

# NOTES ON SOCIAL CHOICE THEORY

Arunava Sen\*

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## 1 BINARY RELATIONS AND ORDERINGS

Let  $A = \{a, b, c, \dots, x, y, z, \dots\}$  be a finite set of alternatives. Let  $N = \{1, \dots, n\}$  be a finite set of agents. Every agent has a preference over alternatives. The preference relation of agent  $i$  over alternatives is denoted by  $R_i$ , where  $aR_ib$  denotes that preference  $a$  is at least as good as  $b$  for agent  $i$  in preference relation  $R_i$ . It is conventional to require  $R_i$  to satisfy the following assumptions.

1. **ORDERING**: A preference relation  $R_i$  of agent  $i$  is called an **ordering** if it satisfies the following properties:
  - **COMPLETENESS**: For all  $a, b \in A$  either  $aR_ib$  or  $bR_ia$ .
  - **REFLEXIVITY**: For all  $a \in A$ ,  $aR_ia$ .
  - **TRANSITIVITY**: For all  $a, b, c \in A$ ,  $[aR_ib \text{ and } bR_ic] \Rightarrow [aR_ic]$ .

We will denote the set of all orderings over  $A$  as  $\mathbf{R}$ .

2. **BINARY RELATION**: A preference relation  $R_i$  of agent  $i$  is called a **binary relation** if it satisfies completeness and reflexivity. Hence, a binary relation gives unordered pairs of  $A$ . An ordering is a transitive binary relation.

Let  $Q_i$  be a binary relation. The **symmetric component** of  $Q_i$  is denoted by  $\bar{Q}_i$ , and is defined as: for all  $a, b \in A$ ,  $a\bar{Q}_ib$  if and only if  $aQ_ib$  and  $bQ_ia$ . The asymmetric component of  $Q_i$  is denoted by  $\hat{Q}_i$ , defined as: for all  $a, b \in A$ ,  $a\hat{Q}_ib$  if and only if  $aQ_ib$  but  $\sim (bQ_ia)$ . Informally,  $\hat{Q}_i$  is the strict part of  $Q_i$ , whereas  $\bar{Q}_i$  is the weak part of  $Q_i$ . Sometimes, we will refer to the symmetric component of a preference relation  $R_i$  as  $I_i$  and asymmetric

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\*Planning Unit, Indian Statistical Institute, 7 Shahid Jit Singh Marg, New Delhi 110016, India. I am grateful to Debasis Mishra for help in preparing these notes.

component as  $P_i$ . We define transitivity of  $\hat{Q}_i$  and  $\bar{Q}_i$  in the usual way, i.e.  $\hat{Q}_i$  is transitive if for all  $a, b, c \in A$ ,  $[a\hat{Q}_ib \text{ and } b\hat{Q}_ic] \Rightarrow [a\hat{Q}_ic]$ . Similarly for  $\bar{Q}_i$ .

The asymmetric and symmetric components of an ordering  $R_i$  will be denoted by  $P_i$  and  $I_i$  respectively.

**PROPOSITION 1** *Let  $R_i$  be an ordering. Then  $P_i$  and  $I_i$  are transitive. Conversely, suppose  $Q_i$  is a binary relation such that  $\hat{Q}_i$  and  $\bar{Q}_i$  are transitive. Then  $Q_i$  is an ordering.*

*Proof:* Consider  $a, b, c \in A$  and an ordering  $R_i$  such that  $aP_ib$  and  $bP_ic$ . Assume by way of contradiction that  $\sim (aP_ic)$ . Since  $R_i$  is an ordering, it is complete. Hence,  $aR_ic$  or  $cR_ia$  holds. Since  $\sim (aP_ic)$ , we get  $cR_ia$ . But  $aP_ib$ . By transitivity of  $R_i$ , we get  $cR_ib$ . This contradicts  $bP_ic$ .

Similarly, assume  $aI_ib$  and  $bI_ic$ . This implies,  $aR_ib$  and  $bR_ic$ . Also,  $bR_ia$  and  $cR_ib$ . Due to transitivity, we get  $aR_ic$  and  $cR_ia$ . This implies that  $aI_ic$ .

Now consider  $a, b, c \in A$  and a binary relation  $Q_i$  such that  $aQ_ib$  and  $bQ_ic$ . We have to show that  $aQ_ic$ . If  $a\hat{Q}_ib$  and  $b\hat{Q}_ic$ , then  $a\hat{Q}_ic$  holds because of the transitivity of  $\hat{Q}_i$ . Hence  $aQ_ic$ . The argument for the case where  $a\bar{Q}_ib$  and  $b\bar{Q}_ic$  is analogous. The two remaining cases are (i)  $a\hat{Q}_ib$  and  $b\bar{Q}_ic$  and (ii)  $a\bar{Q}_ib$  and  $b\hat{Q}_ic$ . Suppose (i) holds but  $\sim (aQ_ic)$ , i.e.  $cQ_ia$ . If  $c\hat{Q}_ia$ , then the transitivity of  $\hat{Q}_i$  implies  $c\hat{Q}_ib$  which contradicts the assumption that  $b\bar{Q}_ic$ . If  $c\bar{Q}_ia$ , then the transitivity of  $\bar{Q}_i$  implies  $b\bar{Q}_ia$  which contradicts the assumption that  $a\hat{Q}_ib$ .

Case (ii) can be dealt with analogously. ■

**DEFINITION 1** A **quasi-ordering** is a binary relation  $Q_i$  whose asymmetric component is transitive.

**REMARK:** The symmetric component of a quasi-ordering need not be transitive. Hence, a quasi-ordering is not an ordering. Indeed, in many situations it is natural to regard the ‘‘indifference’’ relation to be intransitive - for instance, an agent may be indifferent between Rs  $x$  and Rs  $x + \epsilon$  ( $\epsilon > 0$  and  $\epsilon$  very small). Transitivity would imply the agent is indifferent between  $x$  and  $x + \Delta$  for arbitrarily large  $\Delta$  which is implausible.

**DEFINITION 2** An ordering  $R_i$  is **anti-symmetric** if for all  $a, b \in A$   $aR_ib$  and  $bR_ia$  implies  $a = b$  (i.e., no indifference). An anti-symmetric ordering is also called a **linear ordering**.

**REMARK:** If  $R_i$  is anti-symmetric then its asymmetric component  $P_i$  is complete.

## 2 ARROVIAN SOCIAL WELFARE FUNCTIONS

**DEFINITION 3** An **Arrovian social welfare function (ASWF)**  $F$  is a mapping  $F : \mathbf{R}^n \rightarrow \mathbf{R}$ .

A typical element of the set  $\mathbf{R}^n$  will be denoted by  $R \equiv (R_1, \dots, R_n)$  and will be referred to as a **preference profile**.

We give several examples of well-known social welfare functions.

## 2.1 SCORING RULES

For simplicity assume that individual orderings  $R_i$  are linear. Let  $\#A = p$  and  $s = (s_1, \dots, s_p)$ , where  $s_1 \geq \dots \geq s_p \geq 0$  and  $s_1 > s_p$ . The vector  $s$  is called a scoring vector. For all  $i \in N$ ,  $R_i \in \mathbf{R}$ ,  $a \in A$ , define the rank of  $a$  in  $R_i$  as

$$r(a, R_i) = \#\{b \in A \setminus \{a\} : bP_i a\} + 1$$

The score of rank  $r(a, R_i)$  is  $s_{r(a, R_i)}$ . For every profile  $R \in \mathbf{R}^n$  compute the score of alternative  $a \in A$  as

$$s(a, R) = \sum_{i \in N} s_{r(a, R_i)}$$

The **scoring rule**  $F^s$  is defined as for all  $a, b \in A$ , for all  $R \in \mathbf{R}^n$  we have  $aF^s(R)b$  if and only if  $s(a, R) \geq s(b, R)$ . It is easy to see that  $F^s$  defines an ordering. Here are some special cases of the scoring rule.

- **PLURALITY RULE:** This is the scoring rule when  $s = (1, 0, 0, \dots, 0)$ .
- **BORDA RULE:** This is the scoring rule when  $s = (p - 1, p - 2, \dots, 1, 0)$ .
- **ANTI-PLURALITY RULE:** This is the scoring rule when  $s = (1, 1, \dots, 1, 0)$ .

