

EFFICIENT ITERATIVE COMBINATORIAL AUCTIONS *

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Abstract

This survey considers the following setting. A seller wants to sell a set of indivisible goods. A set of buyers are interested in buying the goods. Buyers have private values for bundles of goods they are interested in buying. The seller's objective is to allocate the goods to the buyers efficiently (maximizing the total utility of buyers). Since the values of buyers are private information, this creates an incentive problem. Economic theory prescribes the Vickrey-Clarke-Groves (VCG) mechanism to solve this problem. However, the VCG suffers from various drawbacks because of its centralized implementation. An alternative decentralized way of implementing the VCG is the iterative combinatorial auctions. We survey the foundations of iterative combinatorial auctions that implement the VCG mechanism. We also describe a particular iterative combinatorial auction that implements the VCG mechanism.

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1 INTRODUCTION

Auctions are a popular way of selling goods - some examples include auctioning of flowers in the Netherlands, auctioning of advertising slots on search engines such as Yahoo! and Google, and auctioning of various goods on eBay. Typically, auctions come in two variants - (a) sealed-bid, where bidders submit bids only once and (b) iterative auctions, which are characterized by monotonic price adjustments given the demands of buyers at every price. Iterative auctions are preferred over their sealed-bid counterparts because of price discovery, decentralized nature, valuation privacy, and better revenue and efficiency properties (Cramton, 1998).

Traditionally, auction theory in economics looks at the theory of selling a single indivisible good to multiple buyers (Krishna, 2002). But this changed dramatically with the success of spectrum auctions in US, where multiple spectrums were allocated to buyers simultaneously. This also inspired many applications of auctions in which multiple objects were sold simultaneously to buyers - auction of airport landing slots, auction of truckloads in transportation industry, auction of bus routes in London, and auction of industrial procurement (Ball et al., 2006). Such auctions where multiple indivisible objects were sold simultaneously is commonly called the **combinatorial auctions** because bidders have values on *bundles* of objects as opposed to individual objects. Because objects may be either complements or substitutes, the necessity to elicit bids on bundles of objects is clear - for instance, if there is a computer and a keyboard for sale, a bidder may have value 10 and 5 respectively on each of them, but may have a value of 20 on the bundle of computer and keyboard.

With the success of combinatorial auctions in practice, researchers started investigating research issues in this area. Two strands of literature started - (1) computational, which looked at various computational problems associated with combinatorial auctions and (2) auction theoretic, which looked at designing combinatorial auctions assuming participants are rational selfish agents. This survey tries to touch on some issues which can be described at the interface of these two strands of literature.

This survey is *not about* “optimal auction design” literature, which asks the question how to design auctions for selling multiple objects so that the expected revenue of the auctioneer is maximized. This problem was addressed by Myerson (1981) in his seminal paper for the single object case, but its solution for the multiple object case remains illusive till date. As we explain later, this survey is about designing auctions for selling multiple objects that achieve economic efficiency.

A standard model in auction theory to model the sale of a single indivisible object is that of private values, where each bidder (buyer/agent) has a value which is his private information, but the seller does not know the value of buyers. The seller would like to sell the object to the highest-valued bidder (economic efficiency). But unless the auction procedure is properly designed, bidders may lie about their value to get more profit. To remedy this, auction theory prescribes the second-price auction, where the good is sold to

the bidder with the highest bid but charged a price equal to the second highest bid. It can be shown that bidding the true value is a dominant strategy for every bidder ¹.

However, this auction is impractical for many reasons (Rothkopf et al., 1990), and is rarely used in practice. However, its iterative variant (sometimes referred to as the *English auction*) is widely used in practice (for example, Sotheby's auctions, auctions on eBay website). The auction is iterative and maintains a price in each iteration. It asks buyers if they are willing to buy the good at the current price. If more than one buyer is interested in the good at the current price, then the price is raised. Else, the auction terminates with the only interested buyer winning the good at the current price. Because of its transparency in price discovery, this auction is more practical than the second-price auction, though both the auctions achieve the same outcome in equilibrium.

This inspired researchers to think of the generalization of the English auction for the combinatorial auctions setting. The first such paper was in mid-80s, by Demange et al. (1986) on a specific combinatorial auction setting. We investigate some recent work on general combinatorial auction setting. We will mainly focus on the works of Mishra and Parkes (2007), Bikhchandani and Ostroy (2002), Ausubel and Milgrom (2002), Parkes and Ungar (2000), de Vries et al. (2007), and Ausubel (2004).

The reason for surveying this branch of literature is twofold: (a) the fundamentals of this literature is linear and integer programming, and someone with a good operations research background will enjoy this application; (b) the motivation of this literature is economic, in the sense that the auctions we will talk about have simple equilibrium bidding strategies for bidders to follow. In the process, we will leave out a large literature on iterative auctions which did not consider the economic motivations, and was purely inspired by computational and practical considerations.

Critical to the survey is the presence of strategic agents who are payoff maximizers. What we mean by efficient is that in every state of the world we achieve efficiency. This is possible only by considering the appropriate incentive schemes as we discuss. There are many iterative auctions which achieve efficiency in the presence of truthful bidders (i.e. bidders are assumed to behave truthfully even in the absence of appropriate incentive schemes), for example the Clock auctions in Ausubel and Milgrom (2006b). However, we do not treat this as efficient auctions in this survey. The readers are referred to the classic book Ausubel and Milgrom (2006a) on this topic.

¹There is a significant literature in auction theory which considers interdependent valuations - see Krishna (2002). In such a model, every bidder is not sure of his valuations and it depends on the private signals of all the bidders. There are results in this model for simple settings, but not much is known for multiple object auction case. Interested readers may find Krishna (2002) useful.

2 THE MODEL AND BACKGROUND

The abstract model of the survey is defined in this section. We denote $N = \{1, \dots, n\}$ to be the set of n buyers (bidders/agents) and $M = \{1, \dots, m\}$ to be the set of m indivisible goods. Note that the goods in M can be anything: they can be identical (homogeneous) or heterogenous or both. Let $\mathbb{S} = \{S : S \subseteq M\}$ be the set of all bundles of good. Every buyer has a valuation function. The valuation function of buyer i is $v_i : \mathbb{S} \rightarrow \mathbb{R}_+$. The valuation function of buyer i is completely known to buyer i but not known to the seller who is selling the goods. We make the following assumptions about the valuation function.

