λ^4) is an admissible pair. Other admissible pairs can be similarly defined.

We now define restrictions on preferences.

DEFINITION 4 *The domain \mathbb{D} is semi-single-peaked with respect to the admissible pair (G, λ) such that for all $P_i \in \mathbb{D}$ and all $p_t \in \mathbb{P}(G)$ such that $r_1(P_i) \in p_t$, we have*

$$(i) [a_r \in p_t \text{ such that } \lambda(p_t) \in \langle r_1(P_i), a_r \rangle] \implies [\lambda(p_t) P_i a_r].$$

$$(ii) [a_r, a_s \in p_t \text{ such that } a_r, a_s \in \langle r_1(P_i), \lambda(p_t) \rangle \text{ and } a_r \in \langle r_1(P_i), a_s \rangle] \implies [a_r P_i a_s].$$

We say that \mathbb{D} is semi-single-peaked if there exists an admissible pair (G, λ) with respect to which it is semi-single-peaked.

Let P_i be a semi-single peaked ordering. Let a be the peak of P_i . Pick a maximal path p_t containing a . Let b be the threshold for this path, i.e. $\lambda(p_t) = b$. Then, it must be the case preferences “decline” on the path from a to b . Moreover b is better (according to P_i) than any alternative c which is further along p_t than b in the direction “away” from a . These restrictions are shown in the diagram below.

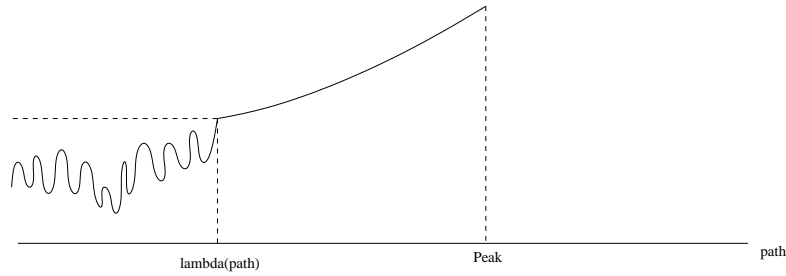


Figure 5: Semi-Single Peaked Preferences

Consider the admissible pair (G_L, λ^3) , i.e the λ for the unique path is a_3 . Consider the preference orderings below.

P_1	P_2	P_3	P_4	P_5
a_5	a_5	a_5	a_5	a_5
a_6	a_4	a_4	a_3	a_4
a_4	a_3	a_3	a_4	a_1
a_3	a_6	a_1	a_6	a_3
a_1	a_2	a_2	a_2	a_2
a_2	a_1	a_6	a_1	a_6

Table 2: Preferences in the case (G_L, λ^3)

In Table 2, preference orderings P_1, P_2 and P_3 are semi-single-peaked for (G_L, λ^3) . However P_4 and P_5 are not; P_4 and P_5 violate parts (ii) and (i) of Definition 4 respectively.

Consider semi-single-peaked preferences with respect to the admissible pair (G_S, λ^j) , for any $j = 1, \dots, 6$. All such preference orderings are subject to the same restriction: whenever $a_j, j = 2, \dots, 6$ is ranked first, a_1 must be ranked second. No other restrictions are implied.

Finally, consider semi-single-peaked preferences with respect to (G_0, λ^4) . Suppose a_1 is ranked first in P_i . Then semi-single-peakedness would require (i) a_2 to be ranked above a_3 (ii) a_2 should be ranked above a_5 which in turn should be ranked above both a_4 and a_6 .

P_1	P_2	P_3	P_4	P_5
a_1	a_5	a_2	a_5	a_2
a_2	a_6	a_3	a_3	a_4
a_5	a_2	a_5	a_4	a_1
a_3	a_4	a_1	a_6	a_3
a_6	a_3	a_4	a_2	a_5
a_4	a_1	a_6	a_1	a_6

Table 3: Preferences in the case (G_0, λ^4)

In Table 3, P_1, P_2 and P_3 are semi-single-peaked with respect to (G_0, λ^4) but P_4 and P_5 are not.

Semi-single-peaked preferences are clearly related to “single-peaked orders on a tree” introduced by Demange (1982). A preference ordering is single-peaked on a tree G if and only if it is single-peaked on every path of G . This notion was introduced in the context of aggregation theory. In particular it was shown that a non-empty core is guaranteed for every simple game defined on the set of players N and profiles of single-peaked orders on G . Moreover, such preferences were the “largest” set which had the non-emptiness of the core property. We identify semi-single-peaked domains as a salient domain in a different context; we show that domains which admit well-behaved SCFs are semi-single-peaked provided they satisfy some richness conditions.

It is important to point out that single-peaked orders on a tree are a subset of semi-single-peaked preferences. In fact the set of semi-single-peaked preferences is significantly larger than the set of single-peaked orders on a tree. This is because semi-single-peaked preferences are restricted *only on one side of the peak*. In contrast single-peaked preferences are restricted on *both sides of the peak*. Consider the simplest case where G is a line (Figure 2). Suppose a_3 is the peak. Then single-peakedness would require a_2 to be ranked above a_1 and a_4 to be ranked above a_5 and a_5 to be ranked above a_6 . Semi-single-peakedness specifies an additional alternative, the threshold, say a_4 . Suppose a_2 is the peak. We only impose restrictions on alternatives in the “increasing” direction from a_2 ; in particular a_3 should be better than a_4 and a_5 and a_6 must be worse than a_4 .

We can make the relationship between semi-single-peaked and single peaked preference on a tree, precise. Let G be a tree and let $\mathbb{D}^{SP}(G)$ denote the set of single-peaked preferences

on G . Let (G, λ) be an admissible pair and let $\mathbb{D}(G, \lambda)$ denote the set of semi-single-peaked preferences with respect to (G, λ) . Finally, let $\Lambda(G)$ denote the set of TAMs λ such that (G, λ) is admissible. The following proposition establishes the connection between single-peakedness and semi-single-peakedness.

PROPOSITION 1 $\mathbb{D}^{SP}(G) = \bigcap_{\lambda \in \Lambda(G)} \mathbb{D}(G, \lambda)$.

Proof: It is easy to check that if $P_i \in \mathbb{D}^{SP}(G)$, then $P_i \in \bigcap_{\lambda \in \Lambda(G)} \mathbb{D}(G, \lambda)$. Now suppose that $P_i \in \bigcap_{\lambda \in \Lambda(G)} \mathbb{D}(G, \lambda)$ but $P_i \notin \mathbb{D}^{SP}(G)$. There must therefore exist alternatives b, c and a path p_t containing $r_1(P_i)$ and b, c such that $b \in \langle r_1(P_i), c \rangle$ and cP_ib . Let λ be a TAM where the alternative a_k in Definition 3 is the alternative c . Then P_i violates part (ii) in Definition 4, contradicting the assumption that $P_i \in \mathbb{D}(G, \lambda)$. ■

Proposition 1 implies that the domain of single-peaked preferences on a tree G , is the *largest* domain of semi-single-peaked domains consistent with *all specifications of thresholds*. Single-peaked preferences are therefore, “neutral” within the class of semi-single-peaked preferences, i.e. one where it is not necessary to specify thresholds. We note that a meaningful notion of a neutral domain requires restrictions on the set of admissible permutations on alternatives with respect to which the neutrality property is defined. If neutrality is required with respect to all permutations of alternatives then the only neutral domain is the complete domain \mathbb{P} . According to Proposition 1, single-peaked preferences is the neutral domain where admissible permutations can relabel thresholds arbitrarily. However the ordering generated by such a permutation must be semi-single-peaked with respect to the relabeled threshold.

Can one offer a behavioural justification of semi-single-peaked preferences? For every path $p_t \in \mathbb{P}(G)$, one can think of the threshold $\lambda(p_t)$, as a focal point “beyond” (in the direction “away” from the peak) which preferences are comparatively vague. Consider a tax-payer’s preferences over tax rates from 0 to a 100 percent. If she has a threshold of 25 percent and a peak of 10 percent, then her preferences decline till 25. Beyond this threshold, she has no views (say whether 45 percent is better than 50 percent) except that everything is worse than 25 percent. On the other hand, if her peak is 40 percent, then her preferences decline till 25 percent with everything below 25 percent less-well preferred to 25 percent.

We now state one of our main results.

THEOREM 1 *Let \mathbb{D} be a strongly path-connected domain and let n be an even integer. If there exists an anonymous, tops-only, unanimous and strategy-proof SCF $f : \mathbb{D}^n \rightarrow A$, then \mathbb{D} is semi-single peaked. Conversely, if \mathbb{D} is a semi-single-peaked domain, then there exists an anonymous, strategy-proof, tops-only and unanimous SCF $f : \mathbb{D}^n \rightarrow A$ for all integers n .*

Proof: We first prove the first part of the Theorem. We begin with a Proposition which is of some independent interest.

PROPOSITION 2 *Let \mathbb{D} be an arbitrary domain and let n be a positive even integer. Suppose there exists an anonymous, tops-only, unanimous and strategy-proof SCF $f : \mathbb{D}^n \rightarrow A$. Then there exists an anonymous, tops-only, unanimous and strategy-proof SCF $g : \mathbb{D}^2 \rightarrow A$.*

Proof: Let $f : \mathbb{D}^n \rightarrow A$ be a anonymous, tops-only and strategy-proof SCF and suppose n is even. Let $N^1 = \{1, \dots, \frac{n}{2}\}$ and let $N^2 = \{\frac{n}{2} + 1, \dots, n\}$. Define $g : \mathbb{D}^2 \rightarrow A$ as follows. Pick an arbitrary pair $P_1, P_2 \in \mathbb{D}$. Then $g(P_1, P_2) = f(P)$ where $P \in \mathbb{D}^n$ and $P_j = P_1$ for all $j \in N_1$ and $P_j = P_2$ for all $j \in N_2$. In other words, the value of G at the two-individual profile (P_1, P_2) is the value of the f at the n -individual profile P where all individuals in the set N_1 have the same ordering P_1 and all individuals in N_2 have the same ordering P_2 . It is easy to verify that g is unanimous and tops-only (these properties are inherited from the corresponding properties in f). We show below that g is anonymous and strategy-proof.