## 2.2 MAJORITY RULES

For every  $R \in \mathbf{R}^n$  define the binary relation  $Q^{maj}(R)$  as follows: for all  $a, b \in A$  we have  $aQ^{maj}(R)b$  if and only if  $\#\{i \in N : aR_i b\} \geq \#\{i \in N : bR_i a\}$ .

**PROPOSITION 2 (Condorcet Paradox)** *There exists  $R$  for which  $Q^{maj}(R)$  is not a quasi-ordering, and hence not an ordering.*

*Proof:* Let  $N = \{1, 2, 3\}$  and  $A = \{a, b, c\}$ . Consider the preference profile in Table 1, where every agent has a linear ordering. Verify that  $\{i \in N : aR_i b\} = \{1, 2\}$ ,  $\{i \in N : bR_i c\} = \{1, 3\}$ , and  $\{i \in N : cR_i a\} = \{2, 3\}$ . Hence,  $a\hat{Q}^{maj}(R)b$ ,  $b\hat{Q}^{maj}(R)c$ , and  $c\hat{Q}^{maj}(R)a$ . This means that  $\hat{Q}^{maj}(R)$  is not an ordering. ■

The proposition above demonstrates that the majority rule procedure (the map which associates  $Q^{maj}(R)$  with every profile  $R$ ) is not a ASWF.

$R_1$	$R_2$	$R_3$
$a$	$c$	$b$
$b$	$a$	$c$
$c$	$b$	$b$

Table 1: Condorcet Cycle

### 2.3 OLIGARCHIES

Let  $R \in \mathbf{R}^n$  be a preference profile and let  $\emptyset \neq G \subseteq N$  be a group of agents. The binary relation  $Q_G^{OL}(R)$  is defined as: for all  $a, b \in A$  we have  $aQ_G^{OL}(R)b$  if and only if there exists  $i \in G$  such that  $aR_i b$ . In other words,  $a\hat{Q}_G^{OL}(R)b$  if and only if for all  $i \in G$  we have  $aP_i b$  and  $a\bar{Q}_G^{OL}(R)b$  otherwise.

**PROPOSITION 3** *For all profiles  $R$ , the binary relation  $Q_G^{OL}(R)$  is a quasi-ordering. Moreover, when  $\#G = 1$ ,  $Q_G^{OL}(R)$  is an ordering.*

*Proof:* Consider a preference profile  $R$  and  $a, b, c \in A$ . Let  $a\hat{Q}_G^{OL}(R)b$  and  $b\hat{Q}_G^{OL}(R)c$ . By definition,  $aP_i b$  and  $bP_i c$  for all  $i \in G$ . Since  $P_i$  is transitive (Proposition 1) we have  $aP_i c$  for all  $i \in G$ . This immediately implies that  $a\hat{Q}_G^{OL}(R)c$ . Hence,  $\hat{Q}_G^{OL}(R)$  is transitive. This implies that  $Q_G^{OL}(R)$  is a quasi-ordering.

When  $G = \{i\}$ ,  $a\hat{Q}_G^{OL}(R)b$  if and only if  $aP_i b$  and  $a\bar{Q}_G^{OL}(R)b$  if and only if  $aI_i b$ . This means  $aQ_G^{OL}(R)b$  if and only if  $aR_i b$ . Since  $R_i$  is transitive,  $Q_G^{OL}(R)$  is transitive. Hence,  $Q_G^{OL}(R)$  is an ordering. ■

**REMARK:** The quasi-ordering  $Q_G^{OL}(R)$  is not an ordering if  $\#G \geq 2$ . As an example, consider the preference profile (linear orderings) of two agents with three alternatives in Table 2. Let  $G = N = \{1, 2\}$ . Then  $a\hat{Q}_G^{OL}(R)b$ ,  $b\bar{Q}_G^{OL}(R)c$  and  $c\bar{Q}_G^{OL}(R)a$ . Transitivity would imply that  $a\bar{Q}_G^{OL}(R)b$ , which is not true.

$R_1$	$R_2$
$a$	$c$
$b$	$a$
$c$	$b$

Table 2: Oligarchy is not an ordering if  $\#G \geq 2$

## 3 ARROW'S IMPOSSIBILITY THEOREM

This section states and proves Arrow's impossibility theorem. In what follows,  $F(R)$  is social ordering induced by  $F$  at the profile  $R$  and  $\hat{F}(R)$  and  $\bar{F}(R)$  denote its asymmetric

and symmetric components respectively.

### 3.1 THE AXIOMS

The following axioms are used in Arrow's impossibility theorem.

**DEFINITION 4** *The ASWF  $F$  satisfies the **Weak Pareto (WP)** axiom if for all profiles  $R$ ,  $a, b \in A$  we have  $aP_i b$  for all  $i \in N$  implies that  $a\hat{F}(R)b$ .*

For the next axiom, we need some notation. Let  $R, R'$  be profiles and let  $a, b \in A$ . We say that  $R$  and  $R'$  agree on  $\{a, b\}$  if

$$\begin{aligned} aP_i b &\Leftrightarrow aP'_i b \quad \forall i \in N \\ aI_i b &\Leftrightarrow aI'_i b \quad \forall i \in N. \end{aligned}$$

We denote this by  $R|_{a,b} = R'|_{a,b}$ .

**DEFINITION 5** *The ASWF  $F$  satisfies **Independence of Irrelevant Alternatives (IIA)** axiom if for all  $R, R' \in \mathbf{R}^n$  and for all  $a, b \in A$ , if  $R|_{a,b} = R'|_{a,b}$  then  $F(R)|_{a,b} = F(R')|_{a,b}$ .*

**PROPOSITION 4** *Scoring rules violate IIA.*

*Proof:* We show it for Plurality rule and Borda rule. Let  $A = \{a, b, c\}$  and  $N = \{1, 2, 3\}$ . Consider the linear orderings in Table 3. Observe that  $R|_{a,b} = R'|_{a,b}$ . By IIA, we should

$R_1$	$R_2$	$R_3$	$R'_1$	$R'_2$	$R'_3$
$a$	$c$	$b$	$a$	$c$	$c$
$b$	$a$	$c$	$b$	$a$	$b$
$c$	$b$	$a$	$c$	$b$	$a$

Table 3: Scoring rules violate IIA

have  $F(R)|_{a,b} = F(R')|_{a,b}$ . Also,  $a\bar{F}(R)b$  but  $a\hat{F}(R')b$  in Plurality and Borda. This proves the claim. ■

**DEFINITION 6** *The ASWF  $F$  is **dictatorial** if there exists an agent  $i \in N$  such that for all  $a, b \in A$  and for all profiles  $R$  we have  $[aP_i b \Rightarrow a\hat{F}(R)b]$ . Voter  $i$  is called a **dictator** in this case.*

**REMARK:** Notice that if  $F$  is dictatorial, it is not the case that there exists a voter  $i$  such that  $F(R) = R_i$  for all profiles  $R$ . For example, the following rule is still dictatorial. For all  $R \in \mathbf{R}^n$ , there exists an agent  $i$  such that  $aP_i b$  implies  $a\hat{F}(R)b$ . But if  $aI_i b$  then  $a\hat{F}(R)b$  if  $aP_j b$  for some  $j \neq i$ . But  $F(R) = R_i$  is true if  $R_i$  is anti-symmetric. Check that  $F(R)$  is an ordering for all profiles  $R$ .

### 3.2 ARROW'S THEOREM

Arrow's theorem demonstrates that the consequence of requiring ASWFs to satisfy WP and IIA is extremely restrictive.

**THEOREM 1 (Arrow's Impossibility Theorem)** *Suppose  $\#A \geq 3$ . A ASWF which satisfies IIA and WP must be dictatorial.*

*Proof:* Consider an ASWF  $F$  that satisfies IIA and WP. We say a group of agents  $\emptyset \neq G \subseteq N$  is **decisive** for  $a, b \in A$  (denoted by  $D_G(a, b)$ ) if for all  $R \in \mathbf{R}^n$

$$[aP_i b \forall i \in G] \Rightarrow [a\hat{F}(R)b].$$

We say a group of agents  $\emptyset \neq G \subseteq N$  is **almost decisive** for  $a, b \in A$  (denoted by  $\bar{D}_G(a, b)$ ) if for all  $R \in \mathbf{R}^n$

$$[aP_i b \forall i \in G, bP_i a \forall i \in N \setminus G] \Rightarrow [a\hat{F}(R)b].$$

Clearly,  $D_G(a, b) \Rightarrow \bar{D}_G(a, b)$  for all  $\emptyset \neq G \subseteq N$  and for all  $a, b \in A$ . We prove the following two important lemmas.