- **ZERO OUTSIDE OPTION.** For every $i \in N$, $v_i(\emptyset) = 0$.
- **BUNDLE MONOTONICITY (FREE DISPOSAL).** For every $i \in N$ and every $S, T \subseteq \mathbb{S}$ with $S \subseteq T$, we have $v_i(S) \leq v_i(T)$.

Such valuation functions are very general. In particular settings, there are succinct ways to represent a valuation function (Lehmann et al., 2006a). We give two such examples below. Let $v : \mathbb{S} \rightarrow \mathbb{R}_+$ be a general valuation function which satisfies zero outside option and bundle monotonicity.

1. **UNIT DEMAND.** A valuation function v is a **unit demand** valuation function if for every $S \in \mathbb{S}$ we have $v(S) = \max_{j \in S} v(\{j\})$. In this case, a buyer is interested in at most one good. So, when presented with a bundle of goods, he picks the one with the highest value amongst those goods. For example, consider a potential buyer is choosing to buy a house amongst a set of houses. Such a valuation function can be very succinctly expressed: every valuation vector can only take m values - one for each good.
2. **HOMOGENEOUS GOODS.** A valuation function v is a **homogeneous goods** valuation function if for every $S, T \in \mathbb{S}$ with $|S| = |T|$, we have $v(S) = v(T)$. In this case, goods are identical. So, only the size of the bundle dictates the value. Here also, the value function can only take m values, and thus can be succinctly represented.

Below, we give an example to illustrate how valuation functions may look like in a combinatorial auction model. Suppose there are three agents $\{1, 2, 3\}$ and two goods $\{1, 2\}$. A possible valuation profile is shown in Table 1.

There are two parts to the problem: (a) an allocation rule (who should get which bundles) (b) a payment rule (who should pay how much). While there are many allocation rules possible, the one we discuss here (and which is the center of attention in the literature) is the efficient allocation rule. An allocation $X = (X_0, X_1, \dots, X_n)$ is partitioning of the set of goods where $X_i \in \mathbb{S}$ denotes the bundle allocated to bidder i and X_0 is the unallocated

	\emptyset	$\{1\}$	$\{2\}$	$\{1, 2\}$
v_1	0	8	9	12
v_2	0	6	8	14
v_3	0	6	5	6

Table 1: Valuations in a combinatorial auction

set of goods. Let \mathbb{X} be the set of all allocations. Given a profile of valuation functions $v = (v_1, \dots, v_n)$, an allocation X is efficient if

$$\sum_{i=1}^n v_i(X_i) \geq \sum_{i=1}^n v_i(Y_i) \quad \forall Y \in \mathbb{X}.$$

An efficient allocation rule chooses an efficient allocation for every profile of valuations. Computing an efficient allocation for a given valuation profile is a computationally hard problem (Rothkopf et al., 1998), but some special cases can be solved easily (Lehmann et al., 2006b).

A payment rule on the other hand decides how much a bidder pays. If a bidder i has valuation function v_i and he is allocated bundle X_i and pays p_i , then his net utility is given by $v_i(X_i) - p_i$. Such form of utility function is called the **quasi-linear** utility function.

The problem is that the seller has no idea about the valuation functions of the buyer. So, he elicits this information from the buyers in various forms. For example, he may ask the buyers directly the valuation function. This is referred to as the *direct* mechanism in auction theory. For many purposes, it is without loss of generality to focus on the direct mechanisms. This is known as the *revelation principle* - for a formal definition of revelation principle, the readers are referred to any standard book in microeconomics, e.g., (Mas-Colell et al., 1995).

However, there is an incentive problem here. The buyer may get higher net utility by lying about his valuation function. For this, the payment and allocation rules need to be designed carefully. Fortunately, for the efficient allocation rule, there is a family of payment functions which can overcome this incentive problem. These are called the **Groves payment rules** (Groves, 1973). Suppose $v = (v_1, \dots, v_n)$ is the valuation profile of the buyers. Denote by v_{-i} the valuation profile of buyers other than i . Then, payment of buyer i in a Groves payment rule is given by

$$p_i^g(v) = h_i(v_{-i}) - \sum_{j \neq i} v_j(X_j^*),$$

where X^* is an efficient allocation and h_i is any arbitrary function from valuations of other buyers to real numbers. A particular choice of h_i function stands out. Suppose $h_i(v_{-i}) = \max_X \sum_{j \neq i} v_j(X_j)$. Then the payment function becomes

$$p_i^{vcg}(v) = \max_X \sum_{j \neq i} v_j(X_j) - \sum_{j \neq i} v_j(X_j^*).$$

This particular payment rule is popularly known as the **pivotal** payment rule or the **Vickrey-Clarke-Groves (VCG)** payment rule (Vickrey, 1961; Clarke, 1971; Groves, 1973). The VCG payment rule along with the efficient allocation rule is called the **VCG mechanism**. Note that the net utility of buyer i in the VCG mechanism is given by

$$\pi_i^{vcg}(v) = v_i(X_i^*) - p_i^{vcg}(v) = \max_X \sum_{j \in N} v_j(X_j) - \max_X \sum_{j \neq i} v_j(X_j).$$

In other words, every buyer gets a net utility equal to his marginal contribution - the increase in total value of all the buyers when he participates in the auction. In the example in Table 1, the efficient allocation X^* is to allocate good 1 to buyer 1 and good 2 to buyer 2. So the externality of buyer 3 is zero. The externality of buyer 1 is $(8 + 6) - 8 = 6$ and that of buyer 2 is $(9 + 6) - 8 = 7$. Hence, the VCG payments of buyers 1, 2, and 3 are 6, 7, and zero respectively.

The VCG mechanism is central in economic theory due to various reasons (Moulin, 1986). For example, it satisfies the following properties.