In order to show that g is anonymous, pick $P_1, P_2 \in \mathbb{D}$. We will show that $g(P_1, P_2) = g(P_2, P_1)$. Let P be the n -individual profile where individuals in N_1 and N_2 have the orderings P_1 and P_2 respectively and let \bar{P} be the n -individual profile where individuals in N_1 and N_2 have the orderings P_2 and P_1 respectively. Consider the permutation $\eta : N \rightarrow N$ defined by $\eta(i) = (i + \frac{n}{2}) \bmod n$. Observe that \bar{P} is the image of P under η , i.e. $P^\eta = \bar{P}$. Since f is anonymous, $f(P) = f(\bar{P})$. But $g(P_1, P_2) = f(P)$ and $g(P_2, P_1) = f(\bar{P})$, so that g is anonymous.

We now show that g is strategy-proof. Pick arbitrary orderings $P_1, P'_1, P_2 \in \mathbb{D}$. Once again let P be the n -individual profile where individuals in N_1 and N_2 have the orderings P_1 and P_2 respectively. Let $g(P_1, P_2) = f(P) = a$. Let $f(P'_1, P_1, \dots, P_1, P_2, \dots, P_2) = b$. We must have either $b = a$ or aP_1b ; otherwise individual 1 would manipulate in P via P'_1 contradicting the strategy-proofness of f . Now let $f(P'_1, P'_1, P_1, \dots, P_1, P_2, \dots, P_2) = c$. In order to prevent individual 2 from manipulating in the profile $(P'_1, P_1, \dots, P_1, P_2, \dots, P_2)$ via P'_1 , we must have $c = b$ or bP_1c , i.e either $c = a$ or aP_1c . Progressively switching individual preferences in the set N_1 from P_1 to P'_1 , we obtain that if $g(P'_1, P_2) = x$, then either $x = a$ or aP_1x . Therefore g is strategy-proof. ¹ ■

Let \mathbb{D} be a strongly path-connected domain. Suppose that there exists n even, such that there exists $f : \mathbb{D}^n \rightarrow A$ which is anonymous, tops-only, unanimous and strategy-proof. In view of Proposition 2, we can assume without loss of generality that $n = 2$. We will show that \mathbb{D} is semi-single-peaked.

Let f be a two-person anonymous, tops-only, unanimous and strategy-proof SCF. We shall denote the two individuals by i and j and their typical preference orderings by P_i and P_j respectively. Since f is tops-only we can also represent the profile (P_i, P_j) by (a_k, a_r)

¹It is clear from our proof that the following more general statement is true. Suppose that there exists an anonymous, tops-only, unanimous, strategy-proof SCF $f : \mathbb{D}^n \rightarrow A$. Then there exists an anonymous, tops-only, unanimous, strategy-proof SCF $f : \mathbb{D}^m \rightarrow A$ whenever m divides n .

where a_k and a_r are the peaks of P_i and P_j respectively. Thus $f(a_k, a_{k'})$ will denote the social choice for a profile of preferences where individual i has a preference ordering whose peak is a_k and j has a preference ordering whose peak is $a_{k'}$. We will also interchangeably use the notation $(a_k a_l \dots)$ to signify a preference ordering whose (i) peak is a_k (ii) whose second ranked element is a_l and (iii) the order of the remaining alternatives is not specified. Finally, a preference profile where individual i has the preference ordering $(a_k a_l \dots)$ and j has the preference ordering $(a_{k'} a_{l'} \dots)$ is denoted $(a_k a_l \dots, a_{k'} a_{l'} \dots)$, and $f(a_k a_l \dots, a_{k'} a_{l'} \dots)$ will denote the outcome of f at this profile.

We begin with two important properties of the \approx relation associated with \mathbb{D} .

LEMMA 1 *Let $a_r, a_s \in A$ such that $a_r \approx a_s$. Then $f(a_r, a_s) \in \{a_r, a_s\}$.*

Proof: Suppose the lemma is false, i.e let $f(a_r, a_s) = a_k \neq a_r, a_s$. Since f is tops-only $f(a_r a_s \dots, a_s) = a_k$ (we are using the fact that there exists a feasible ordering $a_r a_s \dots$ since $a_r \approx a_s$). But then individual i manipulates via $(a_s \dots)$ thereby obtaining a_s since f is unanimous. ■

LEMMA 2 *Let $a_r, a_s \in A$ and suppose $f(a_r, a_s) = a_r$. Let a_t be an alternative distinct from a_r and a_s .*

(i) *If $a_t \approx a_s$, then $f(a_r, a_t) = a_r$.*

(ii) *If $a_t \approx a_r$ and $a_t \approx a_s$, then $f(a_t, a_s) = a_t$.*

Proof: We consider (i) first. Since $a_s \approx a_t$, we can find $P_j, P'_j \in \mathbb{D}$ such that (i) $r_1(P_j) = a_s = r_2(P'_j)$ (ii) $r_2(P_j) = a_t = r_1(P'_j)$ and (iii) $r_l(P_j) = r_l(P'_j)$, $l = 3, \dots, m$. Since f is tops-only $f(a_r, P_j) = a_r$. Suppose $f(a_r, P'_j) \neq a_r$. If $f(a_r, P'_j) P_j a_r$, then j manipulates at (a_r, P_j) via P'_j . If $a_r P_j f(a_r, P'_j)$, then $a_r P'_j f(a_r, P'_j)$ as well by construction, so that j manipulates at (a_r, P'_j) via P_j . Hence $f(a_r, P'_j) = a_r$. Since f is tops-only $f(a_r, a_t) = a_r$.

We now show that (ii) holds. Since $a_t \approx a_r$, we can find $P_i, P'_i \in \mathbb{D}$ such that $r_1(P_i) = a_r = r_2(P'_i)$ and (ii) $r_2(P_i) = a_t = r_1(P'_i)$. Since f is tops-only $f(P_i, a_s) = a_r$. Since f is strategy-proof, it also follows from standard arguments that $f(P'_i, a_s) \in \{a_r, a_t\}$. Since $a_t \approx a_s$, Lemma 1 that $f(P'_i, a_s) \in \{a_t, a_s\}$. Hence $f(P'_i, a_s) = a_t$. Since f is tops-only $f(a_t, a_s) = a_t$ as required. ■

We can now demonstrate further important properties regarding the \approx relation.

LEMMA 3 *The \approx relation does not admit cycles i.e. there does not exist a sequence $a_{k(j)}$, $j = 0, \dots, T$ such that $a_{k(j)} \approx a_{k(j+1)}$, $j = 0, \dots, T - 1$ and $a_{k(T)} \approx a_{k(0)}$.*

Proof: Since $a_{k(0)} \approx a_{k(1)}$, Lemma 1 implies $f(a_{k(0)}, a_{k(1)}) \in \{a_{k(0)}, a_{k(1)}\}$. Assume without loss of generality that $f(a_{k(0)}, a_{k(1)}) = a_{k(0)}$. Since $a_{k(1)} \approx a_{k(2)}$, Lemma 2 (i) implies $f(a_{k(0)}, a_{k(2)}) = a_{k(0)}$. Moreover applying the same argument along the sequence $a_{k(j)}$, $j = 2, \dots, T$, we obtain $f(a_{k(0)}, a_{k(T)}) = a_{k(0)}$. Suppose to the contrary $a_{k(0)} \approx a_{k(T)}$.

Since $f(a_{k(0)}, a_{k(T-1)}) = a_{k(0)}$, $a_{k(0)} \approx a_{k(T)}$ and $a_{k(T)} \approx a_{k(T-1)}$, we can apply Lemma 2 to obtain $f(a_{k(T)}, a_{k(T-1)}) = a_{k(T)}$. Now applying Lemma 2 (i) repeatedly along the sequence $a_{k(j)}$, $j = T-1, \dots, 0$, we obtain $f(a_{k(T)}, a_{k(0)}) = a_{k(T)}$. By anonymity, $f(a_{k(0)}, a_{k(T)}) = a_{k(T)}$. But this contradicts our earlier conclusion that $f(a_{k(0)}, a_{k(T)}) = a_{k(0)}$. ■

We have demonstrated that the strong-connectivity graph induced by \mathbb{D} is a tree. Let this tree be denoted by G . The set of its maximal paths will be denoted by $\mathbb{P}(G)$.

The distance between any pair $a_s, a_r \in A$, denoted by $d(a_s, a_r)$ is defined as $|\{k \neq s, r : a_k \in \langle a_s, a_r \rangle\}|$. It is thus, the number of alternatives not including a_s and a_r that lie on the path between a_s and a_r .

LEMMA 4 $f(a_r, a_s) \in \langle a_r, a_s \rangle$ for all $a_r, a_s \in A$.

Proof: We prove the lemma by induction on $d(a_r, a_s)$. Observe first that the lemma holds in the case where $d(a_r, a_s) = 0$ (i.e. $a_r = a_s$) by virtue of the assumption that f is unanimous. Now suppose that $f(a_r, a_s) \in \langle a_r, a_s \rangle$ whenever $d(a_r, a_s) \leq t$ for some integer $t < m - 1$. Pick a_r, a_s such that $d(a_r, a_s) = t + 1$. Suppose $f(a_r, a_s) = a_z \notin \langle a_r, a_s \rangle$. There must exist $a_k \in \langle a_r, a_s \rangle$ such that $a_r \approx a_k$. Note that $d(a_k, a_s) = t$ so that $f(a_k, a_s) \in \langle a_k, a_s \rangle \subset \langle a_r, a_s \rangle$. Since $a_r \approx a_k$, there exists $P_i, \bar{P}_i \in \mathbb{D}$ such that (i) $r_1(P_i) = r_2(\bar{P}_i) = a_r$ (ii) $r_2(P_i) = r_1(\bar{P}_i) = a_k$ and (iii) $r_j(P_i) = r_j(\bar{P}_i)$ for $j = 3, \dots, m$. Since f is tops-only, $f(P_i, a_s) = a_z$. Moreover, using standard arguments for strategy-proofness, it follows that $f(\bar{P}_i, a_s) = a_z$. Hence $f(a_k, a_s) = a_z$ by tops-onlyness. Since $a_z \notin \langle a_r, a_s \rangle$ by assumption, we have a contradiction to our earlier conclusion that $f(a_k, a_s) \in \langle a_k, a_s \rangle$. ■

Pick an arbitrary pair a_r, a_s be such that $d(a_r, a_s) = 1$. Let $\overline{\langle a_r, a_s \rangle}$ be a maximal path. Suppose $\overline{\langle a_r, a_s \rangle} = \{a_{j(0)}, \dots, a_{j(k)} = a_r, a_{j(k+1)}, a_{j(k+2)} = a_s, a_{j(k+3)} \dots, a_{j(T)}\}$.