**LEMMA 1 (Field Expansion)** *For all  $\emptyset \neq G \subseteq N$  and for all  $a, b, x, y \in A$*

$$\bar{D}_G(a, b) \Rightarrow D_G(x, y).$$

*Proof:* We consider seven possible cases.

C1 Suppose  $x \neq y \neq a \neq b$ . Consider  $R' \in \mathbf{R}^n$  such that  $xP'_i y$  for all  $i \in G$  and  $R \in \mathbf{R}^n$  such that  $xP_i a P_i b P_i y$  for all  $i \in G$ . Also, for all  $i \in N \setminus G$ , impose  $xP_i a, bP_i y, bP_i a, R_i |_{x,y} = R'_i |_{x,y}$ .

Now,  $\bar{D}_G(a, b) \Rightarrow a\hat{F}(R)b$ . By WP,  $x\hat{F}(R)a$  and  $b\hat{F}(R)y$ . By transitivity, we get  $x\hat{F}(R)y$ . But  $R |_{x,y} = R' |_{x,y}$ . By IIA,  $x\hat{F}(R')y$ . Hence,  $D_G(x, y)$ .

C2 Suppose  $x \neq a \neq b$  but  $y = b$ . Consider  $R' \in \mathbf{R}^n$  such that  $xP'_i b$  for all  $i \in G$  and  $R \in \mathbf{R}^n$  such that  $xP_i a P_i b$  for all  $i \in G$ . Also, for all  $i \in N \setminus G$ , impose  $xP_i a, bP_i a$  and  $R_i |_{x,b} = R'_i |_{x,b}$ .

Now,  $\bar{D}_G(a, b) \Rightarrow a\hat{F}(R)b$ . Pareto gives  $x\hat{F}(R)a$ . By transitivity,  $x\hat{F}(R)b$ . By IIA,  $x\hat{F}(R')b$ . Hence,  $D_G(x, b)$ .

C3 Suppose  $x = a$  and  $y \neq a \neq b$ . Consider  $R' \in \mathbf{R}^n$  such that  $aP'_i y$  for all  $i \in G$  and  $R \in \mathbf{R}^n$  such that  $aP_i b P_i y$  for all  $i \in G$ . Also, for all  $i \in N \setminus G$ , impose  $bP_i y, bP_i a$ , and  $R_i |_{a,y} = R'_i |_{a,y}$ .

Now,  $\bar{D}_G(a, b) \Rightarrow a\hat{F}(R)b$ . Pareto give  $b\hat{F}(R)y$ . By transitivity,  $a\hat{F}(R)y$ . By IIA,  $a\hat{F}(R')y$ . Hence,  $D_G(a, y)$ .

- C4 Suppose  $x = b$  and  $y \neq a \neq b$ . From (C3), we get  $\bar{D}_G(a, b) \Rightarrow D_G(a, y) \Rightarrow \bar{D}_G(a, y)$ . From (C2), we get  $\bar{D}_G(a, y) \Rightarrow D_G(b, y)$ .
- C5 Suppose  $y = a$  and  $x \neq a \neq b$ . From (C2), we get  $\bar{D}_G(a, b) \Rightarrow D_G(x, b) \Rightarrow \bar{D}_G(x, b)$ . From (C3), we get  $\bar{D}_G(x, b) \Rightarrow D_G(x, a)$ .
- C6 Suppose  $x = a$  and  $y = b$ . Consider some  $y \neq a \neq b$  (since  $\#A \geq 3$ , this is possible). From (C3)  $\bar{D}_G(a, b) \Rightarrow D_G(a, y) \Rightarrow \bar{D}_G(a, y)$ . Apply (C3) again to get  $\bar{D}_G(a, y) \Rightarrow D_G(a, b)$ .
- C7 Suppose  $x = b$  and  $y = a$ . Consider some  $y \neq a \neq b$ . From (C5), we get  $\bar{D}_G(a, b) \Rightarrow D_G(y, a) \Rightarrow \bar{D}_G(y, a)$ . From (C2), we get  $\bar{D}_G(y, a) \Rightarrow D_G(b, a)$ .

■

As a consequence of Field Expansion Lemma, we can speak of a decision group of agents without reference to any pair of alternatives. We now prove the other important lemma.

**LEMMA 2 (Group Contraction)** *Suppose  $\emptyset \neq G \subseteq N$  is decisive. If  $\#G \geq 2$ , then there exists a proper non-empty subset of  $G$  which is also decisive.*

*Proof:* Let  $G = G_1 \cup G_2$  with  $G_1 \cap G_2 = \emptyset$  and  $G_1, G_2 \neq \emptyset$ . Let  $a, b, c \in A$  and let  $R \in \mathbf{R}^n$  be a preference profile as in Table 4. Since  $aP_i b$  for all  $i \in G$  and  $G$  is decisive, we get that  $a\hat{F}(R)b$ . We consider two possible cases.

$G_1$	$G_2$	$N \setminus G$
$a$	$c$	$b$
$b$	$a$	$c$
$c$	$b$	$a$

Table 4: A preference profile

- C1 Suppose  $a\hat{F}(R)c$ . But  $aP_i c$  for all  $i \in G_1$  and  $cP_i a$  for all  $i \in N \setminus G_1$ . Hence  $\bar{D}_{G_1}(a, c)$ . By Field Expansion Lemma,  $G_1$  is decisive.
- C2 Suppose  $c\hat{F}(R)a$ . Since  $a\hat{F}(R)b$ , transitivity implies  $c\hat{F}(R)b$ . But  $cP_i b$  for all  $i \in G_2$  and  $bP_i c$  for all  $i \in N \setminus G_2$ . Hence,  $\bar{D}_{G_2}(c, b)$ . By Field Expansion Lemma, we get that  $G_2$  is decisive.

■

By WP, the grand coalition  $N$  is decisive. Repeated application of Group Contraction Lemma gives us that there exists an agent  $i \in N$  such that  $i$  is decisive. By definition an ASWF is dictatorial if there is a single agent who is decisive.

■

## 4 RELAXING THE WEAK PARETO AXIOM: WILSON'S THEOREM

We follow Malawski-Zhou (SCW 1994).

**DEFINITION 7** *The ASWF  $F$  satisfies **Non-Imposition** or **NI** if for all  $a, b \in A$ , there exists a profile  $R$  such that  $aF(R)b$ .*

An example of a ASWF violating NI is the following: for all profiles  $R$ , the social ordering  $F(R)$  is a fixed ordering  $\bar{R}_i$ . Note that it trivially satisfies IIA.

**REMARK:** If a ASWF satisfies WP, it satisfies NI.

**DEFINITION 8** *The ASWF  $F$  is **anti-dictatorial** if there exists a voter  $i$  such that for all  $a, b \in A$  and all profiles  $R$ , we have  $[aP_i b \Rightarrow b\hat{F}(R)a]$ .*

The **null** ASWF  $F^n$  is defined as follows: for all  $a, b \in A$  and for all profiles  $R$ ,  $a\bar{F}^n(R)b$ .

**THEOREM 2 (Wilson's Theorem)** *Assume  $|A| \geq 3$ . A ASWF which satisfies IIA and NI must be null or dictatorial or anti-dictatorial.*

*Proof:* Let  $F$  be a SWF satisfying IIA and NI.

For all  $a, b \in A$ , we write  $PO(a, b)$  if for all profiles  $R$ ,  $[aP_i b \text{ for all } i \in N \Rightarrow a\hat{F}(R)b]$ .

For all  $a, b \in A$ , we write  $APO(a, b)$  if for all profiles  $R$ ,  $[aP_i b \text{ for all } i \in N \Rightarrow b\hat{F}(R)a]$ .

**LEMMA 1:** For all  $a, b, x, y \in A$  we have  $PO(a, b) \Rightarrow PO(x, y)$ .

*Proof:* There are several cases to consider like in the Field Expansion Lemma. We only prove the case  $PO(a, b) \Rightarrow PO(a, y)$  where  $b \neq y$ . Pick an arbitrary profile  $R$  where  $aP_i y$  for all  $i \in N$ . We will show that  $a\hat{F}(R)y$ .