1. **IMPLEMENTS EFFICIENCY.** Every buyer has no incentive to lie about their valuations in the VCG mechanism, i.e., bidding truthfully is a dominant strategy.
2. **INDIVIDUAL RATIONALITY.** The VCG mechanism satisfies the property that every buyer has non-negative net utility. To see this, note that $\max_X \sum_{j \in N} v_j(X_j) \geq \max_X \sum_{j \neq i} v_j(X_j)$, and hence, $\pi_i^{vcg}(v) \geq 0$.
3. **LOSERS PAY NOTHING.** In the VCG mechanism, if a buyer does not win any bundle, then he pays nothing. This follows from the zero outside option and bundle monotonicity assumptions.
4. **NON-NEGATIVE PAYMENTS.** Every buyer pays non-negative amount in the VCG mechanism, i.e., no buyer is awarded or given money. This follows from the fact that $\max_X \sum_{j \neq i} v_j(X_j) \geq \sum_{j \neq i} v_j(X_j^*)$.

However, the VCG mechanism suffers from various practical drawbacks also (Rothkopf et al., 1990; Ausubel and Milgrom, 2006a; Rothkopf, 2007). We list some below.

1. **COMPUTATIONAL.** The VCG mechanism requires one to solve $n + 1$ optimization problems. These optimization problems are computationally hard for arbitrary valuation functions (Rothkopf et al., 1998).
2. **COMMUNICATION.** The VCG mechanism requires that every buyer communicates his entire valuation function to the seller. The fact that every buyer's valuation function can have exponential number of bundle values makes this communication difficult.

3. PRICE DISCOVERY AND PRIVACY. Irrespective of the compelling theoretical property of dominant strategy incentive compatibility, one does not expect buyers to reveal their true values in any mechanism. Privacy of valuations is an important aspect for buyers in practice. Further, the natural process of price discovery is absent in the (sealed-bid) VCG mechanism.

As [Ausubel and Milgrom \(2006a\)](#) point out there are some economic issues with the VCG mechanism too - losing bidders have incentive to collude, the outcome in the VCG mechanism is not *stable* etc. However, these are inherent problems in any Groves mechanism (which implements the efficient allocation rule). For example, [Rastegeri et al. \(2010\)](#) point out that any Groves mechanism will not be *stable* for a large class of valuation functions. Further, whenever these economic issues are not present for any Groves mechanism, the VCG mechanism are the natural choice for use.

It is difficult to come up with an *algorithm* to overcome the computational and communication overheads of the VCG mechanism. However, algorithms can be designed to implement (achieve) the VCG mechanism which are more transparent (and respects privacy to some extent) than the VCG mechanism we just described. These algorithms are precisely the **iterative VCG combinatorial auctions**. Since they implement the same allocation (efficient) and the same payment (VCG) as the VCG mechanism they inherit the incentive properties of the VCG mechanism. One needs some restrictions on the algorithm/iterative auction to inherit the incentive properties of the VCG mechanism - see [Mishra and Parkes \(2007\)](#). We do not discuss this issue in detail here. In brief, we need some activity rules in the auction to ensure that every bidder bids in a *myopically optimal* manner (as we will see later), and this leads to the VCG outcome. With these activity rules in place, it is also optimal (in a global sense) for every bidder to bid in this manner. We will not talk about the incentive issues in these iterative VCG combinatorial auctions, and assume that bidders are always truthfully bidding. But, we keep in mind that truthful bidding is an equilibrium in these auctions under some mild restrictions on bidding activity.

3 FOUNDATIONS OF ITERATIVE VCG COMBINATORIAL AUCTIONS

We start by laying the foundations of designing iterative VCG combinatorial auctions. Let us first recall that for the single object case, the Vickrey (second-price) auction has a counterpart in the iterative auction world - the English auction. This auction sheds light into how one can generalize this to the multiple object case. Suppose $v_1 \geq v_2 \geq \dots \geq v_n \geq 0$ be the values of n buyers for the single indivisible good. The English auction starts from a low price (say zero) and asks every buyer if it is interested in the object at this price. If more than one buyer is interested in the good, then the price is increased, and the process is repeated till exactly one buyer is interested. At this point, this buyer wins the good at the current price. It can be shown the buyers have no incentive to lie in this auction, i.e., buyers should express interest in the good till the price reaches their respective valuations. Further, if the amount

of price increase is low enough, the auction terminates at the second-highest valuation, and hence implements the same outcome as the Vickrey auction. There are other interesting features of this auction.

- a) The buyers see a single price (in \mathbb{R}_+) which is increased in every iteration, and this makes the auction very transparent.
- b) Call a price a *competitive equilibrium* price if exactly one buyer demands the object at that price. Any price between v_1 and v_2 is a competitive equilibrium price. Indeed, the auction terminates at the lowest such competitive equilibrium price (ignoring issues of ties). Moreover, such an equilibrium price satisfies the property that if we remove any one of the buyers, this remains an equilibrium price of the new economy as well: v_2 is a competitive equilibrium price of an economy without any buyer. As we will see next, that such pricing is pivotal in designing iterative VCG combinatorial auctions.

3.1 COMPLEX PRICING

Unfortunately, for general combinatorial auction models, the simple pricing of the English auction for the single object auction model is not enough. One needs richer prices (Mishra and Parkes, 2007). These richer and complex prices are essential even without strategic considerations, i.e., to simply compute efficiency (Blumrosen and Nisan, 2010). We will mainly be concerned with four types of prices.

1. **LINEAR AND ANONYMOUS.** In this pricing, a price is maintained for every good and every buyer sees the same price. The price of a bundle is the sum of prices of goods in that bundle. Formally, a linear and anonymous price p is a vector in \mathbb{R}_+^m , where $p(j)$ indicates the price of good $j \in M$ and price of a bundle of goods S is just $\sum_{j \in S} p(j)$.
2. **LINEAR AND NON-ANONYMOUS.** In this pricing, a price is maintained for every good and every buyer, i.e., every buyer sees a different linear price vector. Formally, a linear and non-anonymous price p is a vector in $\mathbb{R}_+^{m \times n}$, where $p_i(j)$ indicates the price of good $j \in M$ for buyer i , and price of a bundle of goods S for buyer i is just $\sum_{j \in S} p_i(j)$.
3. **NON-LINEAR AND ANONYMOUS.** In this pricing, a price is maintained for every bundle of good and every buyer sees the same price. Formally, a non-linear and anonymous price p is a vector in $\mathbb{R}_+^{|S|}$, where $p(S)$ indicates the price of bundle of goods S .
4. **NON-LINEAR AND NON-ANONYMOUS.** This is the most complex of pricing schemes. In this pricing, a price is maintained for every bundle of good and every buyer sees a different pricing price vector. Formally, a non-linear and non-anonymous price p is a vector in $\mathbb{R}_+^{|S| \times n}$, where $p_i(S)$ indicates the price of bundle of goods S for buyer i .