LEMMA 5 (i) Suppose $f(a_r, a_s) = a_r$, i.e. $f(a_{j(k)}, a_{j(k+2)}) = a_{j(k)}$. Pick integers u, v such that $u \geq 0$, $u < v$ and $k + v \leq T$. Then $f(a_{j(k+u)}, a_{j(k+v)}) = a_{j(k+u)}$.

(ii) Suppose $f(a_r, a_s) = a_s$, i.e. $f(a_{j(k)}, a_{j(k+2)}) = a_{j(k+2)}$. Pick integers u, v such that $v \leq 0$, $u < v$ and $k + u \geq 0$. Then $f(a_{j(k+u)}, a_{j(k+v)}) = a_{j(k+v)}$.

Proof: We first prove (i). Consider the case where $u = 0$. Since $f(a_{j(k)}, a_{j(k+2)}) = a_{j(k)}$ and $a_{j(k+1)} \approx a_{j(k+2)}$, a direct application of Lemma 2(i) yields $f(a_{j(k)}, a_{j(k+1)}) = a_{j(k)}$. An identical argument yields $f(a_{j(k)}, a_{j(k+3)}) = a_{j(k)}$. Moreover, since $a_{j(k+3)} \approx a_{j(k+4)}$ etc till $a_{j(k+v-1)} \approx a_{j(k+v)}$, the same argument applied repeatedly yields $f(a_{j(k)}, a_{j(k+v)}) = a_{j(k)}$.

Now consider the case where $u = 1$. Choose an arbitrary v such that $v > 1$ and $k + v \leq T$. By Lemma 4, $f(a_{j(k+1)}, a_{j(k+v)}) \in \langle a_{j(k+1)}, a_{j(k+v)} \rangle$. Note that since $a_{j(k)} \approx a_{j(k+1)}$, we can argue (like in the proof of Lemma 2 (ii)) that $f(a_{j(k+1)}, a_{j(k+v)}) \in \{a_{j(k)}, a_{j(k+1)}\}$. But $a_{j(k)} \notin \langle a_{j(k+1)}, a_{j(k+v)} \rangle$. Hence $f(a_{j(k+1)}, a_{j(k+v)}) = a_{j(k+1)}$. Applying this argument repeatedly, we obtain $f(a_{j(k+u)}, a_{j(k+v)}) = a_{j(k+u)}$.

The proof of part (ii) is the symmetric counterpart of the proof of part (i) of the Lemma and is therefore omitted. \blacksquare

Lemma 5 says the following. Suppose we can find two alternatives a_r and a_s where there is exactly one alternative other than a_r and a_s in the (unique) path that connects a_r and a_s . Suppose $f(a_r, a_s) = a_r$. Then if one picks a profile, $(a_k, a_{k'})$ where (i) both a_k and $a_{k'}$ lie on a maximal path connecting a_r and a_s (ii) both a_k and $a_{k'}$ lie on the segment of this maximal path which begins at a_r and contains the path from a_r to a_s and (iii) a_k is closer to a_r than $a_{k'}$, then $f(a_k, a_{k'}) = a_k$.

Consider an arbitrary maximal path $p_t \in \mathbb{P}(G)$. Assume without loss of generality that $p_t = \{a_{j(0)}, \dots, a_{j(k)}, \dots, a_{j(T)}\}$. Consider profiles of preferences where the peaks of both individual's preferences lie on p_t , i.e. profiles of the form $(a_{j(k)}, a_{j(k+2)})$, $k = 1, \dots, T - 2$. In view of Lemma 4, the following cases are mutually exhaustive.

Case A: There exists $k \in \{0, \dots, T - 2\}$ such that $f(a_{j(k)}, a_{j(k+2)}) = a_{j(k+1)}$.

Case B: For all $k \in \{0, \dots, T - 2\}$, $f(a_{j(k)}, a_{j(k+2)}) \in \{a_{j(k)}, a_{j(k+2)}\}$.

Suppose *Case A* holds. The following lemma characterizes the SCF in this case.

LEMMA 6 *Suppose Case A holds, i.e. there exists k such that $f(a_{j(k)}, a_{j(k+2)}) = a_{j(k+1)}$. Then for any preference profile $(a_{j(r)}, a_{j(s)})$, $0 \leq r, s \leq T$ (i.e. both individual's peaks lie on p_t),*

$$f(a_{j(r)}, a_{j(s)}) = \begin{cases} a_{j(k+1)} & \text{if } \min\{r, s\} \leq k + 1 \leq \max\{r, s\} \\ a_{j(\max\{r, s\})} & \text{if } k + 1 > \max\{r, s\} \\ a_{j(\min\{r, s\})} & \text{if } k + 1 < \min\{r, s\} \end{cases}$$

Proof: We have assumed that $f(a_{j(k)}, a_{j(k+2)}) = a_{j(k+1)}$. We show that $f(a_{j(k-1)}, a_{j(k+3)}) = a_{j(k+1)}$ as well. Since $a_{j(k)} \approx a_{j(k-1)}$ there exists $P_i, \bar{P}_i \in \mathbb{D}$ such that (i) $r_1(P_i) = a_{j(k)} = r_2(\bar{P}_i)$ (ii) $r_2(P_i) = a_{j(k-1)} = r_1(\bar{P}_i)$ and (iii) $r_j(P_i) = r_j(\bar{P}_i)$ for $j = 3, \dots, m$. By tops-onlyness $f(P_i, a_{j(k+2)} \dots) = a_{j(k+1)}$. Now consider $f(\bar{P}_i, a_{j(k+2)} \dots)$. Since for all a_l , $a_{j(k+1)} P_i a_l$ iff $a_{j(k+1)} \bar{P}_i a_l$, strategy-proofness implies $f(\bar{P}_i, a_{j(k+2)} \dots) = a_{j(k+1)}$. By tops-onlyness, one has $f(a_{j(k-1)}, a_{j(k+2)}) = a_{j(k+1)}$. An analogous argument with respect to $k + 3$ applies to yield $f(a_{j(k-1)}, a_{j(k+3)}) = a_{j(k+1)}$. Repeated application of this procedure yields $f(a_{j(r)}, a_{j(s)}) = a_{j(k+1)}$ whenever $\min\{r, s\} \leq k + 1 \leq \max\{r, s\}$.

Now take $(a_{j(r)}, a_{j(s)})$ with $k + 1 > \max\{r, s\}$. As $f(a_{j(k)}, a_{j(k+3)}) = a_{j(k+1)}$, by strategy-proofness, $f(a_{j(k+1)}, a_{j(k+3)}) = a_{j(k+1)}$ holds. We have $f(a_{j(r)}, a_{j(s)}) = a_{j(\max\{r,s\})}$ by Lemma 5 (ii). An analogous reasoning appealing to Lemma 5 (i) establishes $f(a_{j(r)}, a_{j(s)}) = a_{j(\min\{r,s\})}$ whenever $k + 1 < \min\{r, s\}$. \blacksquare

Consider maximal paths $p_t = \langle a_{j(0)}, \dots, a_{j(T)} \rangle$ where Case A holds, i.e. there exists $k \in \{0, \dots, T - 2\}$ such that $f(a_{j(k)}, a_{j(k+2)}) = a_{j(k+1)}$. Define $\lambda(p_t) = a_{j(k+1)}$. We show that properties (i) and (ii) of Definition 4 hold. Pick P_i such that $r_1(P_i) \in p_t$ and let $a_{j(r)} \in p_t$ be such that $\lambda(p_t) \in \langle r_1(P_i), a_{j(r)} \rangle$. It is evident from Lemma 6 that $f(P_i, a_{j(r)}) = a_{j(k+1)} = \lambda(p_t)$. Suppose that individual deviates to P'_i where $r_1(P'_i) = a_{j(r)}$. Since f satisfies unanimity, $f(P'_i, P_j) = a_{j(r)}$. Since f is strategy-proof, we must have $\lambda(p_t)P_i a_{j(r)}$ as required by part (i) of Definition 4.

Now, pick some $a_{j(r)}, a_{j(s)} \in p_t$ such that $a_{j(r)}, a_{j(s)} \in \langle r_1(P_i), \lambda(p_t) \rangle$ and $a_{j(r)} \in \langle r_1(P_i), a_{j(s)} \rangle$. Let individual j have preference P_j with $r_1(P_j) = a_{j(r)}$. By Lemma 6, $f(P_i, P_j) = a_{j(r)}$. Now consider a deviation by individual i to \bar{P}_i such that $r_1(\bar{P}_i) = a_{j(s)}$. Again by Lemma 6, $f(\bar{P}_i, P_j) = a_{j(s)}$. By strategy-proofness of f , we must have $a_{j(r)}P_i a_{j(s)}$, as required by part (ii) of the Definition 4.

Now consider a path $p_t = \{a_{j(0)}, \dots, a_{j(k)}, \dots, a_{j(T)}\}$ where Case B holds. Note that in this case we must have $f(a_{j(0)}, a_{j(2)}) \in \{a_{j(0)}, a_{j(2)}\}$.

Suppose $f(a_{j(0)}, a_{j(2)}) = a_{j(0)}$. Define $\lambda(p_t) = a_{j(0)}$. Consider any $P_i \in \mathbb{D}$ with $r_1(P_i) \in p_t$. Consider $a_r, a_s \in \langle \lambda(p_t), r_1(P_i) \rangle$ such that $a_r \in \{a_s, r_1(P_i)\}$. From Lemma 5 (i) we have $f(P_i, a_r) = a_r$ and $f(a_s, a_r) = a_s$. Strategy-proofness, yields $a_r P_i a_s$ as required by Definition 4.

Suppose $f(a_{j(0)}, a_{j(2)}) = a_{j(2)}$. There are two subcases to consider.

(a) $f(a_{j(k)}, a_{j(k+2)}) = a_{j(k+2)}$ for all $k \in \{0, \dots, T - 2\}$. Then, in particular $f(a_{j(T-2)}, a_{j(T)}) = a_{j(T)}$. Let $\lambda(p_t) = a_{j(T)}$. Consider any $P_i \in \mathbb{D}$ with $r_1(P_i) \in p_t$. Consider $a_r, a_s \in \langle \lambda(p_t), r_1(P_i) \rangle$ such that $a_r \in \langle a_s, r_1(P_i) \rangle$. By Lemma 5 (ii), $f(P_i, a_r) = a_r$ and $f(a_s, a_r) = a_s$. From strategy-proofness, we must have $a_r P_i a_s$, as required by Definition 4.

