Strategy-proofness and Pareto-Efficiency in Classical Exchange Economies

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Abstract

In this paper we investigate the structure of strategy-proof and efficient social choice functions in classical exchange economies. Using techniques developed by Myerson in the context of auction-design, we show that in a very specific quasi-linear domain, every efficient and strategy-proof social choice functions satisfying non-bossiness and a mild continuity property, is dictatorial. The result holds for arbitrary numbers of players but the two-person version does not require either the non-bossiness or continuity assumptions. It also follows that the dictatorship conclusion holds on any superset of this domain. We also provide a minimum consumption guarantee result in the spirit of Serizawa and Weymark (2003).

1 Introduction

Allocating available resources amongst a given set of agents who have preferences defined over these resources has been a well studied problem. A minimal and uncontroversial criterion for such allocations is *Pareto-efficiency*. However, in order to allocate the resources in a Pareto-efficient way manner, the mechanism designer needs to know the true preferences of the agents concerned. If true preferences of the agents are private information then an added requirement criterion for the allocation procedure is *incentive-compatibility* that is, the allocation rule or social choice function must induce agents to reveal their preferences truthfully. The most attractive incentive-compatibility requirement to impose on a social choice function is *strategy-proofness*; if a social choice function is strategy-proof, then no

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agent can benefit by lying irrespective of her beliefs regarding the announcements of other agents. Unfortunately, strategy-proofness is a stringent requirement. According to the well-known Gibbard-Satterthwaite Theorem (Gibbard (1977) and Satterthwaite (1975)), a social choice function defined over an unrestricted domain with a range of at least three alternatives, is strategy-proof only if it is *dictatorial*. A dictatorial social choice function is a trivial social choice function which always picks the best outcome for a given agent.

In many contexts, it is natural to assume that agent preferences are *restricted*. In such cases the dictatorship result need not hold. A large literature has developed investigating the structure of strategy-proof social choice functions in models such as single-peaked domains, quasi-linear domains with monetary domains and so on. In this paper, we consider a familiar restricted domain model that of a it classical exchange economy. In this model there is a fixed amount of m goods, $m \geq 2$ which have to be distributed amongst n agents $n \geq 2$. Agents's preferences defined over bundles of m goods that are assumed to be *strictly increasing, continuous* and *strictly convex*. Although a large literature exists on this problem, there is as yet no comprehensive characterization of strategy-proof and Pareto-efficient social choice functions on this domain (see literature review below).

The main objective of our paper is to establish the equivalence of strategy-proofness and Pareto-efficiency in the presence of certain mild regularity assumptions on the social choice functions. Our methodological innovation is to consider a narrow class of *quasi-linear* preferences and use the techniques developed in the context of auction design and show that strategy-proofness and Pareto-efficiency in conjunction with *non-bossiness* and *continuity* implies dictatorship. The dictatorship result then extends in a straight forward way to all supersets of this domain including in particular to the domain of all classical preferences. Our approach also allows us to provide relatively simple extensions of existing results in the literature.

The classic paper on incentive-compatibility in exchange economies is Hurwicz (1972) which shows that there does not exist strategy-proof, efficient and individually rational SCFs when there are two-agents. A general characterization of strategy-proof and efficient SCFs however remains an open question and has been the focus of considerable research. Dasgupta et al. (1979) show that in the case of $n = 2$, every efficient and strategy-proof SCF is dictatorial when the set of agent preferences is the set of all (strictly) convex and monotone. Zhou (1991) extends this result to the case where preferences are classical, i.e. (strictly) convex and monotone and continuous. There are several versions of this result (for $n = 2$) on restricted domains, for example Schummer (1997) for linear preferences, Ju (2003) for classical quasi-linear and CES and Hashimoto (2008) for Cobb-Douglas preferences. There are significant difficulties involved in extending these results to the case of general n. Zhou (1991) conjectures that efficient dictatorial SCFs in the case of $n \geq 3$ must be *inversely dictatorial*. However Kato and Ohseto (2002) by means of an example have shown that this conjecture is not true.

The only results that exist for general n for strategy-proof and Pareto-efficient social choice functions are Serizawa (2006) and Serizawa and Weymark (2003). The former shows that every strategy-proof and efficient SCF violates a *minimum consumption guarantee* or MCG assumption. In particular, for every $\epsilon > 0$ but arbitrarily small there exists a profile and an agent whose allocation is less than ϵ in terms of the Euclidean norm. Although this result is illuminating it is far from being a characterization. In particular it says nothing about the value of an efficient and strategy-proof SCF at an arbitrary profile.

Serizawa (2006) proves a dictatorship result by strengthening strategy-proofness to effective pairwise strategy-proofness. Effective pairwise strategy-proofness requires pairs of agents not to have a "self-enforcing manipulation" in addition to strategy-proofness. A manipulation by a pair of agents is self-enforcing if it does not decrease the utility of either agents in the pair, increases utility of at least one and neither of the agents has an incentive to betray his partner. Note that effective pairwise strategy-proofness like notions such as group strategy-proofness, requires coordination between agents. However if information is private, it is not clear how this coordination actually takes place and the notion is therefore somewhat problematic.

Two other papers that deal with the $n \geq 2$ agents case are Barberà and Jackson (1995) and Satterthwaite and Sonnenschein (1981). Neither consider Pareto-efficiency. Barberà and Jackson (1995) show that a strategy-proof, individually rational, anonymous and *non-bossy* SCF must be a fixed-proportion trading rule. Satterthwaite and Sonnenschein (1981) show that a strategy-proof, *non-bossy*, *regular* and *everywhere total* SCF must be serially dictatorial.

We consider a domain of classical quasi-linear preferences of the following kind:

$$u_i(x_{i1}, \ldots, x_{im}; \theta_i) = \theta_i\{\sqrt{x_{i1}} + \ldots + \sqrt{x_{im-1}}\} + x_{im}, \theta_i > 0.$$

We use methods in auction design developed by Myerson (1981) to characterize strategy-proof social choice functions. We prove three main results. First, we provide an elementary proof the MCG result of the Serizawa and Weymark (2003). Second, we show that in two-agent economies, every strategy-proof and Pareto-efficient social choice function, is dictatorial. This result is independent of the existing two-person dictatorship results in the literature because of the specificity of our domain. Finally we show that in the case of three or more agents, strategy-proofness and Pareto-efficiency together with non-bossiness and continuity, imply dictatorship. We believe that the $n \geq 3$ result is the first of its kind in the literature. Note that continuity in this problem is defined in the usual Euclidean sense. Note that all dictatorship results extend to all domains that include this domain.

Our paper is organized as follows. In Section 2 we describe our model. In Section 3 we prove some critical results regarding strategy-proofness and Pareto-efficiency in our domain. In Section 4 we prove the MCG result; in Section 5 we prove dictatorship results first, for the two agents and then for the multi-agent case.

2 Notation and Definitions

We consider an exchange economy with the set of agents $I = \{1, 2, ..., n\}$ and the set of goods $M = \{1, 2, ..., m\}$. We assume that $n \geq 2$ and $m \geq 2$. Let the fixed total endowment of good j be denoted by Ω_j and the total endowment vector to be $\Omega = (\Omega_1, \Omega_2, ..., \Omega_m)$. We assume $\Omega_j > 0$, for all $j \in M$. The set of feasible allocations denoted by Δ is the set $\Delta = \{(x_{i1}, ..., x_{im}) | x_{ij} \geq 0, \text{ for all } j \in M \text{ and } i \in I \text{ and } \sum_{i \in I} x_{ij} = \Omega_j \text{ for all } j \in M\}$.

A preference ordering for agent i, R_i is a complete, reflexive and transitive ordering of the elements of $\Re^m_+$. We say that R_i is *classical* if it is (a) continuous, (b) strictly monotonic in $\Re^m_{++}$ and (c) the upper contour sets are strictly convex in $\Re^m_{++}$ [1]. The asymmetric component of R_i will be denoted by P_i. Let the set of classical orderings be denoted by $\mathbb{D}^c$. A preference profile R is an n-tuple $R \equiv (R_1, R_2, \dots, R_n) \in [\mathbb{D}^c]^n$. We shall let R_{-i} denote the $(n-1)$-tuple $R_{-i} \equiv (R_1, \dots, R_{i-1}, R_{i+1}, \dots, u_n) \in [\mathbb{D}^c]^{n-1}$.

An *admissible* domain $\mathbb{D}$ is a subset of $\mathbb{D}^c$. A *Social Choice Function (SCF)* is a map $F : [D]^n \to \Delta$. We will let $F_i(R_i, R_{-i})$ denote the allocation to agent i at the profile (R_i, R_{-i}) under the SCF F.