Since  $F$  satisfies NI, there exists a profile  $R'$  such that  $bF(R')y$ . Construct the profile  $\tilde{R}$  as follows: for all  $i \in N$ ,  $a\tilde{P}_i b$ ,  $a\tilde{P}_i y$  and  $\tilde{R} |_{b,y} = R' |_{b,y}$ . This is clearly feasible. Since  $PO(a, b)$  we have  $a\hat{F}(\tilde{R})b$ . On the other hand, IIA implies  $bF(\tilde{R})y$ . Since  $F(\tilde{R})$  is transitive, we have  $a\hat{F}(\tilde{R})y$ . Now IIA implies  $a\hat{F}(R)y$ . This completes the proof of Lemma 1.

**LEMMA 2:** For all  $a, b, x, y \in A$  we have  $APO(a, b) \Rightarrow APO(x, y)$ .

*Proof:* Once again there are several cases to consider. We only prove the case  $APO(a, b) \Rightarrow APO(a, y)$  where  $b \neq y$ . Pick an arbitrary profile  $R$  where  $aP_i y$  for all  $i \in N$ . We will show that  $y\hat{F}(R)a$ .

Since  $F$  satisfies NI, there exists a profile  $R'$  such that  $yF(R')b$ . Construct the profile  $\tilde{R}$  as follows: for all  $i \in N$ ,  $a\tilde{P}_i b$ ,  $a\tilde{P}_i y$  and  $\tilde{R} |_{b,y} = R' |_{b,y}$ . This is clearly feasible. Since  $PO(a, b)$  we have  $b\hat{F}(\tilde{R})a$ . On the other hand, IIA implies  $yF(\tilde{R})b$ . Since  $F(\tilde{R})$  is transitive, we have  $y\hat{F}(\tilde{R})a$ . Now IIA implies  $y\hat{F}(R)a$ . This completes the proof of Lemma 2.

LEMMA 3: One of the following statements must hold

- (i)  $F$  is null.
- (ii)  $PO(a, b)$  holds for some pair  $a, b$ .
- (iii)  $APO(a, b)$  holds for some pair  $a, b$ .

Proof: Suppose that neither (i) nor (ii) nor (iii) hold. Since (i) does not hold, there exists a pair  $x, y$  and a profile  $R$  such that  $x\hat{F}(R)y$  holds. Pick  $z \neq x, y$  and let  $R'$  be a profile such that  $xP'_i z, yP'_i z$  for all  $i \in N$  and  $R' \mid_{x,y} = R \mid_{x,y}$ . Again this is clearly feasible. Since neither  $PO(x, z)$  nor  $APO(x, z)$  hold, we must have  $x\bar{F}(R')z$ . Similarly, since neither  $PO(y, z)$  nor  $APO(y, z)$  hold, we must have  $y\bar{F}(R')z$ . Since  $F(R')$  is transitive, we have  $x\bar{F}(R')y$ . Applying IIA, we have  $x\bar{F}(R)y$ . But this contradicts our assumption that  $x\hat{F}(R)y$  and completes the proof of Lemma 3.

Suppose  $F$  is not null. Applying Lemma 3, either  $PO(a, b)$  must hold for some  $a, b$  or  $APO(a, b)$  must hold for some pair  $a, b$ . Suppose the former holds. Then WP holds and the existence of a dictator follows from Arrow's Theorem. If the latter holds, then the proof of Arrow's Theorem can be modified in a straightforward manner to show that  $F$  is anti-dictatorial. ■

## 5 EXISTENCE OF MAXIMAL ELEMENTS

Let  $Q_i$  be a binary relation over the elements of the set  $A$ . Let  $B \subset A$ .

DEFINITION 9 *The set of **maximal elements** of  $B$  according to  $Q$  denoted by  $M(B, Q_i)$  is the set  $\{x \in B \mid \nexists y \in B \text{ and } y\hat{Q}_i x\}$ .*

REMARK: Since  $Q_i$  is complete, we can define the set of maximal elements equivalently as  $M(B, Q_i) = \{x \in B \mid xQ_i y \text{ for all } y \in B\}$ .

DEFINITION 10 *The binary relation  $Q_i$  is **acyclic** if for all  $a_1, a_2, \dots, a_K \in A$ , we have  $[a_1\hat{Q}_i a_2, a_2\hat{Q}_i a_3, \dots, a_{K-1}\hat{Q}_i a_K] \Rightarrow a_1Q_i a_K$ .*

REMARK:  $Q_i$  is transitive  $\Rightarrow Q_i$  is quasi-transitive  $\Rightarrow Q_i$  is acyclic.

PROPOSITION 5 *Let  $Q_i$  be a binary relation over a finite set  $A$ . Then  $[M(B, Q_i) \neq \emptyset] \Rightarrow [Q_i \text{ is acyclic}]$ .*

*Proof:*  $\Rightarrow$  Suppose not, i.e there exists  $a_1, \dots, a_K$  such that  $a_1\hat{Q}_i a_2, \dots, a_{K-1}\hat{Q}_i a_K$  and  $a_K\hat{Q}_i a_1$ . Let  $B = \{a_1, \dots, a_K\}$ . Clearly  $M(B, Q_i) = \emptyset$  which contradicts our hypothesis.

$\Leftarrow$  Suppose  $Q_i$  is acyclic and let  $B$  be an arbitrary subset of  $A$ . Pick an arbitrary element  $a_1 \in B$ . If  $a_1 \in M(B, Q_i)$ , we are done. Suppose  $a_1 \notin M(B, Q_i)$ . There must exist  $a_2 \in B$  such that  $a_2 \hat{Q}_i a_1$ . If  $a_2 \in M(B, Q_i)$ , we are done again. Otherwise there exists  $a_3$  such that  $a_3 \hat{Q}_i a_2$ . Note that acyclicity implies  $a_3 Q_i a_1$ , i.e.  $a_3 \neq a_1$ . If  $a_3 \in M(B, Q_i)$  our algorithm stops; otherwise we find an element  $a_4$  such that  $a_4 \hat{Q}_i a_3$ . Critically acyclicity implies  $a_4 \neq a_2, a_1$ . In general, acyclicity implies that the sequence  $a_1, \dots, a_k, \dots$  constructed in the manner above contains no repetitions. Since  $B$  is finite, the algorithm must stop, i.e.  $M(B, Q_i) \neq \emptyset$ .  $\blacksquare$

REMARK: Acyclicity over triples is not sufficient for maximal elements to exist. Consider the following example:  $A = \{a_1, a_2, a_3, a_4\}$  and  $a_1 \hat{Q}_i a_2, a_2 \hat{Q}_i a_3, a_3 \hat{Q}_i a_4, a_4 \hat{Q}_i a_1, a_1 \bar{Q}_i a_3$  and  $a_2 \bar{Q}_i a_4$ . Then acyclicity over triples is satisfied but  $M(B, Q_i) = \emptyset$ .

REMARK: Acyclicity does not guarantee the existence of maximal elements if  $A$  is not finite. For example, let  $Q_i$  be the natural ordering of the real numbers and let  $A = [0, 1)$ . Then  $M(A, Q_i) = \emptyset$ .

## 6 DOMAIN RESTRICTIONS: SINGLE-PEAKED PREFERENCES

We endow  $A$  with additional structure.

Let  $\geq$  be a linear order over  $A$ . For instance  $A$  could be the unit interval and  $\geq$  the natural ordering over the reals.

DEFINITION 11 *The ordering  $R_i$  is **single-peaked** if there exists  $a^* \in A$  (called the **peak** of  $R_i$ ) such that for all  $b, c \in A$*

$$[a^* \geq b > c \text{ or } c > b \geq a^*] \Rightarrow b P_i c$$

Let  $\mathcal{R}^{SP}(\geq)$  be the set of all single-peaked preferences with respect to the ordering  $\geq$ . Throughout this section we shall keep  $\geq$  fixed so that we shall refer to the set of single-peaked preferences simply as  $\mathcal{R}^{SP}$ . We shall denote the peak of a single-peaked (or any other, for that matter) ordering  $R_i$  as  $\tau(R_i)$ .