(b) Suppose there exists t where $f(a_{j(t)}, a_{j(t+2)}) = a_{j(t)}$. Let t be the lowest index for which $f(a_{j(t)}, a_{j(t+2)}) = a_{j(t)}$ that is, $f(a_{j(l)}, a_{j(l+2)}) = a_{j(l+2)}$, $0 \leq l < t$. Therefore, $f(a_{j(t-1)}, a_{j(t+1)}) = a_{j(t+1)}$. Since $a_{j(t+1)} \approx a_{j(t+2)}$, and $f(a_{j(t)}, a_{j(t+2)}) = a_{j(t)}$, Lemma 2 (i) implies $f(a_{j(t)}, a_{j(t+1)}) = a_{j(t)}$. However, since $a_{j(t-1)} \approx a_{j(t)}$, and $f(a_{j(t-1)}, a_{j(t+1)}) = a_{j(t+1)}$, Lemma 2 (i) implies $f(a_{j(t)}, a_{j(t+1)}) = a_{j(t+1)}$. We have a contradiction. Hence this case cannot arise.

We have shown that there exists a tree G and a function $\lambda : \mathbb{P}(G) \rightarrow A$ such that all orderings in \mathbb{D} satisfy the restrictions in Definition 4. In order to show that \mathbb{D} is semi-single-peaked, we only need to show that the pair (G, λ) is admissible for G (Definition 3).

For every $a_r, a_j \in A$, a_s is the neighbour of a_r on the path $\langle a_j, a_r \rangle$ if (i) $a_s \in \langle a_j, a_r \rangle$ (ii) there does not exist $a_k \neq a_r, a_s$ with $a_k \in \langle a_s, a_r \rangle$. In other words, a_s is a neighbour of a_r on the path $\langle a_j, a_r \rangle$ if a_s lies on the path and there does not exist an alternative a_k on the same path lying “between” a_s and a_r . We say that a_s is a neighbour of a_r if there exists a path containing a_r and a_s is a neighbour of a_r on that path.

Let $p_t \in \mathbb{P}(G)$. Let $\mathbb{P}(p_t)$ denote the set of maximal paths that contain $\lambda(p_t)$. Let $\overline{\mathbb{P}}(p_t) = \{p_l \in \mathbb{P}(p_t) : \lambda(p_l) \neq \lambda(p_t)\}$. Thus $\overline{\mathbb{P}}(p_t)$ are those paths containing $\lambda(p_t)$ with the property that their λ 's do not coincide with $\lambda(p_t)$.

LEMMA 7 *Let p_t be a maximal path such that $\overline{\mathbb{P}}(p_t) \neq \emptyset$. Then there exists a unique neighbour a_s of $\lambda(p_t)$ such that $a_s \in p_l$ for all $p_l \in \overline{\mathbb{P}}(p_t)$. Moreover $\lambda(p_l) \neq \lambda(p_t)$ for all $p_l = \overline{\langle \lambda(p_t), a_s \rangle}$.*

Proof: Pick arbitrary $p_l, p_{l'} \in \overline{\mathbb{P}}(p_t)$. Let a_s be the neighbour of $\lambda(p_t)$ on the path $\langle \lambda(p_l), \lambda(p_t) \rangle$. We claim that $a_s \in p_{l'}$. Suppose not. Then there exists a_r distinct from a_s which is the neighbour of $\lambda(p_t)$ on the path $\langle \lambda(p_t), \lambda(p_{l'}) \rangle$.

Applying Lemma 5 (ii) to the path $p_{l'}$, we have $f(\lambda(p_t), a_s) = a_s$. Applying the same lemma to the path p_l , we have $f(\lambda(p_t), a_r) = a_r$. Now consider the path $\langle a_r, a_s \rangle$. Since G is a tree, this path contains $\lambda(p_t)$. In fact $\langle a_r, a_s \rangle$ can be written as

$\{a_{j(0)}, \dots, a_{j(k-1)}, a_{j(k)}, a_{j(k+1)}, \dots, a_{j(T)}\}$ where $a_{j(k-1)} = a_r, a_{j(k)} = \lambda(p_t)$ and $a_{j(k+1)} = a_s$. Then we have $f(a_{j(k-1)}, a_{j(k)}) = a_{j(k-1)}$ and $f(a_{j(k)}, a_{j(k+1)}) = a_{j(k+1)}$. However we have already shown in dealing with Case B, part (b) above (using Lemma 2 (i)) that f cannot behave like this. This establishes the first part of the Lemma.

Suppose the second part of the Lemma is false, i.e. there exists $p_l = \overline{\langle \lambda(p_t), a_s \rangle}$ such that $\lambda(p_l) = \lambda(p_t)$. Applying Lemma 5 (i) to the path p_l , we have $f(\lambda(p_t), a_s) = \lambda(p_t)$. By assumption and the first part of the Lemma, there exists $p_{l'} = \overline{\langle \lambda(p_t), a_s \rangle}$ such that $\lambda(p_{l'}) \neq \lambda(p_t)$. Applying Lemma 5 (ii) to the path $p_{l'}$ we have $f(\lambda(p_t), a_s) = a_s$, contradicting our earlier conclusion. \blacksquare

We identify an alternative a_k^* by the following algorithm. Start with an arbitrary maximal path p_t^1 . If $\overline{\mathbb{P}}(p_t^1) = \emptyset$, we let $\lambda(p_t^1) = a_k^*$. If $\overline{\mathbb{P}}(p_t^1) \neq \emptyset$, pick an arbitrary $p_t^2 \in \overline{\mathbb{P}}(p_t^1)$. If $\overline{\mathbb{P}}(p_t^2) = \emptyset$, we let $\lambda(p_t^2) = a_k^*$. Otherwise pick $p_t^3 \in \overline{\mathbb{P}}(p_t^2)$ and check if $\overline{\mathbb{P}}(p_t^3) = \emptyset$ etc. The algorithm stops whenever a_k^* has been found.

Consider the r^{th} step of the algorithm, $r > 1$. Since the algorithm has not stopped at step $r-1$, i.e. $\overline{\mathbb{P}}(p_t^{r-1}) \neq \emptyset$. Therefore there must exist a maximal path p_l containing $\lambda(p_t^{r-1})$ such that $\lambda(p_l) \neq \lambda(p_t^{r-1})$. It follows from the construction of the algorithm that p_l contains the alternatives $\lambda(p_t^1), \lambda(p_t^2), \dots, \lambda(p_t^{r-1})$, i.e $p_l = \overline{\langle \lambda(p_t^1), \dots, \lambda(p_t^{r-1}) \rangle}$. It follows from the second part of Lemma 7 that $\lambda(p_t^r) \neq \lambda(p_t^1), \dots, \lambda(p_t^{r-1})$.

For any positive integer r , let $D(p_t^r) = \mathbb{P}(p_t^r) - \overline{\mathbb{P}}(p_t^r)$. These are the maximal paths discarded in the r^{th} step of the algorithm. By definition these are paths p_l which contain $\lambda(p_t^r)$ and satisfy $\lambda(p_l) = \lambda(p_t^r)$. Suppose that $p_t^r \in D(p_t^l)$ for some integer $l < r$. From our

earlier remarks, $p_t^r \in \overline{\langle \lambda(p_t^1), \dots, \lambda(p_t^{r-1}) \rangle}$. Also $\lambda(p_t^r) \neq \lambda(p_t^l)$ contradicting our assumption that $p_t^r \in D(p_t^l)$. Thus the algorithm cannot pick a maximal path in the r^{th} step which has been discarded in an earlier step. Since the number of maximal paths is finite, this implies that the algorithm must terminate, i.e a_k^* exists.

Observe that by construction, $\lambda(p_l) = a_k^*$ for all maximal paths p_l that contain a_k^* . Now pick a maximal path p_t such that $a_k^* \notin p_t$. Since G is a tree, there must exist a (unique) path containing a_k^* that has a unique alternative in common with p_t . Let this alternative be a_j . In order to prove that the pair (G, λ) is an admissible pair, it suffices to prove that $\lambda(p_t) = a_j$. Suppose that this is false, i.e. $\lambda(p_t) \neq a_j$. Let a_r be the neighbour of a_j on the path $\langle a_j, \lambda(p_t) \rangle$ and let a_s be the neighbour of a_j on the path $\langle a_j, a_k^* \rangle$. Applying Lemma 5 to the path p_t , we have $f(a_j, a_r) = a_r$. Applying the same lemma to the path p_l , we have $f(a_s, a_j) = a_s$. Now consider the path $\langle a_r, a_s \rangle$. Since G is a tree, this path contains a_j . In fact $\langle a_r, a_s \rangle$ can be written as $\{a_{j(0)}, \dots, \lambda(p_t), \dots, a_{j(k-1)}, a_{j(k)}, a_{j(k+1)}, \dots, a_k^*, \dots, a_{j(T)}\}$ where $a_{j(k-1)} = a_r$, $a_{j(k)} = a_j$ and $a_{j(k+1)} = a_s$. Then we have $f(a_{j(k-1)}, a_{j(k)}) = a_{j(k-1)}$ and $f(a_{j(k)}, a_{j(k+1)}) = a_{j(k+1)}$. As we have seen earlier, this contradicts the strategy-proofness of f via Lemma 2 (i).

This completes the first part of the proof.

We now show that if \mathbb{D} is semi-single-peaked (with respect to an admissible pair (G, λ)), then it admits an anonymous, tops-only, unanimous and strategy-proof SCF for all n .

For any set $B \subset A$, let $G(B)$ be the minimal subgraph of G that contains B as nodes. More formally, $G(B)$ is the unique graph that satisfies the properties below.

1. The set of nodes in $G(B)$ contains B .
2. Let $a_j, a_k \in B$. The graph $G(B)$ has an edge $\{a_j, a_k\}$ only if $\{a_j, a_k\}$ is an edge in G .
3. $G(B)$ is connected.
4. $a_k \in G(B)$ if and only if $a_k \in \langle a_r, a_j \rangle$ where $a_r, a_j \in B$.

An alternative way to define $G(B)$ would be as the minimal graph satisfying properties 1, 2 and 3 above. It is clear that $G(B)$ exists and is a tree.

For any profile $P \in \mathbb{D}^n$, let $\{r_1(P)\}$ denote the set of all first-ranked alternatives in the profile P , i.e $\{r_1(P)\} = \{a_i \in A | r_1(P_i) = a_i \text{ for some } i \in N\}$. Let $a_k \in A$ be the alternative specified in Definition 3 applied to the admissible pair (G, λ) . Consider the graph $G(\{r_1(P)\})$. Assume $a_k \notin \{r_1(P)\}$. Since G is a tree and contains no cycles, there exists a unique alternative in $G(\{r_1(P)\})$ that belongs to every path from a_k to $\{r_1(P)\}$. Let this alternative be denoted by $\beta(P)$.