We now introduce some important but standard definitions.

Definition **1** *A SCF F is* **Manipulable** *by agent i at profile R via $R_i^{'} \in \mathbb{D}$ if $F(R_i^{'}, R_{-i}) P_i F(R)$. It is* **Strategy-Proof** *if it is not manipulable by any agent at any profile. Equivalently F is strategy-proof if $F_i(R)) R_i F_i(R_i^{'}, R_{-i}))$ for all $R_i, R_i^{'} \in \mathbb{D}$, for all $R_{-i} \in [\mathbb{D}]^{n-1}$ and for all $i \in I$.*

In the usual strategic voting model, an agent's preference ordering is private information and F represents the mechanism designer's objectives. If F is strategy-proof, all agent has dominant-strategy incentives to reveal their private information truthfully.

Definition **2** *An allocation $x \in \Delta$ is* **Pareto-Efficient** *at profile R if there does not exist another allocation $x' \in \Delta$ such that $x'_i R_i x_i$ for all $i \in I$ and $x'_j P_j x_j$ for some $j \in I$.*

Let $PE(R)$ denote the collection of Pareto-Efficient allocations at the profile R.

Definition **3** *A SCF F is* **Pareto-Efficient** *if $F(R) \in PE(R)$ for all $R \in [\mathbb{D}]^n$.*

[1] For a preference ordering R_i and a vector $x \in \Re^m_+$, the upper contour set of R_i at x is denoted by $UC(R_i, x)$ and is the set $\{z \in \Re^m_+ | z R_i x\}$. Similarly the lower contour set of R_i at x is denoted by $LC(R_i, x)$ and is the set $\{z \in \Re^m_+ | x R_i z\}$. A preference ordering R_i is continuous if $UC(R_i, x)$ and $LC(R_i, x)$ are both closed for all $x \in \Re^m_+$. A preference ordering R_i is strictly convex if $UC(R_i, x)$ is strictly convex for all $x \in \Re^m_{++}$. For $x, z \in \Re^m_+$ by $x > z$ we mean $x_k \geq z_k$ for all $k \in M$ and $x_k > z_k$ for some k. A preference ordering is strictly monotonic in $\Re^m_{++}$ if $x > z$ implies $x P_i z$.

Definition 4 *A SCF F is* **Non-Bossy** *if, for all $R_i, R_i' \in \mathbb{D}$, $R_{-i} \in [\mathbb{D}]^{n-1}$ and $i \in I$,*

$$[F_i(R_i, R_{-i}) = F_i(R_i^{'}, R_{-i})] \Rightarrow [F(R_i, R_{-i}) = F(R_i^{'}, R_{-i})]$$

The non-bossiness axiom was introduced by Satterthwaite and Sonnenschein (1981). In a non-bossy SCF, an agent who is unable to change her allocation by a unilateral deviation from a preference profile, is also unable to change the allocation of any other agent by the same deviation. The non-bossiness axiom is particularly useful in characterizing strategy-proof SCFs in environments where agent can be indifferent across allocations. It has been widely used in the literature [2].

An important and familiar SCF is *dictatorship*.

Definition 5 *A SCF F is* **Dictatorial** *if there exists an agent i such that for all $R \in [\mathbb{D}]^n$*

$$F_i(R) = \Omega$$

.

The dictatorial SCF gives all resources to the same agent at all preference profiles. It is of course, both strategy-proof and Pareto-efficient but ethically unsatisfactory. Serizawa and Weymark (2003) introduce a condition that ensures that all agents receive a minimal bundle of goods. In the definition below $||\cdot||$ denotes the Euclidean norm.

Definition 6 *A SCF F satisfies the* **Minimum Consumption Guarantee (MCG)** *axiom if there exists an $\epsilon > 0$ such that for all profiles $R \in [\mathbb{D}]^n$ and all $i \in I$,*

$$||F_i(R)|| \geq \epsilon$$

3 Quasi-linear Domains

Quasi-linear preferences are preference orderings R_i that can be represented by utility functions of the form $u_i(x) = v_i(x_{i1}, \ldots, x_{im-1}) + x_m$ [3]. The use of quasi-linear preferences is pervasive in economic theory. We note that quasi-linear preferences are classical.

In this paper, we restrict attention to a small sub-class of quasi-linear preferences. These preferences are represented by utility functions of the following form:

$$u_i(x_i, y_i; \theta_i) = \theta_i\{\sqrt{x_{i1}} + \ldots + \sqrt{x_{i,m-1}}\} + y_i. \tag{1}$$

[2]See for example S.Pápai (2000), Svensson (1999), Barberà and Jackson (1995). For a review see Barberà (2010)

[3]A preference ordering R_i is represented by the utility function $u_i : \Re^m_+ \rightarrow \Re$ if, for all $x, x' \in \Re^m_+$, $xR_ix' \Leftrightarrow u_i(x) \geq u_i(x')$

For notational convenience we denote the m^{th} good by y. We denote the set of preferences above by $\mathbb{D}^q$. Note that all preferences from $\mathbb{D}^q$ are represented by a parameter θ_i. Hence a preference profile in $[\mathbb{D}^q]^n$ can be represented by an n-tuple $\theta \equiv (\theta_1, \theta_2, \ldots, \theta_n)$. The definitions stated in the preceding section are applicable for $\mathbb{D}^q$ with the profiles being written as θ. An allocation is denoted by $\{(x_i, y_i)\}_{i \in I}$, where $x_i \equiv x_{i1}, \ldots, x_{im-1}$ for all $i \in I$. The set of Pareto-efficient allocations at profile θ will now be denoted by $PE(\theta)$.

All the results presented in this paper are first proved for the domain $\mathbb{D}^q$ and then extended to any domain that includes $\mathbb{D}^q$. In particular the results holds for $\mathbb{D}^c$.

In the next two subsections, we present some critical results relating to Pareto-efficiency and strategy-proofness of F over the domain $\mathbb{D}^q$.

3.1 Pareto-Efficiency in $\mathbb{D}^q$

We make an observation regarding Pareto-Efficient allocations in $\mathbb{D}^q$.

Proposition 1 If $(x^*(\theta), y^*(\theta)) \in PE(\theta)$ then it satisfies the following properties hold:

- If $x^*_{ij}(\theta) = 0$ for some $j \in \{1, \ldots, m-1\}$ then $x^*_{ij}(\theta) = 0$ for all $j \in \{1, \ldots, m-1\}$
- If $x^*_{ij}(\theta) > 0$ for some $j \in \{1, \ldots, m-1\}$ then $\frac{x^*_{ij}(\theta)}{x^*_{ij'}(\theta)} = \frac{\Omega_j}{\Omega_{j'}}$ for all $j' \in \{1, \ldots, m-1\}$.

The proof of the result is contained in the Appendix. According to it, every Pareto-efficient allocation has the feature that every agent i receives all goods from 1 through $m-1$ in fixed proportions independently of θ_i. This suggests a reduction of the problem from an m-good to a two-good model. The utility of agent i from a Pareto-efficient allocation $(x^*_1, \ldots, x^*_{m-1}, y)$ in the m-good model is

$$u_i((x^*_{i1}, \ldots, x^*_{im-1}, y^*_i; \theta_i)) = \theta_i \Big[1 + \sum_{j \in M \setminus \{1\}} \sqrt{\frac{\Omega_j}{\Omega_1}}\Big] \sqrt{x^*_{i1}} + y^*_i.$$

Now consider a two-good model with goods x_1 and y with endowments Ω_1 and Ω_m respectively. Since $\theta_i[1+\sum_{j \in M \setminus \{1\}} \sqrt{\frac{\Omega_j}{\Omega_1}}]$ is a positive real number, it follows that the allocation (x^*_1, y^*) is Pareto-efficient in the two-good economy in the domain $\mathbb{D}^q$ for the profile δ where $\delta_i = \theta_i[1 + \sum_{j \in M \setminus \{1\}} \sqrt{\frac{\Omega_j}{\Omega_1}}]$. Now consider a SCF F is strategy-proof and Pareto-efficient in the m-good economy. We can construct a two-good SCF $\bar{F}$ from F as follows: for every m-good profile θ,

$$[F(\theta) = (x_1, \ldots, x_{m-1}, y)] \Rightarrow [\bar{F}(\delta) = (x_1, y)]$$

where δ is defined as above. By our earlier arguments, $\bar{F}$ is Pareto-efficient. It is easily verified that $\bar{F}$ is strategy-proof. For every strategy-proof and Pareto-efficient SCF in the m-good model, there is an "equivalent" (in the sense above) strategy-proof and Pareto-efficient

SCF in the two-good model. Henceforth, we restrict attention to the two-good model and the results generalize in an obvious way to the m-good case.