Such different types of pricing schemes were first discussed in [Bikhchandani and Ostroy \(2002\)](#). As we will discuss, most combinatorial auction models require non-linear and non-anonymous prices for iterative VCG combinatorial auctions. However, there are instances where simpler prices such as linear and anonymous prices is sufficient to design iterative VCG combinatorial auctions.

3.2 COMPETITIVE EQUILIBRIUM PRICES

We will define some notions in this section. These notions are defined for non-linear and non-anonymous prices, but can be defined for other types of prices as well. From now on, we fix a profile of valuation functions $v = (v_1, \dots, v_n)$. For any price $p \in \mathbb{R}^{|\mathbb{S}| \times n}$, define the **(maximum) payoff** of buyer i as

$$\pi_i(p) = \max_{S \in \mathbb{S}} [v_i(S) - p_i(S)]$$

and the **demand set** of buyer i as

$$D_i(p) = \{S \in \mathbb{S} : v_i(S) - p_i(S) = \pi_i(p)\}.$$

Similarly, for any price $p \in \mathbb{R}^{|\mathbb{S}| \times n}$, the **revenue** of the seller is

$$\pi^s(p) = \max_{X \in \mathbb{X}} \sum_{i=1}^n p_i(X_i)$$

and the **supply set** of the seller is

$$F(p) = \{X \in \mathbb{X} : \sum_{i=1}^n p_i(X_i) = \pi^s(p)\}.$$

Like in any economic equilibrium, the supply and demand should match. Here, this translates to saying that we can pick bundles in the demand sets of buyers in such a manner that it leads to an allocation in the supply set of the seller.

DEFINITION 1 *A price vector p and an allocation X form a **competitive equilibrium** if $X_i \in D_i(p)$ for all $i \in N$ and $X \in F(p)$. If (p, X) is a competitive equilibrium then p is called a **competitive equilibrium (CE) price vector**.*

A competitive equilibrium always exists if price vector is chosen to be non-linear and non-anonymous: choose X to be an efficient allocation and p_i to be equal to v_i for all $i \in N$. If price vectors are restricted to be linear and anonymous, then a CE exists under a condition called *gross substitutes* ([Crawford and Kelso, 1982](#); [Gul and Stacchetti, 1999](#)). Gross substitutes condition requires goods to be substitutes of each other. This condition is

is satisfied for unit demand valuations and for homogeneous goods valuations if the marginal value of a unit is non-increasing.

If (p, X) is a CE, then X is an efficient allocation. This fact can be shown by formulating the problem of finding an efficient allocation as a linear program. To do so, define for every $i \in N$ and every $S \in \mathbb{S}$ a binary variable $x_i(S) \in \{0, 1\}$, where $x_i(S) = 1$ indicates that buyer i gets bundle S and $x_i(S) = 0$ indicates that buyer i does not get bundle S . Also, define a binary variable $z(X) \in \{0, 1\}$ for every $X \in \mathbb{X}$ which indicates if allocation X is chosen or not. Then, the problem of finding an efficient allocation can be formulated as follows (Bikhchandani and Ostroy, 2002).

$$\begin{aligned}
 V(N) &= \max_{x,z} \sum_{i=1}^n \sum_{S \in \mathbb{S}} v_i(S) x_i(S) \\
 \text{s.t.} & \\
 \sum_{S \in \mathbb{S}} x_i(S) &= 1 \quad \forall i \in N \\
 \sum_{X \in \mathbb{X}} z(X) &= 1 \\
 x_i(S) &= \sum_{X \in \mathbb{X}: X_i=S} z(X) \quad \forall i \in N, \forall S \in \mathbb{S} \\
 x_i(S) &\in \{0, 1\} \quad \forall i \in N, \forall S \in \mathbb{S} \\
 z(X) &\in \{0, 1\} \quad \forall X \in \mathbb{X}.
 \end{aligned} \tag{IP}$$

It is easy to see that (IP) finds an efficient allocation. The constraints of (IP) ensure that an allocation is chosen and the objective function maximizes the total value of buyers over all allocation. As we will show later, there is another simpler formulation than formulation (IP) possible - see formulation (IP-E). However, formulation IP has some nice features as we discuss below.

The linear programming relaxation of (IP) admits integer solution (Bikhchandani and Ostroy, 2002). Hence, the problem of finding an efficient allocation reduces to the following linear

program.

$$\begin{aligned}
V(N) &= \max_{x,z} \sum_{i=1}^n \sum_{S \in \mathbb{X}} v_i(S) x_i(S) \\
\text{s.t.} & \\
\sum_{S \in \mathbb{S}} x_i(S) &= 1 \quad \forall i \in N \\
\sum_{X \in \mathbb{X}} z(X) &= 1 \\
x_i(S) &= \sum_{X \in \mathbb{X}: X_i=S} z(X) \quad \forall i \in N, \forall S \in \mathbb{S} \\
x_i(S) &\geq 0 \quad \forall i \in N, \forall S \in \mathbb{S} \\
z(X) &\geq 0 \quad \forall X \in \mathbb{X}.
\end{aligned} \tag{P}$$

This linear program has exponential (in number of goods and buyers) number of variables and constraints. So, in practice, it is not useful. However, it serves as an ideal tool to design iterative VCG combinatorial auction. For this, it is useful to look at its dual given below.

$$\begin{aligned}
V(N) &= \min_{p,\pi,\pi^s} \sum_{i \in N} \pi_i + \pi^s \\
\text{s.t.} & \\
p_i(S) + \pi_i &\geq v_i(S) \quad \forall i \in N, \forall S \in \mathbb{S} \\
\pi^s - \sum_{i \in N} p_i(X_i) &\geq 0 \quad \forall X \in \mathbb{X}
\end{aligned} \tag{DP}$$

The interpretation of the dual variables are clear: p is the price vector, π is the payoff vector of buyers, and π^s is the revenue of the seller. The following result establishes the connection between CE and these formulations.