Consider the following example. Suppose $N = \{1, 2, 3\}$. Consider the admissible pair (G_0, λ^4) (Figure 4) and let P be a profile such that $\{r_1(P)\} = \{a_1, a_2, a_3\}$. Then $\beta(P) = a_2$.

Define the SCF $f : \mathbb{D}^n \rightarrow A$ as follows.

$$f(P) = \begin{cases} a_k & \text{if } a_k \in G(\{r_1(P)\}) \\ \beta(P) & \text{if } a_k \notin G(\{r_1(P)\}) \end{cases}$$

It follows immediately from the construction that f is anonymous, unanimous and tops-only. We now show that f is strategy-proof, which will conclude the proof.

Fix a profile P . Observe that whether $a_k \in G(\{r_1(P)\})$ or $a_k \notin G(\{r_1(P)\})$, there exist individuals i and j such that $f(P) \in \langle r_1(P_i), r_1(P_j) \rangle$ (since $G(\{r_1(P)\})$ only considers nodes that belong to an interval of the form $\langle r_1(P_i), r_1(P_j) \rangle$). Moreover these individuals can be chosen such that there does not exist an individual i' such that $r_1(P_{i'}) \in \langle r_1(P_i), f(P) \rangle$ and $r_1(P_{i'}) \in \langle r_1(P_j), f(P) \rangle$ (i.e i and j are the closest peaks on either ‘‘side’’ of $f(P)$ on a maximal path containing $f(P)$). Note that these individuals need not be unique; let the set of these individuals be N' .

We first show that an individual $i \notin N'$ cannot manipulate at P . Observe first that if $a_k \in G(\{r_1(P)\})$ (i.e. $f(P) = a_k$), then i cannot change the outcome by any deviation. Suppose therefore that $a_k \notin G(\{r_1(P)\})$ so that $f(P) = \beta(P)$. By deviating to P'_i where $r_1(P'_i) = a_k$, i can change the outcome to a_k . We claim that this is, in fact, the only outcome different from $\beta(P)$ that i can obtain by deviating from P_i . Suppose this is false. Then it must be the case that there exists P'_i which induces the sub-tree $G(\{r_1(P'_i), P_{-i}\})$ and the outcome $\beta(P'_i, P_{-i})$ which is distinct from both $\beta(P)$ and a_k . Consider $i' \in N'$. It follows that there exists a path in G from $r_1(P_{i'})$ to a_k via $\beta(P)$ and another distinct one from $r_1(P_{i'})$ to a_k via $\beta(P'_i, P_{-i})$. This contradicts our assumption that G is a tree.

Suppose therefore that $f(P'_i, P_{-i}) = a_k$ for some $P'_i \in \mathbb{D}$. In that case $\beta(P) \in \langle r_1(P_i), a_k \rangle$. Since $\lambda(\overline{\langle r_1(P_i), a_k \rangle}) = a_k$ (since (G, λ) is an admissible pair), it follows from Definition 4 (ii) of semi-single-peakedness that $\beta(P) P_i a_k$. Hence i cannot manipulate.

Now consider deviations by individuals $i \in N'$. Suppose j is the other individual such that $f(P) \in \langle r_1(P_i), r_1(P_j) \rangle$. Consider $\lambda(p_t)$ where p_t is any path $\overline{\langle r_1(P_i), r_1(P_j) \rangle}$. There are several possibilities enumerated below.

- 1 $a_k \in p_t$ where $p_t \in \overline{\langle r_1(P_i), r_1(P_j) \rangle}$. Clearly $\lambda(p_t) = a_k$ (part (i) of Definition 3).
- 2 $a_k \notin p_t$ where $p_t \in \overline{\langle r_1(P_i), r_1(P_j) \rangle}$. Let the unique path from a_k to $G(\{r_1(P)\})$ intersect a path p_t in $\overline{\langle r_1(P_i), r_1(P_j) \rangle}$ at a_k^* . Clearly $\lambda(p_t) = a_k^*$ (part (ii) of Definition 3).

Case 1 can be sub-categorized into the cases below.

- 1(i) $a_k \in \text{int}\langle r_1(P_i), r_1(P_j) \rangle$. Then $f(P) = a_k$.
- 1(ii) $r_1(P_i) \in \langle r_1(P_j), a_k \rangle$. Then $f(P) = r_1(P_i)$.

1(iii) $r_1(P_j) \in \langle r_1(P_i), a_k \rangle$. Then $f(P) = r_1(P_j)$.

Similarly Case 2 can be sub-categorized into the cases below.

2(i) $a_k^* \in \text{int}\langle r_1(P_i), r_1(P_j) \rangle$. Then $f(P) = a_k^*$.

2(ii) $r_1(P_i) \in \langle r_1(P_j), a_k^* \rangle$. Then $f(P) = r_1(P_i)$.

2(iii) $r_1(P_j) \in \langle r_1(P_i), a_k^* \rangle$. Then $f(P) = r_1(P_j)$.

In Cases 1(ii) and 2(ii), individual i is getting her best alternative and will clearly not manipulate.

Suppose either Case 1(i) or 2(i) occurs. In each case $f(P) = \lambda(p_t)$ where $p_t = \overline{\langle r_1(P_i), r_1(P_j) \rangle}$. If i announces P'_i such that $r_1(P'_i) \in \langle r_1(P_j), f(P) \rangle$, then $f(P'_i, P_{-i}) = r_1(P'_i)$. By semi-single-peakedness, $f(P)P_i f(P'_i, P_{-i})$ (Definition 4 (i)) so that i does not manipulate. Suppose i deviates to P'_i such that $r_1(P'_i) \notin \langle r_1(P_j), f(P) \rangle$. It follows from the construction of f that $f(P'_i, P_{-i}) = r_1(P_j)$ or else $f(P'_i, P_{-i}) = a_k = f(P)$. This case is clearly covered by our earlier argument (Definition 4 (i)).

Now suppose either Cases 1(iii) or 2(iii) hold. By deviating to P'_i such that $r_1(P'_i) \in \langle r_1(P_j), a_k \rangle$ (in Case 1(iii)) or $r_1(P'_i) \in \langle r_1(P_j), a_k^* \rangle$ (in Case 2(iii)), i can obtain the outcome $f(P'_i, P_{-i}) = r_1(P'_i)$. By part (ii) of Definition 4 of semi-single-peakedness, we have $f(P)P_i f(P'_i, P_{-i})$. Once again i cannot manipulate. \blacksquare

We show below that we can dispense with the tops-only requirement in Theorem 1 by strengthening the richness requirement for admissible preferences. This additional requirement ensures that every strategy-proof, unanimous and anonymous SCF defined over such a domain is tops-only.

DEFINITION 5 *A domain \mathbb{D} satisfies Property \widehat{T} if for every $P_i \in \mathbb{D}$, every $a \neq r_1(P_i)$, there exists $b \in A$ such that (i) $bP_i a$ and (ii) $b \approx a$.*

We observe that strongly path-connected domains do not necessarily satisfy Property \widehat{T} . The property is an additional restriction on the domain. The domains specified in Examples 1, 2 and 3 in the paper do satisfy Property \widehat{T} . Semi-single-peaked domains however need not satisfy Property \widehat{T} . An example is the instance of the line in Figure 2 with a_4 as the threshold. Indeed consider a preference ordering P_i where a_2 is the peak. Semi-single-peakedness requires that a_3 be ranked higher than a_4 and a_4 be ranked higher than both a_5 and a_6 . However, it is possible that a_6 is ranked higher than a_5 . In this case, a_6 is not strongly connected to some alternative that is preferred to it under P_i . However, by ruling out such preference orderings, one may obtain semi-single-peaked domains that do satisfy Property \widehat{T} .

Property \widehat{T} is a strengthening of a similar condition (Property T) that was used in [Chatterji and Sen \(2010\)](#) to prove that every two-person strategy-proof, unanimous SCF must have the tops-only property. The difference between the two conditions is that Property T uses the weaker notion of connectedness specified in Definition 6. It is however not known whether Property T is sufficient to ensure tops-onlyness for general N person SCFs. We establish here that if Property T is strengthened to Property \widehat{T} , one obtains a tops-onlyness result for N person SCFs. This result complements the general N person tops-onlyness result (Theorem 2) of [Chatterji and Sen \(2010\)](#) and is of some independent interest.

PROPOSITION 3 *If a minimally rich domain \mathbb{D} satisfies Property \widehat{T} , then it satisfies the tops-only property, that is, every SCF $f : \mathbb{D}^n \rightarrow A$ which is strategy-proof and satisfies unanimity, also satisfies the tops-only property.*

The proof of Proposition 3 is contained in the Appendix. The Proposition together with Theorem 1 immediately yields the following result.

THEOREM 2 *Let \mathbb{D} be a strongly path-connected domain satisfying Property \widehat{T} and let n be an even integer. If there exists an anonymous, unanimous and strategy-proof SCF $f : \mathbb{D}^n \rightarrow A$, then \mathbb{D} is semi-single-peaked.*

OBSERVATION 2 The proofs of Theorems 1 and 2 also provide a characterization of strategy-proof, anonymous and unanimous SCFs defined over semi-single-peaked domains when there are two voters. These SCFs are median voter rules on every maximal path with a phantom located at the induced threshold on that path.

We now discuss semi-single-peakedness and related literature.

3.1 DISCUSSION

In this section, we discuss the relationship of our work with that of [Nehring and Puppe \(2007b\)](#) and [Nehring and Puppe \(2007a\)](#). They define a ternary relation B over A with the following interpretation: if $(x, y, z) \in B$, then y is “between” x and z . They say that a linear order P_i is *generalized single-peaked* with respect to B iff $(r_1(P_i), y, z) \in B \Rightarrow yP_i z$. They define the notion of a *property space* and use it to construct a natural “betweenness” relationship. According to Theorem 4 in [Nehring and Puppe \(2007b\)](#), if there exists a strategy-proof and neutral social choice function defined on a *rich* domain of generalized single-peaked preference induced by a property space, then this property space must, in fact, be a *median space*. They go on to characterize strategy-proof and neutral social choice functions on these domains. The necessity part of this result is similar in spirit to our analysis. However our analysis and results are quite different in view of the following observations.