In what follows, we consider two goods x and y and utility functions of the form $u_i(x_i, y_i; \theta_i) = \theta_i\sqrt{x_i} + y_i$. For each profile $\theta \in \Re^n_{++}$, $\{(x_i^*(\theta), y_i^*(\theta))\}_{i\in I} \in \Re^{2n}_+$ represents an allocation in $PE(\theta)$. Without loss of generality we set total endowments of both the goods at 1. We note that the domain we consider is "narrow" in a specific technical sense. In particular, it is a single-crossing domain and therefore it does not admit concavification. We do not employ concavification arguments used extensively in this literature (Serizawa and Weymark (2003), Serizawa (2006), Zhou (1991), Hashimoto (2008), Barberà and Jackson (1995)). Thus when θ_i is changed indifference curves can only intersect. The following proposition provides necessary conditions for allocations to be Pareto-efficient in the two-good model described above.

Proposition 2 *If $(x^*(\theta), y^*(\theta)) \in PE(\theta)$ then it satisfies the following conditions.*

(P1) If $x_i^*(\theta) < \frac{\theta_i^2}{\sum\limits_{k\in I}\theta_k^2}$ for some $i \in I$ then $y_i^*(\theta) = 0$.

(P2) If $x_i^*(\theta) > \frac{\theta_i^2}{\theta_i^2 + \min_{k\neq i}\theta_k^2}$ for some $i \in I$ then $y_i^*(\theta) = 1$.

The proof of the Proposition is contained in the Appendix. It is well-known that in quasi-linear domains that if x^* solves $\max_{x_1,\ldots,x_n}\sum_{i\in I}\theta_i\sqrt{x_i}$ subject to the resource constraint on x, then any allocation of good y together with x^*, is a Pareto-efficient allocation. For instance, $\left(\frac{\theta_i^2}{\sum_{k\in I}\theta_k^2}, \ldots, \frac{\theta_n^2}{\sum_{k\in I}\theta_k^2}\right)$ solves $\max_{x_1,\ldots,x_n}\sum_{i\in I}\theta_i\sqrt{x_i}$ subject to $\sum_{i\in} x_i = 1$. We say that agent i is *constrained* at θ if $x_i(\theta) < \frac{\theta_i^2}{\sum_{k\in I}\theta_k^2}$. According to condition $P1$, a constrained agent must not get a positive amount of good y. According to $P2$, any agent i whose x_i exceeds a certain bound, must get the entire amount of good y.

3.2 Strategy-proofness in $\mathbb{D}^q$

In this subsection, we prove some preliminary results regarding strategy-proof and Pareto-efficient SCFs over the domain $\mathbb{D}^q$. The first two results characterize strategy-proofness in this domain and are counterparts of the results in Myerson (1981) in the context of auction design.

Lemma 1 *Consider a strategy-proof SCF, $F : [\mathbb{D}^q]^n \to \Delta$. Then, for each i and for all θ_i, θ_i' with $\theta_i' > \theta_i$ and for all θ_{-i},*

$$x_i(\theta_i', \theta_{-i}) \geq x_i(\theta_i, \theta_{-i})$$

where $F(\theta) = (x(\theta), y(\theta))$ and $F((\theta_i', \theta_{-i})) = (x(\theta_i', \theta_{-i}), y(\theta_i', \theta_{-i}))$.

Proof: By the strategy-proofness of F, we have the following: for all, θ_i, θ_i' and θ_{-i},

$$\theta_i x_i(\theta_i, \theta_{-i})^{\frac{1}{2}} + y_i(\theta_i, \theta_{-i}) \geq \theta_i x_i(\theta_i', \theta_{-i})^{\frac{1}{2}} + y(\theta_i', \theta_{-i}) \text{ and}$$

$$\theta_i' x_i(\theta_i', \theta_{-i})^{\frac{1}{2}} + y_i(\theta_i', \theta_{-i}) \geq \theta_i' x_i(\theta_i, \theta_{-i})^{\frac{1}{2}} + y(\theta_i, \theta_{-i}).$$

Adding the two inequalities above, we obtain

$$\left[x_i(\theta_i', \theta_{-i})^{\frac{1}{2}} - x_i(\theta_i, \theta_{-i})^{\frac{1}{2}}\right][\theta_i' - \theta_i] \geq 0.$$

Therefore, $\theta_i' > \theta_i$ implies $x_i(\theta_i', \theta_{-i})^{\frac{1}{2}} \geq x_i(\theta_i, \theta_{-i})^{\frac{1}{2}}$, for all θ_{-i} from which the Lemma follows. ■

LEMMA **2** *Consider a strategy-proof SCF,* $F : [\mathbb{D}^q]^n \rightarrow \Delta$. *Then, for each* i *and for all* $\theta_i \in [a_i,\ b_i] \subset \Re_{++}$ *and* θ_{-i},

$$u_i(F_i(\theta_i, \theta_{-i}); \theta_i) = u_i(F_i(a_i, \theta_{-i}); a_i) + \int_{a_i}^{\theta_i} x_i(t_i, \theta_{-i})^{1/2} dt_i.$$

Proof: Using strategy-proofness of F it follows that for all $i \in I$ and for all $\theta_i \in \Re_{++}$,

$$\begin{aligned} u_i(F_i(\theta_i, \theta_{-i}); \theta_i) &= \max_{z \in \Re_{++}} \left\{\theta_i x_i(z, \theta_{-i})^{1/2} + y_i(z, \theta_{-i})\right\} \\ &= \max_{z \in [a_i,\ b_i]} \left\{\theta_i x_i(z, \theta_{-i})^{1/2} + y_i(z, \theta_{-i})\right\}. \end{aligned}$$

Since $u_i(F_i(\theta_i, \theta_{-i}); \theta_i)$ is a maximum of a family of affine functions $u_i(F_i(\theta_i, \theta_{-i}); \theta_i)$ is a convex function in θ_i for all θ_{-i}. Therefore $u_i(F_i(\theta_i, \theta_{-i}); \theta_i)$ is an absolutely continuous function in $[a_i,\ b_i]$. Therefore, $\frac{d}{d\theta_i} u_i(F(\theta_i, \theta_{-i}); \theta_i) = x_i(\theta_i, \theta_{-i})^{1/2}$ almost everywhere. The result follows by applying the Fundamental Theorem of Calculus. ■

In the next proposition, we show that if F is strategy-proof, then there cannot exist a $S \subset I$, $|S| \geq 2$ and a neighborhood of profiles where F picks allocations such that no agent from S is constrained and agents not in S are allocated zero of both the goods in that neighborhood. We let $N_\epsilon(\theta)$ denote a open neighborhood of θ with radius ϵ.

PROPOSITION **3** *Consider a strategy-proof SCF* $F : [\mathbb{D}^q]^n \rightarrow \Delta$. *Then there does not exist* $S \subset I$, $|S| \geq 2$, *and a neighborhood* $N_\epsilon(\theta')$ *such that,* $F(\theta) = (x(\theta), y(\theta))$ *satisfies the following properties: for all* $\theta \in N_\epsilon(\theta)$,

$$
\begin{aligned}
x_i(\theta) &= \frac{\theta_i^2}{\sum_{k \in S} \theta_k^2} \;\forall i \in S \\
\sum_{i \in S} y_i(\theta) &= 1
\end{aligned}
$$

Proof: Suppose that the lemma is false, i.e. there exists $S \subset I$, $|S| \geq 2$ and a neighborhood $N_\epsilon(\theta')$ such that for all $\theta \in N_\epsilon(\theta)$, $x_i(\theta) = \frac{\theta_i^2}{\sum_{k \in S} \theta_k^2}$ for all $i \in S$ and $\sum_{i \in S} y_i(\theta) = 1$.