THEOREM 1 (Bikhchandani and Ostroy (2002)) *A pair (p, X) is a competitive equilibrium if and only if X corresponds to the optimal solution of (P) and p corresponds to the optimal solution of (DP).*

The proof of this result involves writing down the complementary slackness conditions and showing their equivalence to the CE conditions. A consequence of this result is that finding an efficient allocation is equivalent to finding a CE price vector. Of course, a CE price vector may be an exponential (in number of goods) sized vector. But the theorem also works in settings where simpler (anonymous and linear) CE price vectors exist. For example, in unit-demand settings, anonymous and linear CE price vectors exist, and there finding an efficient allocation is equivalent to finding such a CE price vector.

The foundations of iterative auctions are CE price vectors, in the sense that every iterative auction (appropriately defined) searches for a CE price vector, and finds an efficient

allocation. However, the incentive properties of the VCG mechanism is not inherited by every CE price vector. As we will show next, there is a large class of CE prices from which VCG payments can be deduced. However, there are CE price vectors which correspond directly to the VCG payments if a condition on valuation function is satisfied. We discuss two such conditions. For every $K \subseteq N$, let $V(K)$ denote the objective function value of the optimal solution of **(P)**, when **(P)** is solved by considering agents in K only.

DEFINITION 2 Agents are substitutes if $V(N) - V(K) \geq \sum_{j \in N \setminus K} [V(N) - V(N \setminus \{j\})]$ for all $K \subseteq N$. **Agents are submodular** if $V(T) - V(T \setminus \{j\}) \geq V(K) - V(K \setminus \{j\})$ for all $T \subseteq K \subseteq N$ and for all $j \in N$.

It is easy to see that if agents are submodular, then agents are substitutes. However, the converse is not true ([Mishra and Parkes \(2007\)](#) contains examples).

We say that a CE price vector is **minimal** if it is a solution to the following linear program (**DPMIN**).

$$\begin{aligned} \Pi^{max} &= \max_{p, \pi, \pi^s} \sum_{i \in N} \pi_i \\ \text{s.t.} & \\ & \sum_{i \in N} \pi_i + \pi^s = V(N) \\ & p_i(S) + \pi_i \geq v_i(S) \quad \forall i \in N, \forall S \in \mathcal{S} \\ & \pi^s - \sum_{i \in N} p_i(X_i) \geq 0 \quad \forall X \in \mathbb{X} \end{aligned} \tag{DPMIN}$$

Formulation (**DPMIN**) finds a CE price vector which gives the maximum total payoff to agents over all CE price vectors. By maximizing the total payoff to agents, we also minimize their total payments over all CE price vectors. This results in finding the minimal CE price vector.

THEOREM 2 ([Bikhchandani and Ostroy \(2002\)](#)) *Agents are substitutes if and only if for every $i \in N$ we have*

$$p_i^{vcg}(v) = p_i^{min}(X_i),$$

where p^{min} is an optimal solution of (**DPMIN**) and X corresponds to an optimal solution of **(P)**, i.e., p^{min} is a minimal CE price vector and X is an efficient allocation.

In general, the VCG mechanism requires one to solve the linear program **(P)** $n + 1$ times. The implication of this theorem is that under agents are substitutes condition, one needs to solve only the linear programs **(P)** and (**DPMIN**) to find the outcomes of the VCG mechanism. The agents are substitutes condition is satisfied if goods are somewhat

substitutes (and not complements). For example, it is satisfied in the unit demand case and in the homogeneous goods case if the marginal value of a unit is non-increasing for buyers.

It is worth commenting that there is an integer programming formulation of **(P)**. Let $x_i(S) \in \{0, 1\}$ denote if agent i gets bundle S or not. Then an alternative formulation of finding an efficient allocation is as follows.

$$\begin{aligned}
 V(N) &= \max_x \sum_{i \in N} \sum_{S \in \mathbb{S}} v_i(S) x_i(S) \\
 \text{s.t.} & \\
 \sum_{i \in N} \sum_{S \in \mathbb{S}: j \in S} x_i(S) &\leq 1 \quad \forall j \in M \\
 \sum_{S \in \mathbb{S}} x_i(S) &= 1 \quad \forall i \in N \\
 x_i(S) &\in \{0, 1\} \quad \forall i \in N, \forall S \in \mathbb{S}.
 \end{aligned} \tag{IP-E}$$

The constraints in **(IP-E)** ensures that no object is assigned to more than one buyer and every agent is assigned to exactly one bundle (possibly the empty bundle). The objective function maximizes total value from the allocation specified by the feasibility constraints. Though this is an integer programming formulation of the efficient allocation problem, it involves less number of constraints and variables than formulation **(P)**.

Theorem 2 is silent on two accounts with regard to the power of CE price vectors in terms of extracting the VCG mechanism information: (a) it does not say how *informative* non-minimal CE price vectors can be; (b) it does not say what happens when the agents are substitutes condition does not hold. These two difficulties can be overcome by generalizing the notion of CE prices.

3.3 UNIVERSAL COMPETITIVE EQUILIBRIUM PRICES

We introduce the notion of universal competitive equilibrium prices. It is a useful concept to analyze and design iterative VCG auctions. For any non-linear and non-anonymous price vector p , for every $K \subseteq N$ ($K \neq \emptyset$) let p^K denote the restriction of price vector p to the set of agents K . Define for every $i \in N$, $N_{-i} = N \setminus \{i\}$ and $\mathbb{N} = N \cup_{i \in N} N_{-i}$. An **economy** is defined by a set of agents N , a set of goods M , and the valuation functions $v \equiv \{v_i\}_{i \in N}$ of agents.