EXAMPLE 1 Let  $A = [0, 1]$  denote the fraction of the Central Government's budget that is spent on education. According to voter  $i$  the optimal fraction is 0.1. If her preferences are single-peaked, she strictly prefers 0.2 over 0.3 and 0.08 over 0.05. Note that single-peakedness places no restrictions on alternatives on different "sides" of the peak, i.e. the voter can either prefer 0.05 to 0.2 or vice-versa.

REMARK: Let  $A = \{a, b, c\}$  and consider the set of linear orders  $(R_1, R_2, R_3)$  which constitute the following Condorcet in Table 1. It is easy to check that there does not exist an ordering  $\geq$  over  $A$  such that  $(R_1, R_2, R_3)$  are single-peaked with respect to  $\geq$ . Suppose for instance  $a > b > c$ . Then  $R_2$  is not single-peaked because if  $c$  is the peak, then  $b$  must be strictly better than  $a$ .

REMARK: Let  $|A| = m$ . Then  $|\mathcal{R}^{SP}| = 2^{m-1}$ .

DEFINITION 12 *Let  $R \in \mathcal{R}^{SP}$  be a profile of single-peaked preferences. The **median voter** in the profile  $R$  is the voter  $h$  such that  $|\{i \in N : \tau(R_h) \geq \tau(R_i)\}| \geq \frac{n}{2}$  and  $|\{i \in N : \tau(R_i) \geq \tau(R_h)\}| \geq \frac{n}{2}$ .*

REMARK: The median voter exists for all profiles although she may not be unique. However if  $n$  is odd, the median peak  $\tau(R_h)$  will be unique.

THEOREM 3 (**Median Voter Theorem**) *Let  $R \in \mathcal{R}^{SP}$  be a profile of single-peaked preferences. Then  $M(A, Q^{maj}) \neq \emptyset$ . In particular  $\tau(R_h) \in M(A, Q^{maj})$ .*

*Proof:* Pick an arbitrary profile  $R \in \mathcal{R}^{SP}$ . We will show that  $\tau(R_h)Q^{maj}b$  for all  $b \neq \tau(R_h)$ . We consider two cases.

Case 1.  $\tau(R_h) > b$ . Let  $i \in N$  be such that  $\tau(R_i) \geq \tau(R_h)$ . Since  $R_i$  is single-peaked and  $\tau(R_i) \geq \tau(R_h) > b$ , we have  $\tau(R_h)P_i b$ . Since  $|\{i \in N : \tau(R_i) \geq \tau(R_h)\}| \geq \frac{n}{2}$  since  $h$  is a median voter, it follows that  $\tau(R_h)Q^{maj}b$ .

Case 2.  $b > \tau(R_h)$ . Let  $i \in N$  be such that  $\tau(R_h) \geq \tau(R_i)$ . Since  $R_i$  is single-peaked and  $b > \tau(R_h) \geq \tau(R_i)$ , we have  $\tau(R_h)P_i b$ . Since  $|\{i \in N : \tau(R_h) \geq \tau(R_i)\}| \geq \frac{n}{2}$  since  $h$  is median voter, it follows that  $\tau(R_h)Q^{maj}b$ .

This covers all possible cases. ■

Is  $Q^{maj}(R)$  transitive for all single-peaked profiles? No, as the following example shows.

EXAMPLE 2 Let  $A = [0, 1]$ ,  $N = \{1, 2\}$ . Let  $R_1$  and  $R_2$  be the following single-peaked orderings:

- $\tau(R_1) = 0.4$  and  $xP_1y$  whenever  $0.4 > x$  and  $y > 0.4$ , i.e voter 1 prefers all alternatives to the “left” of 0.4 to everything on the “right” of 0.4.
- $\tau(R_2) = 0.5$  and  $xP_2y$  whenever  $x > 0.5$  and  $0.5 > y$ , i.e voter 2 prefers all alternatives to the “right” of 0.5 to everything on the “left” of 0.5.

Now consider the alternatives  $a = 0.1$ ,  $b = 0.2$  and  $c = 0.6$ . Note that  $bP_1c$  and  $cP_2b$  so that  $b\bar{Q}^{maj}c$ . Similarly,  $aP_1c$  and  $cP_2a$  so that  $a\bar{Q}^{maj}c$ . However single-peakedness of  $R_1$  and  $R_2$  imply  $bP_1a$  and  $bP_2a$  so that  $b\hat{Q}^{maj}a$ . Clearly  $Q^{maj}$  is not transitive. Note that all alternatives in the interval  $[0.4, 0.5]$  are maximal according to  $Q^{maj}$  in  $A$ .

The binary relation  $Q^{maj}$  defined over single-peaked preferences is transitive in special cases.

**PROPOSITION 6** *Assume that  $n$  is odd and that voter preferences are linear and single-peaked. Then for all profiles  $R$ ,  $Q^{maj}(R)$  is an ordering.*

*Proof:* We only need to show that for all profiles  $R$ ,  $Q^{maj}(R)$  is transitive. Since  $n$  is odd and voter preferences do not admit indifference,  $Q^{maj}(R)$  admits no indifferences, i.e. for all  $a, b \in A$ , either  $a\hat{Q}^{maj}(R)b$  or  $b\hat{Q}^{maj}(R)a$  holds. Now pick  $a, b, c \in A$  and a profile  $R$  and assume w.l.o.g. that  $a\hat{Q}^{maj}(R)b$  and  $b\hat{Q}^{maj}(R)c$ . Observe that for all voters  $i$ ,  $R_i$  induces single-peaked preferences over  $\{a, b, c\}$  (prove!). Applying Theorem 3 to the set  $\{a, b, c\}$ , it follows that  $M(\{a, b, c\}, R) \neq \emptyset$ . Therefore  $c\hat{Q}^{maj}a$  is impossible, i.e.  $a\hat{Q}^{maj}c$  holds and  $Q^{maj}$  is transitive. ■

## 7 INTERPERSONAL COMPARABILITY

We now turn our attention to models where voters are endowed with “richer information” which can be used for aggregation.

Voter  $i$  will be assumed to have a utility function  $u_i : A \rightarrow \mathfrak{R}$ . We shall let  $\mathcal{U}$  denote the set of all such utility functions. A utility profile  $u$  is an  $n$ -tuple  $(u_1, \dots, u_n) \in \mathcal{U}^n$ .

**DEFINITION 13** *A Social Welfare Functional (SWFL)  $F$  is a mapping  $F : \mathcal{U}^n \rightarrow \mathcal{R}$ .*

Let  $F$  be a SWFL. For all utility profiles  $u$  we shall let  $R_u$  denote the social ordering  $F(u)$ .

We now restate some axioms that we had introduced earlier for this environment and also introduce some new ones.

**DEFINITION 14** *A SWFL  $F$  satisfies **Binary Independence of Irrelevant Alternatives (BIIA)** if for all profiles  $u, u'$  and  $a, b \in A$ ,*

$$[u_i(a) = u'_i(a) \text{ and } u_i(b) = u'_i(b) \quad \forall i \in N] \Rightarrow [R_u \upharpoonright_{a,b} = R_{u'} \upharpoonright_{a,b}]$$

let  $a, b, c, d \in A$  and let  $R_i$  be an ordering. We say  $R_i \upharpoonright_{a,b} = R_i \upharpoonright_{c,d}$  if  $[aP_ib \Leftrightarrow cP_id]$  and  $[aI_ib \Leftrightarrow cI_id]$ .

A stronger version of BIIA is Strong Neutrality defined below.

**DEFINITION 15** *A SWFL  $F$  satisfies **Strong Neutrality (SN)** if for all profiles  $u, u'$  and  $a, b, c, d \in A$ ,*

$$[u_i(a) = u'_i(c) \text{ and } u_i(b) = u'_i(d) \quad \forall i \in N] \Rightarrow [R_u \upharpoonright_{a,b} = R_{u'} \upharpoonright_{c,d}]$$

In other words, if the utilities associated with  $a$  and  $b$  in profile  $u$  agree with those of  $c$  and  $d$  respectively in profile  $u'$ , then  $a$  and  $b$  must be ranked in exactly the same way under  $R_u$  as  $c$  and  $d$  under  $R_{u'}$ . Note that while  $a$  and  $b$  are distinct and  $c$  and  $d$  are also distinct, it may be the case that  $b = c$  and  $a = d$  etc.

We introduce some some Pareto type axioms.