Applying Lemma 2, it follows that for each $\theta \in N_\epsilon(\theta')$ and each $i \in S$,

$$
u_i(F_i(\theta_i, \theta_{-i}); \theta_i) = u_i(F_i(a_i, \theta_{-i}); a_i) + \int_{a_i}^{\theta_i} \left[\frac{t_i^2}{\sum\limits_{k \in S \setminus \{i\}} \theta_k^2 + t_i^2} \right]^{\frac{1}{2}} dt_i \tag{2}
$$

Substituting $u_i(F_i(\theta_i, \theta_{-i}); \theta_i)$ with $\theta_i \left[\frac{\theta_i^2}{\sum\limits_{k \in S} \theta_k^2} \right]^{\frac{1}{2}} + y_i(\theta_i, \theta_{-i})$ on the LHS and letting $h_i(\theta_{-i}) \equiv a_i \left[\frac{a_i^2}{a_i^2 + \sum\limits_{k \in S \setminus i} \theta_k^2} \right]^{\frac{1}{2}} + y_i(a_i, \theta_{-i})$ on the RHS of Equation 2, we obtain,

$$
\theta_i \left[\frac{\theta_i^2}{\sum\limits_{k \in S} \theta_k^2} \right]^{\frac{1}{2}} + y_i(\theta_i, \theta_{-i}) = h_i(\theta_{-i}) + \int_{a_i}^{\theta_i} \left[\frac{t_i^2}{\sum\limits_{k \in S \setminus \{i\}} \theta_k^2 + t_i^2} \right]^{\frac{1}{2}} dt_i \tag{3}
$$

Now summing Equation 3 across i and noting that $\sum\limits_{i \in S} y_i(\theta) = 1$ we obtain,

$$
\left[\sum_{i \in S} \theta_i^2 \right]^{\frac{1}{2}} + 1 - \sum_{i \in S} \int_{a_i}^{\theta_i} \frac{t_i}{\left[\sum\limits_{k \in S \setminus \{i\}} \theta_k^2 + t_i^2 \right]^{\frac{1}{2}}} dt_i = \sum_{i \in S} h_i(\theta_{-i}). \tag{4}
$$

Solving for the integrals in the LHS of Equation 4 and simplifying further, we get

$$
(1-n) \left[\sum_{i \in S} \theta_i^2 \right]^{\frac{1}{2}} + 1 + \sum_{i \in S} \left[\sum_{k \in S \setminus \{i\}} \theta_k^2 + a_i^2 \right]^{\frac{1}{2}} = \sum_{i \in S} h_i(\theta_{-i}). \tag{5}
$$

The LHS of Equation 5 is an infinitely differentiable function in $\Re_{++}^{|S|}$. Notice that it's $|S|^{\text{th}}$ order cross-partial derivative is $c(|S|)\,(-1)^{(|S|)} \left(\prod\limits_{i \in S} \theta_i \right) \left(\sum\limits_{i \in S} \theta_i^2 \right)^{\frac{-(2|S|-1)}{2}}$ where $c(|S|)$ is a constant not equal to zero for any value of $|S|$. However, the $|S|^{\text{th}}$ order cross-partial derivative of the right hand side of vanishes at all θ. We have a contradiction. ■

Proposition 3 rules out the existence of strategy-proof and Pareto-efficient SCFs and $S \subset I$ and a neighborhood that give all agents strictly positive amounts of both goods to all agents is S and zero of both the goods to all agents outside S. We formalize this below.

Definition 7 *The SCF $F : \mathbb{D}^n \to \Delta$ satisfies* **S-interiority** *for $S \subset I$, $|S| \geq 2$, if there exists a neighborhood of profiles $N_\epsilon(\theta')$ such that for all $\theta \in N_\epsilon(\theta')$, we have $x_i(\theta), y_i(\theta) > 0$ for all $i \in S$ and $(x_i(\theta), y_i(\theta)) = (0, 0)$ for all $i \notin S$, where $F(\theta) = (x(\theta), y(\theta))$.*

Proposition 4 *Let $F : [\mathbb{D}^q]^n \to \Delta$ be a strategy-proof and Pareto-efficient SCF. Then F does not satisfy S-interiority for any S.*

Proof: Let F be strategy-proof, Pareto-efficient. Suppose F satisfies S-interiority, i.e. there exists a neighborhood of profiles $N_\epsilon(\theta')$ such that for all $\theta \in N_\epsilon(\theta')$, we have $x_i(\theta), y_i(\theta) > 0$ for all $i \in S$ where $F(\theta) = (x(\theta), y(\theta))$. According to $P1$ in Proposition 2, we must have $x_i(\theta) = \frac{\theta_i^2}{\sum\limits_{k \in S} \theta_k^2}$ for all $i \in S$ and all $\theta \in N_\epsilon(\theta')$. But now we have a contradiction to Proposition 3. ■

REMARK 1: Hurwicz and Walker (1990) prove a result (their Theorem 3) to our Proposition 4. They consider a more general class of quasi-linear utility functions.

4 Minimum Consumption Guarantees

In this section we provide a simple proof of a logically independent variant of the main result of Serizawa and Weymark (2003). In particular we show that any strategy-proof and Pareto-efficient SCF defined on a domain that is a superset of $\mathbb{D}^q$ violates the MCG axiom. Thus a strategy-proof and Pareto-efficient SCF defined on the domain of classical preferences violates the MCG axiom.

Theorem 1 *Let $\mathbb{D}$ be an arbitrary domain such that $\mathbb{D}^q \subset \mathbb{D}$. Let $F : \mathbb{D}^n \to \Delta$ be a strategy proof and Pareto-efficient SCF. Then F does not satisfy MCG.*

Proof: It suffices to prove the result for a strategy-proof and Pareto-efficient SCF $F : [\mathbb{D}^q]^n \to \Delta$. Let F be such a SCF. We first establish the following result.

Lemma 3 *Let θ be a profile such that $x_i(\theta) < \frac{\theta_i^2}{\sum\limits_{k \in I} \theta_k^2}$, i.e agent i is constrained. Then $y_i(\theta_i', \theta_{-i}) < \theta_i$ whenever $\theta_i' < \theta_i$.*

Proof: Suppose not, i.e. let for some $\theta_i^{'} < \theta_i$, $y_i(\theta_i^{'}, \theta_{-i}) \geq \theta_i$. Now, by strategy-proofness and the fact that $y_i(\theta_i, \theta_{-i}) = 0$, we have

$$\theta_i x_i(\theta_i, \theta_{-i})^{\frac{1}{2}} \geq \theta_i x_i(\theta_i^{'}, \theta_{-i})^{\frac{1}{2}} + y_i(\theta_i^{'}, \theta_{-i}),$$

Hence

$$\theta_i[x_i(\theta_i, \theta_{-i})^{\frac{1}{2}} - x_i(\theta_i^{'}, \theta_{-i})^{\frac{1}{2}}] \geq y_i(\theta_i^{'}, \theta_{-i})$$

Since $\theta_i > 0$,

$$[x_i(\theta_i, \theta_{-i})^{\frac{1}{2}} - x_i(\theta_i^{'}, \theta_{-i})^{\frac{1}{2}}] \geq \frac{y_i(\theta_i^{'}, \theta_{-i})}{\theta_i} \geq 1 \tag{6}$$

Since $x_i(\theta_i, \theta_{-i}), x_i(\theta_i^{'}, \theta_{-i}) \leq 1$, Inequality 6 can be satisfied only if $x_i(\theta_i, \theta_{-i}) = 1$. However $x_i(\theta) < \frac{\theta_i^2}{\sum\limits_{k \in I} \theta_k^2} < 1$ leading to a contradiction. ■

Returning to the proof of the Theorem, pick $0 < \epsilon < \sqrt{2}$. We show existence of an an agent i and a profile $(\theta_i^{'}, \theta_{-i})$ such that

$$||(x_i(\theta_i^{'}, \theta_{-i}), y_i(\theta_i^{'}, \theta_{-i}))|| < \epsilon.$$

Consider the open set $O = \prod_{j=1}^{N}(0, \ \frac{\epsilon}{\sqrt{2}})$. By Proposition 3 we know that there is a profile $(\theta_i, \theta_{-i}) \in O$ and an agent i such that $y_i(\theta_i, \theta_{-i}) = 0$ and $x_i(\theta_i, \theta_{-i}) < \frac{\theta_i^2}{\sum_{k \in I} \theta_k^2}$. By Lemma 3 and the choice of ϵ, we have the following: for any $\theta_i^{'} < \theta_i$, $y_i(\theta_i^{'}) < \theta_i < \frac{\epsilon}{\sqrt{2}} < 1$. Applying $P2$ in Proposition 2, we infer that $x_i(\theta_i^{'}, \theta_{-i}) \leq \frac{\theta_i^{'2}}{\theta_i^{'2} + \min_{j \neq i} \theta_j^2}$ for all $\theta_i^{'} < \theta_i$. Observe that the RHS of the inequality above converges to zero and $\theta_i^{'}$ converges to zero. Hence, $\lim_{\theta_i \to 0} \ x_i(\theta_i, \theta_{-i}) = 0$. Therefore, there exists $\theta_i^{'} < \theta_i$ such that $x_i(\theta_i^{'}, \theta_{-i}) < \frac{\epsilon}{\sqrt{2}}$ and $y_i(\theta_i^{'}, \theta_{-i}) < \frac{\epsilon}{\sqrt{2}}$.