DEFINITION 3 *A competitive equilibrium price vector p of economy (N, M, v) is a **universal competitive equilibrium (UCE)** price vector of this economy if p^K is a competitive equilibrium price vector for economy $(K, M, \{v_i\}_{i \in K})$ for every $K \in \mathbb{N}$.*

Note that the definition does not require the price vector to be non-linear and non-anonymous. A UCE price vector can be defined for any type of price vector. Similarly, the

	\emptyset	$\{1\}$	$\{2\}$	$\{1, 2\}$
v_1	0	3	0	3
v_2	0	0	6	6
v_3	0	0	2	4

Table 2: Valuations in a combinatorial auction

result we state relating the UCE price vector to the VCG mechanism is independent of the type of UCE price vector (i.e., the representation of the UCE price vector). A non-linear and non-anonymous UCE price vector always exists: let $p_i := v_i$ for all $i \in N$.

We give three examples of UCE prices below. We note that anonymous and linear UCE prices are available in the first two examples below.

1. **SINGLE GOOD.** In the single indivisible good case, there is a unique anonymous UCE price. Take an example with four buyers with values 10, 8, 6, 4 for the good. Then, any price in $[8, 10]$ is an anonymous CE price. However, the price 8 is the unique anonymous UCE price. To see this, note that 8 continues to be a CE price even if we remove any one of the buyers from the economy. As we know, 8 is also the price paid by the highest bidder in the second-price Vickrey auction.
2. **UNIT DEMAND.** In the unit demand valuation case, there is a unique linear and anonymous UCE price vector (Mishra and Parkes, 2009). Consider an example with two goods and three buyers. The valuations of the buyers on goods are given as follows: $v_1(1) = 5, v_1(2) = 3, v_2(1) = 3, v_2(2) = 4, v_3(1) = 2, v_3(2) = 2$. Since this is a unit demand setting, the valuations of buyers on bundles can be derived from this. It is easy to verify that the unique efficient allocation for this example is buyer 1 gets good 1 and buyer 2 gets good 2. The unique linear and anonymous UCE price vector is $(2, 2)$. At this price vector, the demand set of buyer 3 includes all possible bundles of goods, including \emptyset . The demand set of buyer 1 consists of good 1 and that of buyer 2 consists of good 2. Using these facts, one can easily verify that $(2, 2)$ is a linear and anonymous UCE price vector.
3. **GENERAL VALUATION.** Consider the example in Table 2 with three buyers and two goods. Consider the following non-linear and non-anonymous price vector for this example: $p_1(\{1\}) = 2 = p_1(\{1, 2\}), p_1(\{2\}) = 0, p_2(\{1\}) = 0, p_2(\{2\}) = 4 = p_2(\{1, 2\}), p_3(\{1\}) = 0, p_3(\{2\}) = 2, p_3(\{1, 2\}) = 4$. The efficient allocation for this example is buyer 1 gets good 1 and buyer 2 gets good 2. The reader can verify that this is a UCE price vector.

The following important theorem illustrates the relationship between UCE price vectors and VCG payments.

THEOREM 3 (Mishra and Parkes (2007)) *Suppose (p, X) is a CE of economy (N, M, v) . The VCG payments of agents in this economy can be computed from (p, X) if and only if p is a UCE price vector of this economy. Moreover, if p is a UCE price vector then for every $i \in N$,*

$$p_i^{vcg}(v) = p_i(X_i) - [\pi^s(p) - \pi^s(p^{N-i})]$$

This result says that UCE prices (with an efficient allocation) are necessary and sufficient for computing VCG payments from CE prices and efficient allocation. Moreover, it says that the bidders need to be given discounts from their CE prices to obtain VCG payments. Note that these discounts are zero if the UCE price vector is anonymous - this is the case for single good setting and the unit demand setting, where anonymous UCE price vectors are known to exist.

For the example in Table 2, a UCE price vector p is

$$\begin{aligned} p_1(\{1\}) &= 2 = p_1(\{1, 2\}), p_1(\{2\}) = 0 \\ p_2(\{1\}) &= 0, p_2(\{2\}) = p_2(\{1, 2\}) = 4 \\ p_3(\{1\}) &= 0, p_3(\{2\}) = 2, p_3(\{1, 2\}) = 4. \end{aligned}$$

The efficient allocation for this example is to allocate good 1 to buyer 1 and good 2 to buyer 2. We can compute $\pi^s(p) = 6, \pi^s(p^{N-1}) = 4, \pi^s(p^{N-2}) = 4, \pi^s(p^{N-3}) = 6$. Hence, the discounts of buyers 1, 2, and 3 are respectively, 2, 2, and zero. So, the VCG payments of buyers 1, 2, and 3 are $2 - 2 = 0, 4 - 2 = 2$, and zero respectively.

The result in Theorem 3 also gives hints on how to achieve the outcome of the VCG mechanism using CE prices - we need to search for UCE prices. Mishra and Parkes (2007) give a class of auctions using non-linear and non-anonymous prices for general valuations of bidders which search for UCE prices and achieves the VCG outcome. We revisit one of the auctions.

4 *i*BUNDLE - EXTEND AND ADJUST (*i*BEA) AUCTION

We present an iterative auction for general valuations of combinatorial auctions. The auction is an extension of the *i*Bundle auction in Parkes and Ungar (2000) and the auction in Ausubel and Milgrom (2002). This description follows along the lines of Mishra and Parkes (2007).

In every iteration of the auction, a non-linear and non-anonymous price vector is maintained. Denote such a price vector in round t of this auction as p^t . Given such a price vector, the building block of the auction is to solve an integer programming problem, commonly referred to as the *winner determination problem*.