**DEFINITION 16** *The SWFL  $F$  satisfies **Pareto Indifference (PI)** if, for all  $a, b \in A$  and profiles  $u$ ,  $[u_i(a) = u_i(b) \text{ for all } i \in N] \Rightarrow aI_u b$ .*

**DEFINITION 17** *The SWFL  $F$  satisfies **Strong Pareto (SP)** if, for all  $a, b \in A$  and profiles  $u$ ,  $[u_i(a) \geq u_i(b) \text{ for all } i \in N] \Rightarrow aR_u b$ . Moreover if there exists  $k \in N$  such that  $u_k(a) > u_k(b)$ , then  $aP_u b$ .*

## 7.1 MEASURABILITY AND COMPARABILITY AXIOMS

Let  $\phi \equiv (\phi_1, \dots, \phi_n)$  be an  $n$ -tuple of strictly increasing functions  $\phi_i : \mathfrak{R} \rightarrow \mathfrak{R}$ . Let  $\Phi$  be an arbitrary set of such  $n$ -tuples.

Let  $u$  be a profile. The profile  $\phi.u$  denote the profile  $(\phi_1.u_1, \dots, \phi_n.u_n)$ , i.e the utility for alternative  $a$  for voter  $i$  is  $\phi_i(u_i(a))$ .

**DEFINITION 18** *The SWFL  $F$  satisfies invariance with respect to  $\Phi$  if for all profiles  $u$ ,  $F(u) = F(\phi.u)$ .*

The idea is as follows. Divide the set of all profiles  $\mathcal{U}^n$  into equivalence classes. Two profiles  $u, u'$  belong to the same equivalence class if there exists  $\phi \in \Phi$  such that  $u = \phi.u'$ . A SWFL  $F$  which is invariant with respect to  $\Phi$  if  $f(u) = f(u')$ . In other words, two profiles in the same equivalence class have the same “information” permissible for aggregation from the viewpoint of  $F$ . Observe that the finer the partition of  $\mathcal{U}^n$  into equivalence classes or partitions, the greater is the information that is being allowed for aggregation.

We now consider various assumptions on  $\phi$ .

**DEFINITION 19** *A SWFL satisfies **Ordinally Measurable, Non-Comparable Utilities (OMNC)** if  $\Phi$  consists of all  $n$ -tuples of increasing functions  $(\phi_1, \dots, \phi_n)$ .*

**REMARK:** In the OMNC, only ordinal information is being allowed for aggregation. This is the Arrovian case.

**DEFINITION 20** *A SWFL satisfies **Cardinally Measurable, Non-Comparable Utilities (CMNC)** if  $\phi \in \Phi$  if for all  $i \in N$ ,  $\phi_i(t) = \alpha_i + \beta_i t$  with  $\beta_i > 0$ .*

**REMARK:** In CMNC we allow for independent affine transformations of utilities for voters.

**DEFINITION 21** A SWFL satisfies *Ordinally Measurable, Fully-Comparable Utilities (OMFC)* if  $\phi_i \in \Phi$  if, for all  $i \in N$ ,  $\phi_i = \phi_0$  for some increasing function  $\phi_0 : \mathfrak{R} \rightarrow \mathfrak{R}$ .

**DEFINITION 22** A SWFL satisfies *Cardinally Measurable, Fully-Comparable Utilities (CMUC)* if  $\phi_i \in \Phi$  if, for all  $i \in N$ ,  $\phi_i = \alpha + \beta t$  with  $\beta > 0$ .

**QUESTION:** What are the SWFLs which satisfy a certain class of measurability and comparability restriction together with the classical Arrowian assumptions?

## 7.2 WELFARISM

Our goal in this subsection is to show that the questions raised in the previous subsection can be reduced to problems of ranking vectors in  $\mathfrak{R}^n$ .

**PROPOSITION 7 (Welfarism)**  $SN \Rightarrow BIIA$ . If  $|A| \geq 3$ , then  $BIIA + PI \Rightarrow SN$ .

*Proof:* The first proof of the proposition is trivial. There are several cases to deal with like in the Field Expansion Lemma. Consider the case where  $a, b, c \in A$ , and profiles  $u, u'$  are such that  $u_i(a) = u'_i(a)$  and  $u_i(b) = u'_i(c)$  for all  $i \in N$ . We have to show that  $R_u \upharpoonright_{a,b} = R_{u'} \upharpoonright_{a,c}$ . Construct a profile  $\tilde{u}$  such that  $\tilde{u}_i(a) = u_i(a) = u'_i(a)$  and  $\tilde{u}_i(b) = \tilde{u}_i(c) = u_i(b) = u'_i(c)$  for all  $i \in N$ . By BIIA,  $R_u \upharpoonright_{a,b} = R_{\tilde{u}} \upharpoonright_{a,b}$  and  $R_{u'} \upharpoonright_{a,c} = R_{\tilde{u}} \upharpoonright_{a,c}$ . By PI,  $bI_{\tilde{u}}c$  so that the transitivity of  $R_{\tilde{u}}$  implies  $R_{\tilde{u}} \upharpoonright_{a,b} = R_{\tilde{u}} \upharpoonright_{a,c}$ . Hence  $R_u \upharpoonright_{a,b} = R_{u'} \upharpoonright_{a,c}$ .

Similar arguments can be used to prove all cases. Note that in the case where  $u, u'$  are such that  $u_i(a) = u'_i(b)$  and  $u_i(b) = u'_i(a)$  for all  $i \in N$  we need a third alternative  $c$ , i.e we need to use the assumption that  $|A| \geq 3$ . ■

We shall often use the following notation: for all  $a \in A$  and profile  $u$ ,  $u(a) \equiv (u_1(a), \dots, u_n(a))$ .

**PROPOSITION 8** Assume  $|A| \geq 3$ . A SWFL satisfies PI and BIIA if and only if there exists an ordering  $\succeq$  on  $\mathfrak{R}^n$  such that for all  $a, b \in A$  and for all profiles  $u$ ,  $R_u \upharpoonright_{a,b} = \succeq_{|\alpha, \beta}$  where  $u(a) = \alpha$  and  $u(b) = \beta$ .

*Proof:* Let  $\succeq$  be an ordering on  $\mathfrak{R}^n$ . Construct a SWFL  $F$  as follows: for all profiles  $u$  and  $a, b \in A$ ,  $R_u \upharpoonright_{a,b} = \succeq_{|u(a), u(b)}$ . The transitivity of  $R_u$  is a direct consequence of the transitivity of  $\succeq$  while BIIA and PI of  $F$  follows directly from its definition.

Let  $F$  satisfy BIIA and PI. Define  $\succeq$  as follows: for all  $\alpha, \beta \in \mathfrak{R}^n$ ,  $\succeq_{|\alpha, \beta} = R_u \upharpoonright_{a,b}$  for some  $a, b \in A$  and profile  $u$  such that  $u(a) = \alpha$  and  $u(b) = \beta$ . Since  $F$  satisfies PI and BIIA, it satisfies SN (Proposition 7). This implies that the ranking of vectors  $\alpha, \beta \in \mathfrak{R}^n$  according to  $\succeq$  does not depend on the alternatives  $a, b$  and profile  $u$  chosen in the construction (i.e. so that  $u(a) = \alpha$  and  $u(b) = \beta$ ). In other words,  $\succeq$  is well-defined. It is transitive because  $R_u$  is transitive for all  $u$ . ■

Proposition 8 reduces the problem of finding an SWFL satisfying PI and BIIA to the problem of finding an appropriate ordering of utility vectors. We only need to reinterpret the measurability and comparability requirement in this environment.

Let  $F$  be an SWFL satisfying PI, BIIA and invariance with respect to  $\Phi$ . Let  $\succeq$  be the ordering over  $\mathfrak{R}^n$  induced by  $F$ . Let  $\alpha, \beta \in \mathfrak{R}^n$  and  $\phi \in \Phi$ . Let  $\phi.\alpha$  and  $\phi.\beta$  denote the  $n$ -tuples  $(\phi_1(\alpha_1), \dots, \phi_n(\alpha_n))$  and  $(\phi_1(\beta_1), \dots, \phi_n(\beta_n))$  respectively. By invariance on  $F$ , we have  $R_u|_{a,b} = R_{\phi.u}|_{a,b}$ . From the construction of  $\succeq$  we know that  $R_u|_{a,b} = \succeq|_{\alpha,\beta}$  and  $R_{\phi.u}|_{a,b} = \succeq|_{\phi.\alpha,\phi.\beta}$ . Therefore  $\succeq|_{\alpha,\beta} = \succeq|_{\phi.\alpha,\phi.\beta}$ . This motivates the following definition.