Hence,

$$\sqrt{(x_i(\theta_i^{'}, \theta_{-i}))^2 + (y_i(\theta_i^{'}, \theta_{-i}))^2} < \sqrt{\left(\frac{\epsilon}{\sqrt{2}}\right)^2 + \left(\frac{\epsilon}{\sqrt{2}}\right)^2} = \epsilon$$

i.e.

$$||(x_i(\theta_i^{'}, \theta_{-i}), y_i(\theta_i^{'}, \theta_{-i}))|| < \epsilon.$$

■

REMARK 2: Our result is different from its counterpart in Serizawa and Weymark (2003) because of the differences in the domains considered. They use the homothetic preference domain while we use a sub-domain of quasi-linear preferences. The arguments in

Serizawa and Weymark (2003) are geometric while ours are analytical. Moreover, we are able to explicitly construct a set of profiles where minimum consumption guarantee fails.

5 Dictatorship in Classical Exchange Economies

In this section, we prove dictatorship results. In Section 5.1, we show dictatorship in the case of two agents. In Section 5.2 we extend the result to the case of more than two agents with additional hypotheses on the social choice function.

5.1 The $n = 2$ case

Our main result in this subsection is the following.

Theorem 2 *Let $\mathbb{D}$ be an arbitrary domain with $\mathbb{D}^q \subset \mathbb{D}$. Let $F : \mathbb{D}^2 \to \Delta$ be a strategy-proof and Pareto-efficient SCF. Then F is dictatorial.*

Proof: We first show that any strategy-proof and Pareto-efficient SCF $F : [\mathbb{D}^q]^2 \to \Delta$ is dictatorial.

Let $I = \{i, j\}$ and let $F : [\mathbb{D}^q]^2 \to \Delta$ be a strategy-proof and Pareto-efficient SCF defined over this society. We will show that either agent i or agent j is a dictator.

It follows from Proposition 3 that there exists a profile, say $\theta^* \in [\mathbb{D}^q]^2$ and an agent, say j such that j is constrained at θ^*, i.e. $x_j(\theta^*) < \frac{{\theta_j^*}^2}{{\theta_i^*}^2+{\theta_j^*}^2}$ and $y_j(\theta^*) = 0$. Suppose $x_j(\theta^*) = 0$. We claim that in this case i is a dictator. To see this consider an arbitrary profile θ. By strategy-proofness of F, $F_j(\theta_i^*, \theta_j) = (0, 0)$; otherwise j manipulates at θ^* via θ_j. Again by strategy-proofness of F, $F_j(\theta_i, \theta_j) = (0, 0)$; otherwise i manipulates at (θ_i, θ_j) via θ_i^*. Hence i is a dictator.

Assume therefore that $x_j(\theta^*) > 0$. We know that $x_i(\theta^*) > \frac{{\theta_i^*}^2}{{\theta_i^*}^2+{\theta_j^*}^2}$ and $y_i(\theta^*) = 1$. Now consider θ_i such that $x_i(\theta^*) = \frac{{\theta_i}^2}{{\theta_i}^2+{\theta_j^*}^2}$. Note that $\theta_i > \theta_i^*$. It follows from Lemma 1 that $x_i(\theta_i, \theta_j^*) \geq x_i(\theta^*)$. Suppose $x_i(\theta_i, \theta_j^*) > x_i(\theta^*)$. In order to prevent i from manipulating F at θ^* via θ_i, we must have $y_i(\theta_i, \theta_j^*) < 1$. Therefore $x_i(\theta_i, \theta_j^*) > \frac{{\theta_i}^2}{{\theta_i}^2+{\theta_j^*}^2}$ and $y_i(\theta_i, \theta_j^*) < 1$. Since $F(\theta_i, \theta_j^*)$ is Pareto-efficient at (θ_i, θ_j^*), we have a contradiction to $P2$ in Proposition 2. Therefore $x_i(\theta_i, \theta_j^*) = \frac{{\theta_i}^2}{{\theta_i}^2+{\theta_j^*}^2}$ and $y_i(\theta_i, \theta_j^*) = 1$. In fact, we will assume without loss of generality that $x_i(\theta^*) = \frac{{\theta_i^*}^2}{{\theta_i^*}^2+{\theta_j^*}^2}$ and $y_i(\theta^*) = 1$.

Since $x_j(\theta^*) > 0$, we can pick $\bar{x}_j$ such that $0 < \bar{x}_j < x_j(\theta^*)$. Moreover, by the Intermediate Value Theorem, we can find θ'_j such that $\theta'_j x_j(\theta^*)^{\frac{1}{2}} = \theta'_j \bar{x}_j^{\frac{1}{2}} + 1$. Clearly $\theta'_j > \theta_j^*$. Let θ'_i be such that $\bar{x}_i = \frac{\theta_i'^2}{\theta_i'^2+\theta_j'^2}$. (Note that $\bar{x}_i + \bar{x}_j = 1$).

Let $F(\theta') = z$. We shall argue all choices for z lead to contradiction.

Case 1: Suppose $z = ((x_i, y_i), (x_j, y_j))$ such that $x_j \leq \frac{\theta_j'^2}{\theta_i'^2 + \theta_j'^2} = \bar{x}_j$ and $y_j < 1$. By our choice of θ_j', we have $\theta_j' x_j(\theta^*)^{\frac{1}{2}} = \theta_j' \bar{x}_j^{\frac{1}{2}} + 1 > \theta_j' x_j^{\frac{1}{2}} + y_j$. Hence $u_j(F_j(\theta^*); \theta_j') > u_j(F_j(\theta'); \theta_j')$. Now choose $\theta_j'' > \theta_j'$ such that $\frac{\theta_j''^2}{\theta_i'^2 + \theta_j''^2} > x_j(\theta^*) = \frac{\theta_j^{*2}}{\theta_i^{*2} + \theta_j^{*2}}$. Note that $\theta_j'' > \theta_j' > \theta_j^*$. Now Lemma 1 implies that $x_j(\theta_i^*, \theta_j'') \geq x_j(\theta^*)$. If this inequality is strict, then j will manipulate at θ^* via θ_j'' since $y_j(\theta^*) = 0$ by assumption. Hence $x_j(\theta_i^*, \theta_j'') = x_j(\theta^*)$. Now $\theta_i' > \theta_i^*$, so that $x_i(\theta_i', \theta_j'') \geq x_i(\theta_i^*, \theta_j'')$ (Lemma 1). But $x_i(\theta_i^*, \theta_j'') = x_i(\theta^*) = \frac{\theta_i^{*2}}{\theta_i^{*2} + \theta_j^{*2}} > \frac{\theta_i'^2}{\theta_i'^2 + \theta_j''^2}$. Now suppose $x_i(\theta_i', \theta_j'') > x_i(\theta_i^*, \theta_j'')$. By $P2$ in Proposition 2, $y_i(\theta_i', \theta_j'') = 1$. But then agent i manipulates F at (θ_i^*, θ_j'') via θ_i'. Hence $x_i(\theta_i', \theta_j'') = x_i(\theta_i^*, \theta_j'') = x_i(\theta^*)$ and $x_j(\theta_i', \theta_j'') = x_j(\theta_i^*, \theta_j'') = x_j(\theta^*)$. Moreover strategy-proofness also implies $y_j(\theta_i', \theta_j'') = y_j(\theta_i^*, \theta_j'') = y_j(\theta^*) = 0$.

Truth-telling at θ' gives player j (x_j, y_j). Lying gives her $F_j(\theta^*)$. Since $u_j(F_j(\theta^*); \theta_j') > u_j(F_j(\theta'); \theta_j')$ by construction, j will manipulate.

Case 2: $z = ((x_i, y_i), (x_j, y_j))$ such that $x_j(\theta^*) > x_j \geq \bar{x}_j$ and $y_j = 1$.