4.1 THE WINNER DETERMINATION PROBLEM

In this section, we formalize the **winner determination problem (WDP)**. In a WDP, we are given a non-linear and non-anonymous price vector $p \in \mathbb{R}_+^{n \times |S|}$ and the demand sets of buyers at this price vector. The objective is to determine an allocation which *maximizes* the revenue of the seller at this price such that every buyer is either assigned a bundle from his demand set or the \emptyset bundle, i.e., find an allocation X at price vector p , such that $\pi^s(p) = \sum_{i \in N} p_i(X_i)$ subject to $X_i \in D_i(p) \cup \{\emptyset\}$. This problem can be formulated as an integer program as follows. Let $x_i(S) \in \{0, 1\}$ denote whether agent i is allocated a bundle $S \in D_i(p) \cup \{\emptyset\}$. The following integer program solves the WDP at price vector p .

$$\begin{aligned}
 \pi^s(p) &= \max_x \sum_{i \in N} \sum_{S \in D_i(p) \cup \{\emptyset\}} p_i(S) x_i(S) \\
 \text{s.t.} & \\
 \sum_{i \in N} \sum_{S \in D_i(p): j \in S} x_i(S) &\leq 1 \quad \forall j \in M \\
 \sum_{S \in D_i(p) \cup \{\emptyset\}} x_i(S) &= 1 \quad \forall i \in N \\
 x_i(S) &\in \{0, 1\} \quad \forall i \in N, \forall S \in D_i(p) \cup \{\emptyset\}.
 \end{aligned} \tag{WDP}$$

The constraints of formulation **(WDP)** are ensuring that every agent is assigned either a bundle from his demand set or the empty bundle, and every good is assigned to at most one agent. The objective function ensures that we choose an allocation which maximizes the revenue to the seller at this price vector. It is worth noting that formulation **(WDP)** becomes equivalent to formulation **(IP-E)** if the price vector is the same as the valuation vector.

An extensive literature exists on the computational aspects of the WDP - a good reference is [Lehmann et al. \(2006b\)](#). We want to note here that the WDP, in general, is NP-hard to solve, but some tractable instances exists ([Rothkopf et al., 1998](#); [Lehmann et al., 2006b](#)).

In the auction we discuss next, we will solve instances of WDP for any economy $K \in \mathbb{N}$. We will therefore denote WDP^K to denote the winner determination problem of economy with agents $K \in \mathbb{N}$. We denote the allocation chosen by the WDP^K at price vector p^K as X^{p^K} , and call it an **optimal allocation** at p^K . Let $L(p, K) = \{i \in K : X_i^{p,K} = \emptyset \text{ and } \emptyset \notin D_i(p^K)\}$ be the set of agents who do not get any bundle in the WDP^K and their demand sets do not contain the \emptyset . We call such agents **unsatisfied** agents at price vector p . The revenue of the seller from the WDP of economy with agents K at price vector p will be denoted as $\pi^s(p^K)$ - it is the value of the optimal solution of **(WDP)** at price vector p^K for economy with agents K .

4.2 DESCRIPTION OF THE *i*BEA AUCTION

For simplicity of description and analysis, we assume that the valuations of agents are integers. However, the ideas of the auction extends to arbitrary real-valued valuation functions. Assuming any real valuation brings additional convergence problems due to bid increments, for which require cumbersome analysis techniques. These problems disappear if one assumes integer valuations.

The *i*BEA auction orders the elements of the set \mathbb{N} . Suppose the ordering is N , followed by N_{-1} , followed by N_{-2}, \dots, N_{-n} . Denote $N \equiv N_{-0}$.

*i*BEA AUCTION

S0 Initiate the round number $t = 0$ and let p^0 be the zero price vector.

S1 For $i = 0, 1, \dots, n$

S1.1 Ask demand sets of buyers at price vector p^t .

S1.2 Solve WDP^{N-i} at price vector p^t (restricted to N_{-i}).

S1.3 If $L(p^t, N_{-i}) = \emptyset$ (i.e., no unsatisfied agents), then $i = i + 1$ and repeat from Step S1 with $p^{t+1} = p^t$ and $t = t + 1$.

S1.4 If $L(p^t, N_{-i}) \neq \emptyset$, then set $p_j^{t+1}(S) = p_j^t(S) + 1$ for all $j \in L(p^t, N_{-i})$ and for all $S \in D_j(p^t)$. Repeat from Step S1.1 with $t = t + 1$.

S2 Let X be the allocation chosen by WDP^{N-0} in the final round when $i = 0$. Let the final price vector in the auction be p . Then, the final allocation of the auction is X and payment of agent j is $p_j(X_j) - [\pi^s(p) - \pi^s(p^{N-j})]$ (discounts from the terminating prices).

The *i*BEA auction searches for a UCE price vector, and as we will show next, it terminates at a UCE price vector. We illustrate the auction with an example in Table 3. The last columns indicate seller's revenue in economy N (main economy) and *marginal economies* (economies with agents N_{-1} , N_{-2} , and N_{-3} respectively). Prices in parenthesis for a buyer and a bundle indicate that such a bundle is in the demand set of the respective buyer at that price. We see in Table 3 that *i*BEA solves WDP of main and marginal economies in every round and terminates at a UCE price vector. Discounts are then calculated for every bidder to compute the VCG payments.

We say that the *i*BEA auction **terminates after the main economy** if the auction does not see any price increase for $i > 0$ in Step S1, i.e., we reach a UCE price vector after finding a CE price vector of economy with all the agents. The main result is that this auction terminates with a UCE price vector and achieves the VCG outcome. However, under a variety of conditions, it terminates as soon as a CE price vector of main economy (economy with all the agents) is discovered.

Values	Buyer 1			Buyer 2			Buyer 3			Seller revenue in main and marginal economies
	{1}	{2}	{1, 2}	{1}	{2}	{1, 2}	{1}	{2}	{1, 2}	
1	3	0	3	0	6	6	0	2	4	{0,0,0,0}
	WDP of N . WDP selects allocation $\{\{1\}, \{2\}, \emptyset\}$. Buyer $\{3\}$ is unsatisfied.									
2	(0)	0	(0)	0	(0)	(0)	0	0	(1)	{1,1,1,0}
	WDP of N . WDP selects $\{\emptyset, \emptyset, \{1, 2\}\}$. Buyers $\{1, 2\}$ are unsatisfied.									
3	(1)	0	(1)	0	(1)	(1)	0	0	(1)	{2,1,1,2}
	WDP of N . WDP selects $\{\{1\}, \{2\}, \emptyset\}$. Buyer $\{3\}$ is unsatisfied.									
4	(1)	0	(1)	0	(1)	(1)	0	(0)	(2)	{2,2,2,2}
	WDP of N . WDP selects $\{\{1\}, \{2\}, \emptyset\}$. Buyer $\{3\}$ is unsatisfied.									
5	(1)	0	(1)	0	(1)	(1)	0	(1)	(3)	{3,3,3,2}
	WDP of N . WDP selects $\{\emptyset, \emptyset, \{1, 2\}\}$. Buyers $\{1, 2\}$ are unsatisfied.									
6	(2)	0	(2)	0	(2)	(2)	(0)	(1)	(3)	{4,3,3,4}
	WDP of N . WDP selects $\{\{1\}, \{2\}, \emptyset\}$. Buyer $\{3\}$ is unsatisfied.									
7	(2)	0	(2)	0	(2)	(2)	(0)	(2)	(4)	{4,4,4,4}
	CEs of economies with agents N , N_{-2} , and N_{-3} are reached. $\{\{1\}, \{2\}, \emptyset\}$ is an efficient allocation. WDP of N_{-1} . WDP selects $\{\emptyset, \emptyset, \{1, 2\}\}$. Buyer $\{2\}$ is unsatisfied.									
8	(2)	0	(2)	0	(3)	(3)	(0)	(2)	(4)	{5,4,4,5}
	WDP of N_{-1} . WDP selects $\{\emptyset, \emptyset, \{1, 2\}\}$. Buyer $\{2\}$ is unsatisfied.									
9	(2)	0	(2)	0	(4)	(4)	(0)	(2)	(4)	{6,4,4,6}
	A UCE price is reached. Final allocation: $\{\{1\}, \{2\}, \emptyset\}$. Final payment: $p_1(\{1\}) = 2 - [6 - 4] = 0$, $p_2(\{2\}) = 4 - [6 - 4] = 2$, $p_3(\emptyset) = 0$.									