**DEFINITION 23** *The ordering  $\succeq$  over  $\mathfrak{R}^n$  satisfies invariance with respect to  $\Phi$ , if for all  $\alpha, \beta \in \mathfrak{R}^n$  and  $\phi \in \Phi$ , we have  $\succeq|_{\alpha,\beta} = \succeq|_{\phi.\alpha,\phi.\beta}$ .*

**PROPOSITION 9** *Assume  $|A| \geq 3$ . Let  $F$  be a SWFL satisfying PI and BIIA and invariance with respect to  $\Phi$ . Then the induced ordering  $\succeq$  over  $\mathfrak{R}^n$  satisfies invariance with respect to  $\Phi$ .*

Proposition 9 follows from our earlier discussion.

### 7.3 ARROW'S THEOREM: A GEOMETRICAL APPROACH

We restate Arrow's Theorem in this environment.

**THEOREM 4 (Arrow's Theorem for Social Welfare Functionals)** *Assume  $|A| \geq 3$ . If a SWFL satisfies PI, WP, BIIA and OMNC, then it must be dictatorial.*

*Proof:* We will only do the case of  $n = 2$ . Let  $F$  satisfy PI, WP, BIIA and OMNC. Applying Proposition 8, we will show that the induced ordering  $\succeq$  over  $\mathfrak{R}^2$  has the following property: there exists  $i = \{1, 2\}$  such that for all  $\alpha, \beta \in \mathfrak{R}^2$ , we have  $\alpha \succ \beta$  only if  $\alpha_i > \beta_i$ .

Refer to Figure 1. Let  $\alpha$  be an arbitrary point in  $\mathfrak{R}^2$ . We will try to draw an "indifference curve" through  $\alpha$ . Consider Regions *I*, *II*, *III* and *IV* which do not include the dotted lines.

Step 1: All vectors in region *II* must be strictly better than  $\alpha$  according to  $\succeq$ . In other words  $\beta \succ \alpha$  for all  $\beta \in$  Region *I*. This follows from WP. Similarly all vectors in Region *IV* must be worse than  $\alpha$  by WP.

Step 2: Let  $\beta, \gamma \in$  Region *I*. Then  $\succeq|_{\alpha,\beta} = \succeq|_{\alpha,\gamma}$ .

Let  $\phi_1 : \mathfrak{R} \rightarrow \mathfrak{R}$  be a linear function such that  $\phi_1(\beta_1) = \gamma_1$  and  $\phi_1(\alpha_1) = \alpha_1$ . Since  $\beta_1, \gamma_1 < \alpha_1$  it follows that  $\phi_1$  is strictly increasing. Similarly let  $\phi_2 : \mathfrak{R} \rightarrow \mathfrak{R}$  be such that  $\phi_2(\beta_2) = \gamma_2$  and  $\phi_2(\alpha_2) = \alpha_2$ . Since  $\beta_2, \gamma_2 > \alpha_2$   $\phi_2$  is also increasing. Observe the  $\phi(\beta) = \gamma$  and  $\phi(\alpha) = \alpha$ . Since  $\phi_1, \phi_2$  are increasing and  $\succeq$  satisfies OMNC, we must have  $\succeq|_{\alpha,\beta} = \succ|_{\alpha,\gamma}$ .

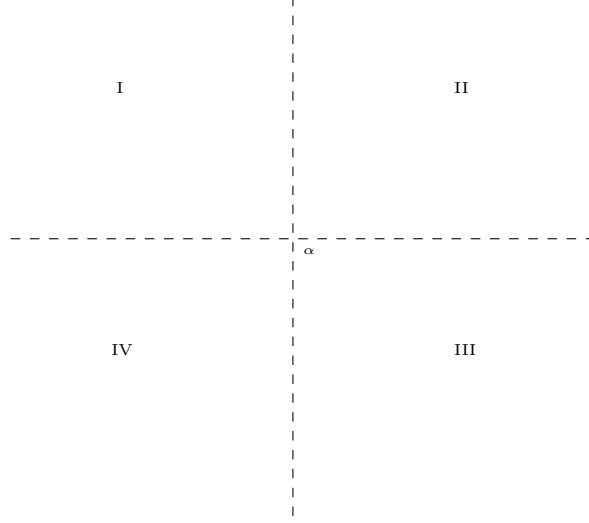


Figure 1: Arrow's Theorem

Step 3: Let  $\beta, \gamma \in \text{Region III}$ . Then  $\succeq|_{\alpha, \beta} = \succeq|_{\alpha, \gamma}$ .

The arguments here are identical to those in Step 2.

Step 4: Let  $\beta \in \text{Region I}$ . Then either  $\beta \succ \alpha$  or  $\alpha \succ \beta$  must hold.

Suppose that the claim above is false, i.e.  $\beta \sim \alpha$ . Since Region II is an open set, we can find  $\gamma \in \text{Region II}$  (sufficiently close to  $\beta$ ) such that  $\gamma \succ \beta$ . From Step 2, we must have  $\gamma \sim \alpha$ , so that  $\beta \sim \gamma$  by transitivity of  $\succeq$ . However  $\gamma \succ \beta$  by WP. Contradiction.

Step 5: Let  $\beta \in \text{Region III}$ . Then either  $\beta \succ \alpha$  or  $\alpha \succ \beta$  must hold.

The arguments here are identical to those in Step 4.

Step 6: Let  $\beta \in \text{Region I}$  and  $\gamma \in \text{Region III}$ . Then  $\beta \succ \alpha \Rightarrow \alpha \succ \gamma$ . Similarly  $\alpha \succ \beta \Rightarrow \gamma \succ \alpha$ .

Suppose  $\beta \succ \alpha$ . Consider the following functions:  $\phi_1(t) = t + (\alpha_1 - \beta_1)$  and  $\phi_2(t) = t - (\beta_2 - \alpha_2)$ . Note that  $\phi_1$  and  $\phi_2$  are strictly increasing. Also  $\phi(\beta) = \alpha$ . Since  $\alpha_1 - \beta_1 > 0$ , we have  $\phi_1(\alpha_1) > \alpha_1$ . Since  $\beta_2 - \alpha_2 > 0$ , we have  $\phi_2(\alpha_2) < \alpha_2$ . Hence  $\phi(\alpha) \in \text{Region III}$ . Since  $\beta \succ \alpha$ , invariance implies  $\phi(\beta) \succ \phi(\alpha)$ , i.e.  $\alpha \succ \gamma$  where  $\gamma \in \text{Region III}$ .

Step 7: Let  $\beta \in \text{Region I}$ . If  $\beta \succ \alpha$ . Let  $\gamma$  be a point on the boundary of Regions I and II and let  $\gamma'$  be a point on the boundary of Regions III and IV. Then  $\gamma \succ \alpha$  and  $\alpha \succ \gamma'$ . This follows from Step 6 and WP. By an identical argument, if  $\alpha \succ \beta$  where  $\beta \in \text{Region I}$ , then all points in on the boundary of Regions I and IV are strictly worse than  $\alpha$  according to  $\succ$  and all points on the boundary of Regions III and IV are strictly better than  $\alpha$  according to  $\succ$ .

Summary: Steps 1 through 7 imply that there are exactly two possibilities: (i) Regions I and II are better than  $\alpha$  and Regions III and IV are worse than  $\alpha$  according to  $\succ$  (ii)

Regions *II* and *III* are better than  $\alpha$  and Regions *I* and *IV* are worse than  $\alpha$  according to  $\succ$ . We say that the *pseudo-indifference curve* through  $\alpha$  is *horizontal* if possibility (i) holds and *vertical* if possibility (ii) holds.

Step 8: If the pseudo-indifference curve is horizontal (resp. vertical) for some  $\alpha$ , it must be horizontal (resp. vertical) *for all*  $\alpha \in \mathfrak{R}^2$ . If this was false, the two pseudo-indifference curves would intersect, contradicting the transitivity of  $\succ$ .