(1) They start with a property space and a rich domain of generalized single-peaked preferences with respect to the betweenness relation induced by the space. The starting point of our analysis is a different and more direct notion of rich domains which has no reference to property spaces or any notion of betweenness. (2) Our notion of richness is specified in terms of the ways in which alternatives are ranked first and second in admissible orderings in the domain. It is this structure that we exploit to obtain the ordering on alternatives which is central to the variant of single-peakedness that we characterize. Although their definition of richness does put restrictions on the way that alternatives are ranked first and second in the domain, the exact specification is different from ours. Nor is the structure of these relationships used in the manner that we do. (3) The notions of generalized single-peakedness and semi-single-peakedness are related but independent of each other. For instance, a single-peaked (in the standard sense) is both generalized single-peaked and single-peaked. However domains that are generalized single-peaked are not necessarily semi-single-peaked and vice-versa. For instance, the complete domain is generalized single-peaked but not semi-single-peaked. Conversely one can construct semi-single-peaked domains with a suitable specification of a threshold that is not generalized single-peaked for any betweenness relation. (4) The axioms on social choice functions, in addition to strategy-proofness, used to characterize domains are different. They focus on neutrality while we look at anonymity and either tops-only ness or tops-selectivity.

A related strand of literature is the literature on *maximal domains* (see for instance, [Ching and Serizawa \(1998\)](#), [Barberà et al. \(1999\)](#), [Berga and Serizawa \(2000\)](#), [Massó and Neme \(2009\)](#)). These papers typically investigate maximal domains on which specific rules, such as the generalized median rule, the uniform rule, the extended rule and so on, retain the property of strategy-proofness. Our paper differs from these papers in two significant ways. Firstly, we focus explicitly on a finite setting. More importantly, as we have remarked earlier (point (2) above), our richness conditions on domains are more “primitive” in nature and not formulated in terms of particular social choice functions. We note that the maximal domain identified by [Massó and Neme \(2009\)](#) in the context of a division problem, bears a distant resemblance to our semi-single-peaked preference. We also observe that the domain of semi-single-peaked preference is the maximal domain in the class of rich preferences which admit strategy-proof, unanimous and tops-only and anonymous SCFs.

4 TOPS-SELECTIVITY AND EXTREME-PEAKED DOMAINS

In this subsection, we explore the consequences of replacing the tops-only requirement by the top-selectivity requirement. In general the two requirements are independent of each other. Note that the SCF f defined in the proof of the previous theorem is tops-only but not top-selective (there exist profiles where the outcome is a_k which is not the peak of any individual). One can also easily construct a SCF where the outcome is always, say either

individual j or k 's peak depending on individual j 's bottom-ranked alternative. Such a SCF would be tops-selective but not tops-only. Observe however that a tops-selective SCF is always unanimous.

Our main result in this subsection states that an appropriately rich domain which admits an anonymous, strategy-proof and tops-selective SCF for an even number of voters must be a special class of semi-single peaked domains which we call extreme-peaked domains. Conversely, every extreme-peaked domain admits an anonymous, strategy-proof and tops-selective SCF for an arbitrary number of voters. An important aspect of this result is that the richness condition required for this result is *weaker* than the richness required for Theorem 1. The reason for this is that though tops-selectivity and tops-onlyness are independent conditions, tops-selectivity in conjunction with strategy-proofness implies tops-onlyness in the case of two voters. Therefore for the case of two voters at least, domains which admit anonymous, strategy-proof and tops-selective SCFs must be semi-single peaked. We are able to identify the exact sub-class of semi-single peaked domains which satisfy this property in the presence of a weaker richness property.

We now describe this richness property.

DEFINITION 6 *Two alternatives a_j, a_k are connected in \mathbb{D} , denoted $a_j \sim a_k$, if there exists $P_i, \bar{P}_i \in \mathbb{D}$ such that $r_1(P_i) = a_j = r_2(\bar{P}_i), r_2(P_i) = a_k = r_1(\bar{P}_i)$.*

This notion was introduced in [Aswal et al. \(2003\)](#). Observe that two alternatives which are strongly connected are also connected.

DEFINITION 7 *The domain \mathbb{D} is weakly path-connected iff for all $a_r, a_s \in A$, there exists a sequence of alternatives $a_{j(k)} \in A, k = 0, \dots, T$ such that*

- $a_{j(0)} = a_r$
- $a_{j(T)} = a_s$
- $a_{j(k)} \sim a_{j(k+1)}, k = 0, \dots, T - 1$.

The notion of weak path-connectedness is analogous to the notion of path-connectedness. It requires every pair of alternatives to be joined by a sequence of pairs of alternatives which are connected. We note that the notion of weak path-connectedness is substantially weaker than that of strong path connectedness. For instance, the domain of separable preferences in Example 4 is weakly path-connected (for details, see [Aswal et al. \(2003\)](#)) although we have shown that it is not strongly path-connected.

DEFINITION 8 *A domain \mathbb{D} is extreme-peaked with respect to the linear order τ iff either (i) or (ii) below hold.*

(i) $[a_s \tau a_r \tau r_1(P_i)] \Rightarrow [a_r P_i a_s]$ for all $P_i \in \mathbb{D}$, and for all $a_r, a_s \in A$.

(ii) $[r_1(P_i) \tau a_r \tau a_s] \Rightarrow [a_r P_i a_s]$ for all $P_i \in \mathbb{D}$, and for all $a_r, a_s \in A$.

We say that \mathbb{D} is extreme-peaked if there exists a linear order with respect to which it is extreme-peaked.

We claim that an extreme-peaked domain is semi-single-peaked. In particular it corresponds to the semi-single-peakedness where the admissible pair (G, λ) is such that G consists of a single maximal path (i.e. $G = G_L$ as in Figure 2) and the TAM λ for the unique path selects one of the extreme alternatives/nodes (terminal nodes) in the maximal path (i.e. either a_1 or a_6 in Figure 2).

Our main result in this subsection is the following.

THEOREM 3 *Let \mathbb{D} be a weakly path-connected domain and let n be an even integer. If there exists an anonymous, tops-selective and strategy-proof SCF $f : \mathbb{D}^n \rightarrow A$, then \mathbb{D} is extreme-peaked. Conversely, if D is an extreme-peaked domain, then there exists an anonymous, strategy-proof and tops-selective SCF $f : \mathbb{D}^n \rightarrow A$ for all integers n .*

Proof: We begin with the first part of the Theorem. Using the same arguments as in Proposition 2, it is straightforward to prove the following.

PROPOSITION 4 *Let \mathbb{D} be an arbitrary domain and let n be an even positive integer. Suppose there exists an anonymous, tops-selective and strategy-proof SCF $f : \mathbb{D}^n \rightarrow A$. Then there exists an anonymous, tops-selective and strategy-proof SCF $g : \mathbb{D}^2 \rightarrow A$.*

Let \mathbb{D} be a weakly path-connected domain. In view of Proposition 4 above, we can assume that there exists $f : \mathbb{D}^2 \rightarrow A$ which is anonymous, tops-selective and strategy-proof.

We first show that f is tops-only. Suppose that this is false, i.e there exists $P_i, P'_i, P_j, P'_j \in \mathbb{D}$ with $r_1(P_i) = r_1(P'_i) = a$, $r_1(P_j) = r_1(P'_j) = b$ but $f(P_i, P_j) \neq f(P'_i, P'_j)$. Since f is tops-selective $f(P_i, P_j) \in \{a, b\}$. Assume without loss of generality that $f(P_i, P_j) = a$. Since f is strategy-proof, it must be the case that $f(P'_i, P_j) = a$; otherwise i will manipulate at (P'_i, P_j) . Since $f(P'_i, P'_j) \in \{a, b\}$ (by tops-selectivity) and since $f(P_i, P_j) \neq f(P'_i, P'_j)$, it must be the case that $f(P'_i, P'_j) = b$. But then j will manipulate at (P'_i, P'_j) . Hence f is tops-only.

Since f is tops-only, we can consider a profile as a pair (a_r, a_s) with the interpretation that i has an ordering P_i with $a_r = r_1(P_i)$ and j has an ordering P_j with $a_s = r_1(P_j)$.

Once again, we establish several facts about the \sim relation.

LEMMA 8 *Let $a_r, a_s \in A$ and suppose $f(a_r, a_s) = a_r$. Let a_t be an alternative distinct from a_r and a_s .*

(i) *If $a_t \sim a_s$, then $f(a_r, a_t) = a_r$.*

(ii) If $a_t \sim a_r$, then $f(a_t, a_s) = a_t$.

Proof: We consider (i) first. Since $a_s \sim a_t$, we can find preferences $(a_s a_t \dots)$ and $(a_t a_s \dots)$. Since f is tops-only $f(a_r, a_s a_t \dots) = a_r$. By tops-selectivity, $f(a_r, a_t a_s \dots) \in \{a_r, a_t\}$. If $f(a_r, a_t a_s \dots) = a_t$, individual j manipulates at $(a_r, a_s a_t \dots)$ via $(a_t a_s \dots)$. Hence $f(a_r, a_t a_s \dots) = a_r$. The result now follows from tops-onlyness.

We now show that (ii) holds. Since $a_t \approx a_r$, we can find preferences $(a_r a_t \dots)$ and $(a_t a_r \dots)$. Since f is tops-only $f(a_r a_t \dots, a_s) = a_r$. Since f is strategy-proof, it also follows from standard arguments that $f(a_t a_r \dots, a_s) \in \{a_r, a_t\}$. But tops-selectivity requires $f(a_t a_r \dots, a_s) \in \{a_t, a_s\}$. Hence $f(a_t, a_r \dots, a_s) = a_t$. Since f is tops-only $f(a_t, a_s) = a_t$ as required. ■

LEMMA 9 *The \sim relation does not admit cycles i.e. there does not exist a sequence $a_{k(j)}$, $j = 0, \dots, T-1$ such that $a_{k(j)} \sim a_{k(j+1)}$, $j = 0, \dots, T-1$ and $a_{k(T)} \sim a_{k(0)}$.*

Proof: By tops-selectivity, $f(a_{k(0)}, a_{k(1)}) \in \{a_{k(0)}, a_{k(1)}\}$. Assume without loss of generality that $f(a_{k(0)}, a_{k(1)}) = a_{k(0)}$. Since $a_{k(1)} \sim a_{k(2)}$, Lemma 8 (i) implies $f(a_{k(0)}, a_{k(2)}) = a_{k(0)}$. Moreover applying the same argument along the sequence $a_{k(j)}$, $j = 2, \dots, T$, we obtain $f(a_{k(0)}, a_{k(T)}) = a_{k(0)}$.