Table 3: Progress of $iBEA$ for an example

THEOREM 4 *Suppose agents submit their demand sets truthfully in $iBEA$ auction. Then, the following statements are true.*

1. *The $iBEA$ auction terminates at a UCE price vector and the outcome is the same as that of the VCG mechanism (Mishra and Parkes, 2007).*
2. *If the agents are substitutes condition holds, then the $iBEA$ auction terminates after the main economy with the VCG outcome (Mishra and Parkes, 2007).*
3. *If the agents are submodular condition holds, then the $iBEA$ auction terminates after the main economy with the VCG outcome, and the discount given to every agent is zero (Ausubel and Milgrom, 2002; Mishra and Parkes, 2007).*

5 CONCLUDING REMARKS

GENERAL CLASS OF ASCENDING AUCTIONS. [Mishra and Parkes \(2007\)](#) discuss a more general class of ascending auctions for general valuation functions. This class contains the *i*BEA auction and generalizations of auctions in [de Vries et al. \(2007\)](#). Results (1) and (2) in [Theorem 4](#) hold for any auction in this general class. Result (3) hold for *i*BEA auction and the auction in [de Vries et al. \(2007\)](#). These auctions can be interpreted to be solving the linear program [\(P\)](#) using a primal dual algorithm. This connection is established explicitly in [de Vries et al. \(2007\)](#) - also see [Bertsekas \(1991\)](#) and [Parkes and Ungar \(2000\)](#)

EXPERIMENTAL EVIDENCE. In a recent paper, [Chen and Takeuchi \(2010\)](#) investigate how the *i*BEA auction performs in practice. They conduct experiments and compare its performance against the sealed-bid VCG auction. The experiment uses humans and robots to study these two auction formats. The main finding of this study is that while the sealed-bid VCG auction achieves greater revenue for the seller, the *i*BEA auction achieves greater bidder profit and efficiency. Further, the *i*BEA auction achieves 100 percent efficiency in significantly large instances than the sealed-bid VCG auction. This substantiates the use of *i*BEA over its sealed-bid counterpart.

SIMPLER PRICES. In general, the *i*BEA auction requires complex non-linear and non-anonymous prices. Such complex prices are necessary for general valuations, as we have argued earlier. For simple valuation functions, one may do away with such complex prices. We discuss two such valuation functions.

One is the case of unit demand valuations. There exists ascending auctions which achieves the VCG outcome using anonymous and linear prices ([Demange et al., 1986](#)).

The other is the case of **homogeneous goods with non-increasing marginal valuations (NIMV)**. In such a case, the goods are homogeneous, but the marginal value of a unit is non-increasing for every buyer. In this case, an ascending auction with simple prices exists, which achieves the VCG outcome ([Ausubel, 2004](#)). This auction implicitly maintains non-linear and non-anonymous prices ([Bikhchandani and Ostroy, 2006](#)). These non-linear and non-anonymous prices find a succinct representation in this case. It is an *open question* if such auctions with simple prices is possible in any other interesting valuation functions.

DESCENDING AUCTIONS. One wonders if the ascending price trajectories can be replaced by descending ones in these auctions. The answer is not trivial as the search for a UCE price vector has to be done carefully, specially if we have auctions which use simpler prices. [Mishra and Parkes \(2009\)](#) demonstrate that the UCE price concept can be used to design descending price auction counterparts of the auctions in [Demange et al. \(1986\)](#) and [Ausubel \(2004\)](#) for unit demand and homogeneous good NIMV settings respectively. These descending auctions maintain simple prices and achieve the VCG outcome. With complex non-linear

and non-anonymous prices, the descending auction counterpart of the i BEA auction can be found in [Mishra and Veeramani \(2007\)](#). However, this auction has the undesirable feature of terminating at “price=value” (i.e., $p_i(S) = v_i(S)$ for all $i \in N$ and for all $S \in \mathbb{S}$) UCE price vector often. It is an *open question* to design descending auction counterpart of the i BEA auction for special cases of valuation functions with simpler prices.

AUCTIONS FOR COMPLEMENTS. Though the i BEA achieves the VCG outcome for general valuations, one needs to run the i BEA auction even after achieving the CE price vector of the main economy if agents are substitutes condition does not hold. This is an undesirable feature since in the i BEA rounds for $i \neq 0$ in Step S1, the demand sets of bidder i is not used in the auction. Further, the bids of bidders in N_{-i} is only used to compute the payment of bidder i . These undesirable features are present if the agents are substitutes condition does not hold.

The agents are substitutes condition almost implies that goods are substitutes and not complements ([Bikhchandani and Ostroy, 2002](#); [Ausubel and Milgrom, 2002](#)). It is an *open question* on how to design iterative auctions which avoids this problem of i BEA when goods are complements. A conjecture is that it may be an auction which is a mixture of ascending and descending prices.

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