We can now complete the proof of the theorem. If all pseudo-indifference curves are horizontal, voter 2 is the dictator; if they are vertical, voter 1 is the dictator. ■

REMARK: The ordering  $\succ$  that we have constructed above is not *complete*. For instance if all the pseudo-indifference curves are vertical, we know the following: for  $\alpha, \beta \in \mathfrak{R}^2$  such that  $\beta_1 > \alpha_1$ , we have  $\beta \succ \alpha$ . But we say nothing in the case  $\beta_1 = \alpha_1$ . In order to characterize  $\succ$ , we need additional axioms.

DEFINITION 24 *The ordering  $\succeq$  satisfies continuity, if for all  $\alpha \in \mathfrak{R}^n$  the sets  $\{\beta : \beta \succeq \alpha\}$  and  $\{\beta : \alpha \succeq \beta\}$  are closed.*

DEFINITION 25 *The ordering  $\succeq$  is strongly dictatorial, if there exists a voter  $i$  such that for all  $\alpha, \beta \in \mathfrak{R}^n$ ,  $[\alpha_i \geq \beta_i] \Leftrightarrow [\alpha \succeq \beta]$*

Suppose  $\succeq$  is strongly dictatorial and that voter  $i$  is the dictator. Then for all  $\alpha, \beta \in \mathfrak{R}^n$ ,  $[\alpha_i > \beta_i] \Rightarrow [\alpha \succ \beta]$ ,  $[\beta_i > \alpha_i] \Rightarrow [\beta \succ \alpha]$  and  $[\alpha_i = \beta_i] \Rightarrow [\alpha \sim \beta]$ .

DEFINITION 26 *The ordering  $\succeq$  is lexicographic, if there exists an ordering of voters  $i_1, i_2, \dots, i_n$  such that for all  $\alpha, \beta \in \mathfrak{R}^n$ ,  $\alpha \succ \beta$  implies that there exists an integer  $K$  lying between 1 and  $n$  such that*

- $\alpha_{i_k} = \beta_{i_k}$  for all  $k = 1, \dots, K - 1$
- $\alpha_{i_K} > \beta_{i_K}$ .

COROLLARY 1 *Assume  $|A| \geq 3$ . If a SWFL satisfies PI, WP, BIIA and OMNC and the induced ordering  $\succeq$  satisfies continuity, then it must be strongly dictatorial.*

COROLLARY 2 *Assume  $|A| \geq 3$ . If a SWFL satisfies PI, SP, BIIA and OMNC then it must be lexicographic.*

## 8 POSITIONAL DICTATORSHIPS

### 9 STRATEGIC VOTING: STRATEGY-PROOFNESS

In this section, we shall limit our attention (for the sake of convenience) to the case where each voters's preference ordering is antisymmetric. Let  $\mathcal{P}$  denote the set of all antisymmetric orderings over the elements of  $A$ . Recall that for all  $P_i \in \mathcal{P}$ , and  $a, b \in A$ ,  $aP_ib$  signifies “ $a$  is strictly better than  $b$ ”. For all  $P_i \in \mathcal{P}$ , we will let  $\tau(P_i)$  denote the maximal element in  $A$  according to  $P_i$ . The existence of such an element is ensured by assuming, for instance that  $A$  is finite. Note that since  $P_i$  is antisymmetric,  $\tau(P_i)$  is unique whenever it exists. A preference profile  $P$  is an  $n$ -tuple  $P \equiv (P_1, \dots, P_n) \in \mathcal{P}^n$ .

**DEFINITION 27** *A social choice function (SCF) or voting rule is a mapping  $f : \mathcal{P}^n \rightarrow A$ .*

For any SCF  $f$ ,  $\text{Range } f = \{a : \exists P \in \mathcal{P}^n \text{ such that } f(P) = a\}$ .

**EXAMPLE 3** Let  $N = \{1, 2\}$ ,  $A = \{a, b, c\}$ . The table below defines a SCF. Voter 1's preferences are specified along rows and voter 2's along columns. Thus the SCF picks  $b$  when 1's preference is  $abc$  and 2's is  $bca$ .

	$abc$	$acb$	$bac$	$bca$	$cab$	$cba$
$abc$	$a$	$a$	$a$	$b$	$a$	$b$
$acb$	$a$	$a$	$a$	$c$	$a$	$c$
$bac$	$b$	$a$	$b$	$b$	$a$	$b$
$bca$	$b$	$c$	$b$	$b$	$c$	$b$
$cab$	$a$	$c$	$a$	$c$	$c$	$c$
$cba$	$b$	$c$	$b$	$c$	$c$	$c$

(1)

In the model, voters know their own preferences but not those of other voters. Given a SCF, sophisticated voters will realize that they may benefit by misrepresenting their preferences, i.e. by voting strategically. Of course their decision to do so will depend critically on *their beliefs* about how other voters will vote.

**DEFINITION 28** *A SCF  $f$  is manipulable at  $P \in \mathcal{P}^n$  by voter  $i$  via  $P'_i$  if  $f(P'_i, P_{-i})P_i f(P_i, P_{-i})$ .*

If  $f$  is manipulable at  $P$  by  $i$  via  $P'_i$ , then  $i$  strictly gains according to her true preference  $P_i$  by misrepresenting her preferences via  $P'_i$  if she believes that the voters are going to announce  $P_{-i}$ . In the example above, the SCF is manipulable at profile  $(acb, cab)$  by voter 2 via  $bca$ .

**DEFINITION 29** *A SCF is strategy-proof (or dominant-strategy incentive compatible) if it is not manipulable by any voter at any profile.*

If a SCF is strategy-proof, then no voter has an incentive to misrepresent *irrespective* of their beliefs about the way that other voters will vote.

QUESTION: What SCFs are strategy-proof?

**DEFINITION 30** *A SCF  $f$  is dictatorial if there exists a voter  $i$  such that for all profiles  $P$ ,  $f(P) = \tau(P_i)$ .*

It is trivial to verify that a dictatorial SCF is strategy-proof. Any others? Yes, for instance the constant SCF which always picks the same alternative  $a$  at all profiles  $P$ . This is clearly not an interesting SCF. We shall therefore assume that the range of  $f$  is non-trivial; in particular that  $\text{Range } f = |A|$ . Before we proceed to our main result, we establish a useful result.

**DEFINITION 31** *A SCF  $f$  satisfies unanimity if, for all profiles  $P$  and alternatives  $a$ , we have  $f(P) = a$  whenever  $\tau(P_i) = a$  for all  $i \in N$ .*

A SCF respects *consensus*. If all voters agree that  $a$  is their maximal alternative at a particular profile, then  $f$  must pick it at that profile.

**PROPOSITION 10** *Let  $f$  be a strategy-proof SCF satisfying the condition  $\text{Range } f = |A|$ . Then  $f$  satisfies unanimity.*

*Proof:* Pick  $a \in A$  and a profile  $\bar{P}$  such that  $\tau(\bar{P}_i) = a$ . Since  $\text{Range } f = |A|$ , there exists a profile  $P$  such that  $f(P) = a$ . We will progressively switch voter preferences from  $P$  to  $\bar{P}$  and argue that the value of  $f$  is  $a$  all along the way. If  $f(\bar{P}_1, P_{-1}) = b \neq a$ , then voter 1 manipulates at  $(\bar{P}_1, P_{-1})$  via  $P_1$  because  $a \bar{P}_1 b$ . Hence  $f(\bar{P}_1, P_{-1}) = a$ . By an identical argument,  $f(\bar{P}_1, \bar{P}_2, P_3, \dots, P_n) = a$ ; otherwise voter 2 manipulates at  $(\bar{P}_1, \bar{P}_2, P_3, \dots, P_n)$  via  $P_2$ . Continuing in this way, it follows that  $f(\bar{P}) = a$ ; i.e.  $f$  satisfies unanimity. ■

**THEOREM 5 (Gibbard-Satterthwaite Theorem)** *Assume  $|A| \geq 3$ . A SCF  $f$  satisfies  $\text{Range } f = |A|$  and strategy-proofness if and only if it is dictatorial.*

*Proof:* The sufficiency part of the Theorem is trivial. We only prove necessity viz. if  $f$  satisfies strategy-proofness and the full range condition, then it must be dictatorial. In view of 10, we can substitute unanimity instead of the range condition in the statement above.

We will prove the result by induction on the number of voters  $n$ . ■