Since $f(a_{k(0)}, a_{k(T-1)}) = a_{k(0)}$, $a_{k(0)} \sim a_{k(T)}$ we can apply Lemma 8 (ii) to obtain $f(a_{k(T)}, a_{k(T-1)}) = a_{k(T)}$. Now applying Lemma 8 (i) repeatedly along the sequence $a_{k(j)}$, $j = T-1, \dots, 0$, we obtain $f(a_{k(T)}, a_{k(0)}) = a_{k(T)}$. By anonymity, $f(a_{k(0)}, a_{k(T)}) = a_{k(T)}$. But this contradicts our earlier conclusion that $f(a_{k(0)}, a_{k(T)}) = a_{k(0)}$. ■

LEMMA 10 *Suppose $a_r, a_s, a_t \in A$ such that $a_r \sim a_s$ and $a_r \sim a_t$. Then there does not exist $a_k \in A$ such that $a_r \sim a_k$.*

Proof: Suppose that the Lemma is false, i.e $a_r \sim a_s$, $a_r \sim a_t$ and $a_r \sim a_k$ where a_r, a_s, a_t and a_k are distinct. We consider two cases.

Case I: $f(a_r, a_s) = a_r$. Since $a_r \sim a_t$, $f(a_t, a_s) = a_t$ (Lemma 8 (ii)). Since $a_s \sim a_r$, $f(a_t, a_r) = a_t$ (Lemma 8 (i)) and since $a_r \sim a_k$, $f(a_t, a_k) = a_t$ (Lemma 8 (i)).

Since $a_r \sim a_k$, $f(a_k, a_s) = a_k$ (Lemma 8 (ii)). Since $a_s \sim a_r$, $f(a_k, a_r) = a_k$ (Lemma 8 (i)) and since $a_r \sim a_t$, $f(a_k, a_t) = a_k$ (Lemma 8 (i)). By anonymity $f(a_t, a_k) = a_k$ which contradicts our earlier conclusion that $f(a_t, a_k) = a_t$. ■

We can now completely characterize the structure of the connectivity graph generated by the \sim relation. Start with an arbitrary alternative say $a_{j(k)}$. By Lemma 10, there exist at most two alternatives, say $a_{j(k-1)}$ and $a_{j(k+1)}$ such that $a_{j(k)} \sim a_{j(k-1)}$ and $a_{j(k)} \sim a_{j(k+1)}$. Also there can be at most one alternative, say $a_{j(k-2)}$ such that $a_{j(k-2)} \sim a_{j(k-1)}$.

From Lemma 9, $a_{j(k-2)} \neq a_{j(k)}, a_{j(k+1)}$. Similarly there can be at most one alternative, say $a_{j(k+2)}$ distinct from $a_{j(k)}, a_{j(k-1)}, a_{j(k-2)}$ such that $a_{j(k+2)} \sim a_{j(k+1)}$. Since A is finite, we can conclude that $A = \{a_{j(0)}, a_{j(1)}, \dots, a_{j(m-1)}\}$ where $a_{j(k)} \sim a_{j(k+1)}$, $k = 1, \dots, m-1$. Let τ be the linear order on A defined by $a_{j(0)}\tau a_{j(1)}\tau \dots \tau a_{j(m)}$. We will show that D is extreme-peaked with respect to τ .

We assume without loss of generality that τ is the order $a_1\tau a_2 \dots \tau a_m$ so that $a_r \sim a_s$ if and only if $s = r + 1$, $r = 1, \dots, m-1$. In view of the assumption of top-selectivity, the following cases are mutually exhaustive.

Case A: $f(a_1, a_3) = a_1$.

Case B: $f(a_1, a_3) = a_3$.

The next two lemmas pave the way for the characterization of f in either case.

LEMMA 11 *If $f(a_1, a_3) = a_1$, then $f(a_i, a_j) = a_{\min\{i,j\}}$ for all $i, j \in \{1, \dots, M\}$.*

Proof: Pick any $i, j \in \{1, \dots, m\}$. As f is anonymous, assume without loss of generality that $i < j$. We first note that $f(a_i, a_j) = a_i$ implies $f(a_i, a_{j+1}) = a_i$ from Lemma 8 (i). Similarly $f(a_i, a_j) = a_i$ implies $f(a_i, a_{j-1}) = a_i$. Since we have $f(a_1, a_3) = a_1$, the previous arguments imply that $f(a_1, a_t) = a_1$ for any $t \in \{1, \dots, m\}$. Now consider $f(a_2, a_t)$ for $t > 2$. We argue that $f(a_2, a_t) = a_2$. If not, by top-selectivity, $f(a_2, a_t) = a_t$. Since tops-onlyness implies $f(a_1 a_2 \dots, a_t) = a_1$, individual j can manipulate at the profile $(a_2 a_1 \dots, a_t)$ via $(a_1 a_2 \dots)$ to obtain a_1 . Thus $f(a_2, a_t) = a_2$. Repeated application of this argument yields $f(a_i, a_j) = a_i$. ■

LEMMA 12 *If $f(a_1, a_3) = a_3$, then $f(a_i, a_j) = a_{\max\{i,j\}}$ for all $i, j \in \{1, \dots, m\}$.*

The proof of this Lemma is the symmetric counterpart of the proof of Lemma 11 and is therefore omitted.

We complete the proof by showing that in both *Case A* and *Case B*, \mathbb{D} is extreme-peaked with respect to τ . Pick any $P_i \in \mathbb{D}$. Suppose *Case A* holds. Pick any two alternatives a_r, a_s such that $a_s\tau a_r\tau r_1(P_i)$. From Lemma 11 we have $f(P_i, a_r) = a_r$ and $f(a_s, a_r) = a_s$. By strategy-proofness, we must have $a_r P_i a_s$ fulfilling Part (i) of Definition 8. Now suppose *Case B* holds. Pick a_r and a_s such that $r_1(P_i)\tau a_r\tau a_s$. By Lemma 12, $f(P_i, a_r) = a_r$ and $f(a_s, a_r) = a_s$. Since f is strategy-proof, $a_r P_i a_s$ fulfilling Part (ii) of Definition 8.

We now prove the second part of the Theorem. Let \mathbb{D} be an extreme-peaked domain with respect to the order τ . Suppose without loss of generality that Part (i) of Definition 8 holds, i.e that $[a_s\tau a_r\tau r_1(P_i)] \Rightarrow [a_r P_i a_s]$ for all $P_i \in \mathbb{D}$, and for all $a_r, a_s \in A$. Let n be an arbitrary positive integer $n \geq 2$. Define the SCF $f_\tau : \mathbb{D}^n \rightarrow A$ as follows. For any profile $P \in \mathbb{D}^n$, let $f_\tau(P) = \max(\{r_1(P_1), \dots, r_1(P_n)\}, \tau)$. It is clear from inspection that f_τ is anonymous and tops-selective. We show that f_τ is strategy-proof. Let P be an arbitrary profile and let

$f(P) = a_r$. Consider an arbitrary individual i . If $r_1(P_i) = a_r$ then i clearly cannot do better by deviation. If $r_1(P_i) \neq a_r$, then $a_r \tau r_1(P_i)$. Moreover according to the definition of f_τ , if $f_\tau(P'_i, P_{-i}) = a_s$ where $a_s \neq a_r$, then $a_s \tau a_r$. It follows from extreme-peakedness that $a_r P_i a_s$ so that f_τ is strategy-proof. ■

Our final example demonstrates that strong path-connectedness cannot be replaced by weak path-connectedness in Theorem 1 and tops-selectivity cannot be replaced by tops-onlyness in Theorem 3.

EXAMPLE 5 Let $A = \{a_1, a_2, a_3, a_4\}$. Let $\hat{\mathbb{D}}$ consist of the orderings below.

a_1	a_2	a_2	a_2	a_3	a_3	a_3	a_4
a_2	a_1	a_3	a_3	a_2	a_4	a_2	a_3
a_4	a_3	a_1	a_4	a_1	a_2	a_4	a_1
a_3	a_4	a_4	a_1	a_4	a_1	a_1	a_2

Table 4: Preferences in the domain $\hat{\mathbb{D}}$

The domain $\hat{\mathbb{D}}$ is weakly path-connected with $a_1 \sim a_2$, $a_2 \sim a_3$ and $a_3 \sim a_4$. However, it is not strongly path-connected because a_1 is not strongly connected to a_2 (or anything else). We claim that $\hat{\mathbb{D}}$ is not semi-single-peaked and hence, not extreme-peaked. It can be verified easily that if a domain is semi-single-peaked with respect to some graph, then it must coincide with the graph of weak path-connectedness associated with the domain (in this case, the line $a_1 \sim a_2$, $a_2 \sim a_3$ and $a_3 \sim a_4$). However, it is not possible to specify an admissible pair (with respect to this line) satisfying conditions (i) and (ii) of Definition 4.

A SCF $f : \hat{\mathbb{D}}^2 \rightarrow A$ is defined by unanimity and the following specifications: $f(a_1, a_2) = f(a_2, a_1) = a_2$, $f(a_1, a_3) = f(a_3, a_1) = a_3$, $f(a_2, a_3) = f(a_3, a_2) = a_3$, $f(a_1, a_4) = f(a_4, a_1) = a_4$, $f(a_2, a_4) = f(a_4, a_2) = a_3$ and $f(a_3, a_4) = f(a_4, a_3) = a_3$. By construction f is anonymous, satisfies tops-onlyness and unanimity. Finally, we claim that f is strategy-proof. When one of the voters has a_3 as her peak, then the the outcome cannot be moved from a_3 . Suppose the profile is (a_1, a_2) or (a_2, a_1) . The outcome is then a_2 . The voter whose peak is a_1 can only obtain his last-ranked alternative a_3 by deviation. If the profile is (a_1, a_4) or (a_4, a_1) , then the outcome is a_4 . Here the voter with peak a_1 can only obtain her worst outcome a_3 by deviating.

5 CONCLUSION

In this paper we have attempted to characterize (subject to certain richness requirements) domains of preferences which admit “well-behaved” strategy-proof social choice functions. According to our results, these domains are closely related to variants of domains of single-peaked preferences.

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6 APPENDIX

We provide a proof of Proposition 3.