Pick $\tilde{x}_j$ such that $x_j(\theta^*) > \tilde{x}_j > x_j$. Choose $\theta_j'' > \theta_j'$ such that $\theta_j'' x_j(\theta^*)^{\frac{1}{2}} = \theta'' \tilde{x}_j^{\frac{1}{2}} + 1$. Once again, the existence of θ_j'' follows from the Intermediate Value Theorem. Now pick θ_i'' such that $x_i < \frac{\theta_i''^2}{\theta_i''^2 + \theta_j''^2}$. It follows from earlier arguments involving Lemma 1 and Proposition 2 that $F(\theta'') = z$. Now observe that by considering θ^* as before, replacing θ' by θ'' and $\bar{x}_j$ by $\tilde{x}_j$, we can apply the arguments of Case 1 to conclude that $F(\theta'') \neq z$. But this contradicts our assumption that $F(\theta') = z$.

Case 3: $z = ((x_i, y_i), (x_j, y_j))$ such that $x_j \geq x_j(\theta^*)$ and $y_j = 1$.

We claim that $F(\theta_i', \theta_j^*) = z$. Since $\theta_j^* < \theta_j'$, Lemma 1 implies that $x_j(\theta_i', \theta_j^*) \leq x_j(\theta')$. Suppose the inequality is strict. Since $y_j(\theta') = 1$, agent will manipulate at (θ_i', θ_j^*) via θ_j'. Hence $F(\theta_i', \theta_j^*) = z$. Now observe that $x_i(\theta_i', \theta_j^*) \leq x_i(\theta^*)$ while $0 = y_i(\theta_i', \theta_j^*) < y_i(\theta^*) = 1$. Hence agent i manipulates at (θ_i', θ_j^*) via θ_i^*.

Cases 1-3 exhaust all possibilities. Therefore F is dictatorial.

Now let $F : \mathbb{D}^2 \to \Delta$ be strategy-proof and Pareto-efficient SCF where $\mathbb{D}^q \subset \mathbb{D}$. We know from our earlier arguments that F restricted to the domain $\mathbb{D}^q$ is dictatorial. Let i be the dictator, i.e. for all $\theta \in [\mathbb{D}^q]^2$, we have $F_i(\theta) = (1, 1)$. Pick an arbitrary profile $R \in \mathbb{D}^2$. If $F_i(R_i, \theta_j) \neq (1, 1)$, i will manipulate F at (R_i, θ_j) via θ_i. If $F_j(R) \neq (0, 0)$, agent j will manipulate F at (R_i, θ_j) via R_j. Therefore i is a dictator in F. ■

5.2 The $n \geq 3$ Case

In this section we consider the case of more than two agents. This case is different from the two agent case because strategy-proof and Pareto-efficient SCFs need not be dictatorial as shown in Kato and Ohseto (2002). We have shown that every strategy-proof and Pareto-efficient SCF for an arbitrary number of agents defined over a quasi-linear domain, must satisfy a highly restrictive property: in every neighborhood of preference profiles, at least one agent must receive a zero amount of good y. However when there are at least three, agents the identity of the agent who does not receive good y, may depend on the announcements of the other agents. This increases the possible complexity in the behavior of a strategy-proof SCF very dramatically. However, by imposing certain familiar regularity assumptions on SCFs, are able to recover the dictatorship result.

Observe that a SCF $F : [\mathbb{D}^q]^n \to \Delta$ is a map $F : \Re^n_{++} \to \Re^{2n}_{+}$. Therefore **continuity** of F can be defined in a standard way.

Definition 8 *Let $\mathbb{D}$ be an arbitrary domain such that $\mathbb{D}^q \subset \mathbb{D}$. We say a SCF $F : \mathbb{D}^n \to \Delta$ is* **q-Continuous** *if the restriction of F to $[\mathbb{D}^q]^n$ is continuous.*

We have already introduced the non-bossiness axiom earlier. Our main result is the following:

Theorem 3 *Let $\mathbb{D}$ be an arbitrary domain with $\mathbb{D}^q \subset \mathbb{D}$. Let $F : \mathbb{D}^n \to \Delta$ be a strategy-proof, Pareto-efficient, non-bossy and q-continuous SCF. Then F is dictatorial.*

Proof: Let $F : [\mathbb{D}^q]^n \to \Delta$ be a strategy-proof, Pareto-efficient, non-bossy and continuous SCF. We will show that F is dictatorial. We will first establish two lemmas.

Lemma 4 *Let θ be an arbitrary profile and let $S = \{j \in I | y_j(\theta) > 0\}$. Let $i \notin S$ be such that $x_i(\theta) > 0$. Then there exists θ_i^* and a neighborhood $N_\epsilon(\theta_i^*, \theta_{-i})$ such that, for all $\theta' \in N_\epsilon(\theta_i^*, \theta_{-i})$, we have $y_k(\theta') > 0$ for all $k \in S \cup \{i\}$.*

Proof: Let θ, i and S be as specified in the statement of the Lemma. By Proposition 2 we know that $x_i(\theta) \leq \frac{\theta_i^2}{\theta_i^2 + \min_{k \neq i} \theta_k^2}$. Consider a decreasing sequence $\theta_i^r \to 0$ as $r \to \infty$. By Lemma 1, $x_i(\theta_i^r, \theta_{-i})) \leq x_i(\theta_i, \theta_{-i})$. Suppose $x_i(\theta_i^r, \theta_{-i}) = x_i(\theta_i, \theta_{-i})$ for all r. Clearly $y_i(\theta_i^r, \theta_{-i}) = y_i(\theta_i, \theta_{-i}) = 0$, otherwise i will manipulate. Observe that $\frac{(\theta_i^r)^2}{(\theta_i^r)^2 + \min_{k \neq i} \theta_k^2} \to 0$ as $r \to \infty$. Therefore, $x_i(\theta_i^r, \theta_{-i}) > \frac{(\theta_i^r)^2}{(\theta_i^r)^2 + \min_{k \neq i} \theta_k^2}$ while $y_i(\theta_i^r, \theta_{-i}) = 0$ for r large enough. This contradicts $P2$ in Proposition 2. Hence $x_i(\theta_i^r, \theta_{-i}) < x_i(\theta_i, \theta_{-i})$ for r large enough which also implies $y_i(\theta_i^r, \theta_{-i}) > 0$ for r large enough. Let $\bar{\theta}_i = \inf_r \{\theta_i^r : y_i(\theta_i^r, \theta_{-i}) = 0\}$. Since $F_i(\bar{\theta}_i, \theta_{-i}) = F_i(\theta_i, \theta_{-i})$, the non-bossiness of F implies that $F(\bar{\theta}_i, \theta_{-i}) = F(\theta_i, \theta_{-i})$. By the continuity of F, there exists $\theta_i^* < \bar{\theta}_i$ and a neighborhood $N_\epsilon(\theta_i^*, \theta_{-i})$ such that for all θ' in the neighborhood, $y_k(\theta') > 0$ for all $k \in S \cup \{i\}$. ■

Lemma 5 *Let θ be an arbitrary profile and let $S = \{j \in I | y_j(\theta) > 0\}$. Then there exists a neighborhood $N_\epsilon(\theta')$ and $S' \subset I$ with $S \subset S'$ such that for all $\tilde{\theta}$ in the neighborhood, we have $x_i(\tilde{\theta}), y_i(\tilde{\theta}) > 0$ for all $i \in S'$ and $\sum_{i \in S'} x_i(\tilde{\theta}) = \sum_{i \in S'} y_i(\tilde{\theta}) = 1$.*

Proof: Let θ be an arbitrary profile and let $S = \{j \in I | y_j(\theta) > 0\}$. Suppose $\sum_{i \in S} x_i(\theta) = 1$, then the Lemma follows by the continuity of F. Suppose there exists an $i \notin S$ and $x_i(\theta) > 0$ but $y_i(\theta) = 0$. Applying Lemma 4, it follows that there exists a profile θ' such that for all θ'' in this neighborhood $y_k(\theta'') > 0$ for all $k \in S \cup \{i\}$. Now suppose there exists an agent i' with $i' \notin S \cup \{i\}$ such that $x_{i'}(\theta'') > 0$ and $y_{i'}(\theta'') = 0$ for some θ'' in this neighborhood. Now applying Lemma 4 again, we can find another neighborhood such that for all profiles θ in this neighborhood $x_k(\theta), y_k(\theta) > 0$ for all $k \in S \cup \{i, i'\}$. Proceeding in this way and noting that the number of agents is finite the desired conclusion follows. ■

We now show that $F : [\mathbb{D}^q]^n \to \Delta$ is dictatorial. In order to see this, suppose that there exists a profile θ and a set S with $|S| \geq 2$ such that $y_i(\theta) > 0$ for all $i \in S$. Then by Lemma 5, there exists a neighborhood and a set of agents S' with $S \subset S'$ with $x_i(\theta), y_i(\theta) > 0$ for all $i \in S'$ and $\sum_{i \in S'} x_i(\theta) = \sum_{i \in S'} y_i(\theta) = 1$. However, this implies that F satisfies S'-interiority contradicting Proposition 4. Therefore $|S| = 1$ for all profiles. By Pareto-efficiency this implies that for all profiles θ there exists an agent i such that $F_i(\theta) = (1, 1)$. A simple argument using non-bossiness establishes that F is dictatorial.

Now let $F : \mathbb{D}^n \to \Delta$ be strategy-proof and Pareto-efficient SCF where $\mathbb{D}^q \subset \mathbb{D}$. We know from our earlier arguments that F restricted to the domain $\mathbb{D}^q$ is dictatorial. Let i be the dictator, i.e. for all $\theta \in [\mathbb{D}^q]^n$, we have $F_i(\theta) = (1, 1)$. Pick an arbitrary profile $R \in \mathbb{D}^n$. If $F_i(R_i, \theta_{-i}) \neq (1, 1)$, i will manipulate F at (R_i, θ_{-i}) via θ_i. Note also that for all $j \neq i$ strategy-proofness implies $Fj(R_i, R_j, \theta_{i,j}) = (0, 0)$. By non-bossiness $F_i(R_i, R_j, \theta_{i,j}) = (1, 1)$. By repeating this argument it follows that $F_i(R) = (1, 1)$ so that i is the dictator in F. ■

REMARK 3: Observe that q-continuity is a relatively mild condition because it imposes continuity only on the quasi-linear sub-domain $\mathbb{D}^q$.

REMARK 4: An open question relates to the role that non-bossiness and q-continuity assumptions play in our result. A reasonable conjecture is that strategy-proofness and Pareto-efficiency imply the *extreme-valuedness* of F, i.e. at all profiles, there exists an agent who receives the entire allocation of all goods.

6 Conclusion

In this paper, we have analyzed the structure of strategy-proof and Pareto-efficient social choice functions in classical exchange economies. Our methodological contribution is to focus on a small class of quasi-linear domains and use techniques developed in the context of auction design. This approach yields sharper results on minimum consumption guarantees. It also allows for a dictatorship characterization for arbitrary numbers of agents provided that social choice functions satisfy mild regularity assumptions. These results immediately extend to supersets of quasi-linear domains and therefore apply to the domain of all classical preferences. There are no existing results for strategy-proof and Pareto-efficient social choice functions in the case of more than two agents and our results are therefore the first of their kind.

7 APPENDIX

We provide a proof of Proposition 1 below. *Proof*: Let $\{x_i^*(\theta), y_i^*(\theta)\}_{i=1}^N$ be a Pareto-efficient allocation. Fix an agent i. We first show that if $x_{ij}^*(\theta) = 0$ for some $j \in \{1, \ldots, m-1\}$, then $x_{ij'}^*(\theta) = 0$ for all $j' \in \{1, \ldots, m-1\}$. Suppose this false, i.e. $x_{ij}^*(\theta) = 0$ but $x_{ij'}^*(\theta) > 0$ for some $j' \in \{1, \ldots, m-1\}$. We argue that this allocation is not Pareto-efficient. There must exist an agent i' with an allocation $(x_{i'}^*(\theta), y_{i'}^*(\theta))$ and $x_{i'j}^*(\theta) > 0$. For agents i and i', define $\Omega_j^{(i,i')} \equiv x_{ij}^*(\theta) + x_{i'j}^*(\theta) > 0$ and $\Omega_{j'}^{(i,i')} \equiv x_{ij'}^*(\theta) + x_{i'j'}^*(\theta) > 0$. Fix the the allocation of other agents and other goods and consider the set of Pareto-efficient allocations in the Edgeworth box of agents i and i' with total endowments of j and j' being $\Omega_j^{(i,i')}$ and $\Omega_{j'}^{(i,i')}$ respectively. In this box Pareto-efficient points lie on the diagonal, i.e. by fixing agent i''s *utility level* at $\theta_{i'}[x_{i'j}^{*1/2}(\theta) + x_{i'j'}^{*1/2}(\theta)]$ agent i can be made better off than at $x_{ij}^*(\theta) = 0, x_{ij'}^*(\theta) > 0$. Hence the initial allocation cannot be Pareto-efficient.

To prove the Proposition consider the following optimization problem for agent i

$$Max_{\{x_i, y_i\}_{i=1}^N} \left[\theta_i \sum_{j=1}^{m-1} x_{ij}^{1/2} + y_i\right]$$

$$subject\ \ to\ \ \left[\theta_k \sum_{j=1}^{m-1} x_{jk}^{1/2} + y_k\right] \geq \bar{u}_k, \forall\ \ k \in N \setminus \{i\}, \qquad \textbf{(P)}$$

$$\sum_{i\in N} x_{ij} = \Omega_j \ \ \forall j \in \{1, \ldots, m-1\}, \sum_{i \in N} y_i = \Omega_m,$$

$$x_{ij} \geq 0 \ \ \forall i \in N, \ \ \forall j \in \{1, \ldots, m-1\} \ \ and \ \ y_i \geq 0 \ \ \forall i \in N.$$

If agent i is the only agent who obtains positive amounts of the first $(m-1)$ goods then we are done. So let $T \subseteq N$ (with $|T| > 1$ and $i \in T$) be the set of agents who obtain positive amount of the first $(m-1)$ goods. Since for any pair of agents in T the marginal rate of substitution between any two goods j and j' (from the first $(m-1)$ goods) must be equal we get $\frac{(x_{ij}^*(\theta))^{1/2}}{(x_{ij'}^*(\theta))^{1/2}} = \frac{(x_{i'j}^*(\theta))^{1/2}}{(x_{i'j'}^*(\theta))^{1/2}}$. Hence

(A) $\frac{x_{ij}^*(\theta)}{x_{ij'}^*(\theta)} = \frac{x_{i'j}^*(\theta)}{x_{i'j'}^*(\theta)}$ for all $i' \in T \setminus \{i\}$.

(B) $\sum_{i' \in T} x_{i'j}^*(\theta) = \Omega_j$ for all $j \in \{1, \ldots, m-1\}$.

Using (A) and (B) we get

$$\frac{\sum\limits_{i'\in T} x^*_{i'j}(\theta)}{\sum\limits_{i'\in T} x^*_{i'j'}(\theta)} = \frac{\Omega_j}{\Omega_{j'}} \Rightarrow \frac{x^*_{ij}(\theta) + x^*_{ij}(\theta)\left(\sum\limits_{i'\in T\setminus\{i\}} \frac{x^*_{i'j'}(\theta)}{x^*_{ij'}(\theta)}\right)}{\sum\limits_{i'\in T} x^*_{i'j'}(\theta)} = \frac{\Omega_j}{\Omega_{j'}} \Rightarrow \frac{x^*_{ij}(\theta)}{x^*_{ij'}(\theta)} = \frac{\Omega_j}{\Omega_{j'}}.$$

■

We now prove Proposition 2.

Proof: We proceed in four steps.

Step 1: Consider a two agent economy with agents i and j and an arbitrary total endowment. We prove the following result Fix a profile $\theta \in [\mathbb{D}^q]^2$. If $y^*_i(\theta) > 0$ then $x^*_i(\theta) > 0$.

Note that a Pareto-efficient allocation is a solution to the following optimization problem:

$$\max_{x_i, y_i} \theta_i x_i^{\frac{1}{2}} + y_i$$

$$\text{s.t. } \theta_j(\Omega_x - x_i)^{\frac{1}{2}} + \Omega_y - y_i \geq \bar{u}_j \text{and}$$

$$x_i \geq 0 \text{ and } y_i \geq 0$$

where $\bar{u}_j$ is a positive number. Now note that by strict monotonicity of the objective function maximum would take place at allocations (x^*_i, y^*_i) such that $\theta_j(\Omega_x - x^*_i)^{\frac{1}{2}} + \Omega_y - y^*_i = \bar{u}_j$. The constraint can be rewritten as, $y_i = \Omega_y - \bar{u}_j + \theta_j(\Omega_x - x_i)^{\frac{1}{2}}$. This is a strictly decreasing function of x_i. Also the level sets of the objective function are strictly decreasing with $\lim_{x_i \to 0} \frac{dy_i}{dx_i} = -\infty$. But the derivative of the function $y_i = \Omega_y - \bar{u}_j + \theta_j(\Omega_x - x_i)^{\frac{1}{2}}$ exists for all $x_i < \Omega_x$. From this it can be argued that the level set of the objective function that meets the constraint at $x_i = 0$ must cut the constraint from below. Thus $y^*_i(\theta) > 0$ and $x^*_i(\theta) = 0$ cannot be a Pareto-efficient allocation. Hence the result follows.