Proof: We begin by introducing some auxiliary concepts that will be used in the proof. Let \mathbb{D} be an arbitrary domain and $f : \mathbb{D}^n \rightarrow A$ be an arbitrary SCF. For any $P_i \in \mathbb{D}$ and every $a \neq r_1(P_i)$, let $\bar{B}(P_i, a) = \{x \in A \mid x P'_i a \text{ for all } P'_i \text{ such that } r_1(P_i) = r_1(P'_i)\}$. For all $i \in N$ and $P_{-i} \in \mathbb{D}^{N-1}$, let $O_i(P_{-i}) = \{a \in A \mid f(P_i, P_{-i}) = a, P_i \in \mathbb{D}\}$. The set $O_i(P_{-i})$ will be referred to as the option set of voter i given P_{-i} and is the restricted range of the SCF f given P_{-i} . (Option sets were introduced in the literature by [Barberà and Peleg \(1990\)](#)). The following facts about option sets will be used in the proof.

1. For any strategy proof SCF $f : \mathbb{D}^n \rightarrow A$, the following holds: For all $(P_i, P_{-i}) \in \mathbb{D}^n$, $f(P_i, P_{-i}) = C(P_i, O_i(P_{-i}))$, where for all $P_i \in \mathbb{D}$, $B \subset A$, $C(P_i, B)$ is the maximal element in the set B according to P_i . (a proof can be found in [Barberà and Peleg \(1990\)](#).)

2. Let $f : \mathbb{D}^n \rightarrow A$ be a strategy-proof SCF. Then f satisfies the tops-only property iff for all $i \in N$ and $P_{-i}, P'_{-i} \in \mathbb{D}^{n-1}$ which are top equivalent (that is, $r_1(P_j) = r_1(P'_j)$ for all $j \neq i$), we have $O_i(P_{-i}) = O_i(P'_{-i})$. (a proof can be found in [Chatterji and Sen \(2010\)](#)).

The proof closely follows the proof of Theorem 2 of [Chatterji and Sen \(2010\)](#). In particular, Steps 1, 2 and 4 below are identical to the corresponding steps of Theorem 2 referred to above and here we will state them but omit their respective verifications.

We will prove the result by induction on the number of voters n . Let \mathbb{D} be a domain satisfying Property \widehat{T} . Since the SCF's under consideration satisfy unanimity, the result is trivially true for the case $n = 1$. Assume now that for some arbitrary integer n , every SCF $f : \mathbb{D}^n \rightarrow A$ which is strategy-proof and satisfies unanimity also satisfies the tops-only property. We will show that an arbitrary SCF $f : \mathbb{D}^{n+1} \rightarrow A$ which is strategy-proof and satisfies unanimity, also satisfies the tops-only property. In view of Fact 2 above, it will suffice to show that for an arbitrary voter i , $O_i(P_{-i}) = O_i(P'_{-i})$ whenever $P_{-i}, P'_{-i} \in \mathbb{D}^{n-1}$ are top equivalent. We proceed in four steps.

Step 1. Pick $j \neq i$, and let $P_i, P'_i \in \mathbb{D}$ be top equivalent. Then $C(P_i, O_i(P_i, P_{-\{i,j\}})) = C(P'_i, O_i(P'_i, P_{-\{i,j\}}))$.

Step 2. Let $x^* = f(P_i, P_i, P_{-\{i,j\}})$ and let $a \in O_i(P_i, P_{-\{i,j\}})$ be such that $a \neq x^*$. Then $x^* \in \overline{B}(P_i, a)$.

Step 3. $O_i(P_j, P_{-\{i,j\}}) = O_i(P'_j, P_{-\{i,j\}})$ whenever $P_j, P'_j \in \mathbb{D}$ are tops-equivalent.

We now verify the statement of Step 3. Suppose that the statement is false. We can assume without loss of generality that there exists $a \in O_i(P_j, P_{-\{i,j\}}) - O_i(P'_j, P_{-\{i,j\}})$. If $a = r_1(P_j) = r_1(P'_j)$, then by a standard argument, voter j will manipulate if the preference ordering P_i of voter i has $a = r_1(P_i)$. Assume therefore that $a \neq r_1(P_j)$. Let $x^* = f(P_j, P_j, P_{-\{i,j\}})$. It follows from Step 2 that either $a = x^*$ or $x^* \in \overline{B}(P_j, a)$. If $a = x^*$, then $a \in O_i(P'_j, P_{-\{i,j\}})$ since $f(P_j, P_j, P_{-\{i,j\}}) = f(P'_j, P'_j, P_{-\{i,j\}})$. Suppose therefore that $a \neq x^*$. Let $a = r_k(P_j)$. Since $x^* \in \overline{B}(P_j, a)$, $x^* = r_l(P_j)$ where $l < k$. We will show by induction on k that $a \in O_i(P'_j, P_{-\{i,j\}})$. We assume that $[a = r_{k'}(P_j) \text{ and } a \in O_i(P_j, P_{-\{i,j\}})] \Rightarrow a \in O_i(P'_j, P_{-\{i,j\}}) \forall k' < k$. We show that $[a = r_k(P_j) \text{ and } a \in O_i(P_j, P_{-\{i,j\}})] \Rightarrow a \in O_i(P'_j, P_{-\{i,j\}})$.

By Property \widehat{T} , there exists a sequence $x_p, p = 0, \dots, T$ such that (i) $x_0 = a$ (ii) $x_p \approx x_{p+1}$ and $x_{p+1} P_j x_p$ for $p = 0, \dots, T - 1$, (iii) $x_T = r_1(P_j)$. We now state and verify the following claim.

Claim. $a(= x_0) \in O_i(P_j, P_{-\{i,j\}}) \Rightarrow x_1 \in O_i(P_j, P_{-\{i,j\}})$.

In order to verify this claim, we first consider the following case. (i) Suppose $x_{p'} \in O_i(P_j, P_{-\{i,j\}})$ for some $p' \in \{1, \dots, T\}$. Then without loss of generality, let p' be the smallest value of p such that $x_{p'} \in O_i(P_j, P_{-\{i,j\}})$. If the claim is not true, $x_1 \notin O_i(P_j, P_{-\{i,j\}})$ and so $f(P_j, P_{-\{i,j\}}, x_1 x_0 \dots) = x_0$. By strategy-proofness, $f(x_1 x_0 \dots, P_{-\{i,j\}}, x_1 x_0 \dots) = x_0$. Define the n voter SCF $g : \mathbb{D}^n \rightarrow A$ by coalescing voters i and j as follows: for all $P_i \in \mathbb{D}$ and $P_{-\{i,j\}} \in \mathbb{D}^{n-2}$, $f(P_i, P_i, P_{-\{i,j\}}) = g(P_i, P_{-\{i,j\}})$. It can be verified [Sen \(2001\)](#) that g is strategy-proof and by the induction hypothesis on the number of agents, g satisfies

the tops-only property. Since g is tops-only, $g(x_1x_0\dots, P_{-\{i,j\}}) = g(x_1x_2\dots, P_{-\{i,j\}}) = x_0$ establishing thereby that $f(x_1x_2\dots, P_{-\{i,j\}}, x_1x_2\dots) = x_0$. Since $x_1 \approx x_2$, strategy-proofness implies that $f(x_2x_1\dots, P_{-\{i,j\}}, x_2x_1\dots) = x_0$. Repeating this sequence of arguments yields $f(x_{p'-1}x_{p'}\dots, P_{-\{i,j\}}, x_{p'-1}x_{p'}\dots) = x_0$. But $f(P_j, P_{-\{i,j\}}, x_{p'-1}x_{p'}\dots) = x_{p'}$ and so voter j manipulates at the profile $(x_{p'-1}x_{p'}\dots, P_{-\{i,j\}}, x_{p'-1}x_{p'}\dots)$ to P_j from the preference ordering $x_{p'-1}x_{p'}\dots$, proving thereby that in this case that $x_1 \in O_i(P_j, P_{-\{i,j\}})$. We now verify the claim for the remaining case (ii) where $x_p \notin O_i(P_j, P_{-\{i,j\}})$ for all $p = 0, \dots, T$. By repeating the arguments of the previous step, we conclude that $f(x_Tx_{T-1}\dots, P_{-\{i,j\}}, x_Tx_{T-1}\dots) = x_0$. We know by Property \widehat{T} , there exists a sequence y_q , $q = 0, \dots, T'$ such that (i) $y_0 = x^*$ (ii) $y_q \approx y_{q+1}$ and $y_{q+1}P_jy_q$ for $q = 0, \dots, T' - 1$ (iii) $y_{T'} = r_1(P_j)$. We know that $y_0 (= x^*) \in O_i(P_j, P_{-\{i,j\}})$ and that $x_q \notin O_i(P_j, P_{-\{i,j\}})$ for all $q = 1, \dots, T'$ (since $x^* = f(P_j, P_j, P_{-\{i,j\}})$ and $x_q P_j x^*$ for all $q = 1, \dots, T'$). Since $x_T = y_{T'} = r_1(P_j)$, here too the tops-only property of g implies that $f(y_{T'}y_{T'-1}\dots, P_{-\{i,j\}}, y_{T'}y_{T'-1}\dots) = x_0$. Repeating the arguments of the previous step imply that $f(y_1x^*\dots, P_{-\{i,j\}}, y_1x^*\dots) = x_0$. But $f(P_j, P_{-\{i,j\}}, y_1x^*\dots) = x^*$ and so voter j manipulates at the profile $(y_1x^*\dots, P_{-\{i,j\}}, y_1x^*\dots)$ to P_j from the preference ordering $y_1x^*\dots$, proving that here too $x_1 \in O_i(P_j, P_{-\{i,j\}})$. This completes the verification of the Claim.

We now establish that $a \in O_i(P'_j, P_{-\{i,j\}})$. Since $x_1 \in O_i(P_j, P_{-\{i,j\}})$ and $a = r_{k'}(P_j)$ for some $k' < k$, our induction hypothesis implies that $x_1 \in O_i(P'_j, P_{-\{i,j\}})$. If $a \notin O_i(P'_j, P_{-\{i,j\}})$, then $f(P'_j, P_{-\{i,j\}}, ax_1\dots) = x_1$ while $f(P_j, P_{-\{i,j\}}, ax_1\dots) = a$. So voter j manipulates at the profile $(P_j, P_{-\{i,j\}}, ax_1\dots)$ from P_j to P'_j . This completes the verification of Step 3.

Step 4. $O_i(P_{-i}) = O_i(P'_{-i})$ whenever $P_{-i}, P'_{-i} \in \mathbb{D}$ are top equivalent.

This follows from the repeated application of Step 3. ■