Step 2: Consider the n-agent economy and suppose $(x^*(\theta), y^*(\theta)) \in PE(\theta)$. Fix an agent i. If $y^*_i(\theta) > 0$, then $x^*_i(\theta) > 0$.

Suppose not, that is let a Pareto-efficient allocation be such that $y^*_i(\theta) > 0$ and $x^*_i(\theta) = 0$. Let $x_{i'}(\theta)^* > 0$. Let agent i and i' share $\Omega_1^{(i,i')}$ and $\Omega_2^{(i,i')}$ of good x and good y respectively. Fix the the allocation of other agents. The utility functions of agent i and i' are now of the form $\theta_i x_i(\theta)^{1/2} + y_i(\theta)$ and $\theta_{i'} x_{i'}(\theta)^{1/2} + y_{i'}(\theta)$ respectively. However, from Step 1, we know that Pareto-efficient allocations in the two-agent, two-good model are such that if $x^*_i(\theta) = 0$ then $y^*_i(\theta) = 0$. Therefore by keeping agent i''s utility level fixed at $\theta_{i'}(x^*_{i'}(\theta))^{1/2} + y^*_{i'}(\theta)$ agent

i can be made better off with a positive amount of good x. This contradicts our assumption that $(x^*(\theta), y^*(\theta))$ is Pareto-efficient. This proves Step 2.

Step 3: If $(x^*(\theta), y^*(\theta)) \in PE(\theta)$ and for agent $i \in I$, $y_i^*(\theta) > 0$, then $x_i^*(\theta) \geq \frac{\theta_i^2}{\sum\limits_{k \in S} \theta_k^2}$ where $S = \{k \in I \mid x_k^*(\theta) > 0\}$.

Let $(x^*(\theta), y^*(\theta)) \in PE(\theta)$ be such that all agents in the set $S (\subseteq I)$ are allocated a positive amount of good x and all agents in the set $S' (\subseteq I)$ are allocated a positive amount of good y. By Step 2, $S' \subseteq S$. Let $i \in S'$. The Lagrangian for agent i's optimization problem (P) is

$$\begin{aligned} L &= u_i(x_i, y_i; \theta_i) + \sum_{k \in I \setminus \{i\}} \alpha_k[-\bar{u}_k + u_k(x_k, y_k; \theta_k)] \\ &+ \sum_{k \in I}(\beta_{k1} x_i + \beta_{k2} y_i) + \gamma_1(1 - \sum_{k \in I} x_k) + \gamma_2(1 - \sum_{k \in I} y_k) \end{aligned}$$

where α_ks',$\beta_{k1}s'$, β_{k2}s', γ_1 and γ_2 are all Lagrange multipliers. The first order conditions and complementary slackness conditions are

$$\frac{\partial L}{\partial x_i} = \frac{\theta_i}{2x_i^{1/2}} + \beta_{i1} - \gamma_1 = 0, \tag{7}$$

$$\frac{\partial L}{\partial x_k} = \frac{\theta_k \alpha_k}{2x_k^{1/2}} + \beta_{k1} - \gamma_1 = 0, \quad \forall k \in S \setminus \{i\}, \tag{8}$$

$$\frac{\partial L}{\partial y_i} = 1 + \beta_{i2} - \gamma_2 = 0, \tag{9}$$

$$\frac{\partial L}{\partial y_k} = \alpha_k + \beta_{k2} - \gamma_2 = 0, \quad \forall k \in S \setminus \{i\}, \tag{10}$$

$$\alpha_k \frac{\partial L}{\partial \alpha_k} = \alpha_k \left[-\bar{u}_k + u_k(x_k, y_k; \theta_k)\right] = 0, \quad \forall k \in S \setminus \{i\}, \tag{11}$$

$$\sum_{i \in I} x_i = 1, \sum_{i \in I} y_i = 1, \tag{12}$$

$$\beta_{k1} x_k = 0, \beta_{k2} y_k = 0, \quad \forall k \in I, \tag{13}$$

$$\alpha_k \geq 0, \quad \forall k \in S \setminus \{i\}, \tag{14}$$

$$\beta_{ij} \geq 0, \quad \forall i \in I, \quad \forall j \in \{1, 2\}. \tag{15}$$

From (13) and $y_i(\theta) > 0$ it follows that $\beta_{i2} = 0$ and hence using (9) we get $\gamma_2 = 1$. Since $\gamma_2 = 1$ from (10) we get $\alpha_k + \beta_{k2} = 1 \; \forall \; k \in S \setminus \{i\}$. By (14) and (15) we obtain $0 \leq \alpha_k \leq 1$ $\forall \; k \in S \setminus \{i\}$. Since by assumption $x_k > 0$ for all $k \in S$, we have $\beta_{k1} = 0$ for all $k \in S$. Now from (7) and (8) we have,

$$\frac{\theta_i}{2x_i^{1/2}} = \frac{\alpha_k \theta_k}{2x_k^{1/2}}, \forall k \in S \setminus \{i\}.$$

By squaring both sides and simplifying we obtain,

$$\frac{\theta_i^2}{x_i} = \frac{\alpha_k^2 \theta_k^2}{x_k}, \forall k \in S \setminus \{i\}$$

Hence,

$$x_k(\theta) = x_i(\theta) \frac{\alpha_k^2 \theta_k^2}{\theta_i^2}, \forall k \in S \setminus \{i\},$$

Now from (12) we obtain,

$$x_i(\theta) + x_i(\theta) \sum_{k \in S \setminus \{i\}} \frac{\alpha_k^2 \theta_k^2}{\theta_i^2} = 1,$$

since, $\alpha_k \leq 1$, $\forall k \in S \setminus \{i\}$,

$$x_i^*(\theta) = \frac{\theta_i^2}{\theta_i^2 + \sum\limits_{k \in S \setminus \{i\}} \alpha_k^2 \theta_k^2} \geq \frac{\theta_i^2}{\sum\limits_{k \in S} \theta_k^2}. \tag{16}$$

This proves Step 3.

Let $(x^*(\theta), y^*(\theta))$ be a Pareto-efficient allocation at θ. If $S' = \{k \in I \mid y_k^*(\theta) > 0\}$ and $S = \{k \in I \mid x_k^*(\theta) > 0\}$ then Step 2 implies $S' \subseteq S$. Also Step 3 implies $x_i^*(\theta) \geq \frac{\theta_i^2}{\sum_{k \in S} \theta_k^2} \geq \frac{\theta_i^2}{\sum_{k \in I} \theta_k^2}$ for all $i \in S'$. Therefore, $y_i^*(\theta) > 0$ implies $x_i^*(\theta) \geq \frac{\theta_i^2}{\sum_{k \in I} \theta_k^2}$ which is equivalent to condition $P1$ of this Proposition.

Step 4: Let $(x^*(\theta), y^*(\theta)) \in PE(\theta)$. If $x_i^*(\theta) > \frac{\theta_i^2}{\theta_i^2 + \min_{k \neq i} \theta_k^2}$ for some $i \in I$, then $y_i^*(\theta) = 1$.

Suppose not, i.e. $x_i^*(\theta) > \frac{\theta_i^2}{\theta_i^2 + \min_{k \neq i} \theta_k^2}$ and $y_i^*(\theta) < 1$. Therefore there is at least one agent $i' (\neq i)$ such that $y_{i'}^*(\theta) > 0$. By Step 2, $x_{i'}^*(\theta) > 0$. Solving the optimization problem in Step 3 for agent i', we obtain $\alpha_i \leq 1$. Suppose agent i' is the only agent other than i who obtains a positive allocation of good x. Then, $x_i^*(\theta) = \frac{\alpha_i^2 \theta_i^2}{\alpha_i^2 \theta_i^2 + \theta_{i'}^2} \leq \frac{\theta_i^2}{\theta_i^2 + \theta_{i'}^2} \leq \frac{\theta_i^2}{\theta_i^2 + \min_{k \neq i} \theta_k^2}$. Note that if we allow more agents to obtain positive allocations of x, then the denominator in the fraction $\frac{\alpha_i^2 \theta_i^2}{\alpha_i^2 \theta_i^2 + \theta_{i'}^2}$ will increase. As a result, the allocation of agent i of good x will decrease further.

Hence we have a contradiction to our assumption that $x_i^*(\theta) > \frac{\theta_i^2}{\theta_i^2 + \min_{k \neq i} \theta_k^2}$. This proves Step 4 and condition $P2$ of the Proposition. ■

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