Instructor: **Debasis Mishra** Email: dmishra@isid.ac.in
Office: Rm 224, Planning Unit Phone: 011-4149 3948
Time: **Tuesdays and Thursdays 2:30 PM - 4:00 PM** Place: Room No. 14
Office hours: **Wednesdays 4:30 PM - 5:30 PM** or by appointment

Mathematical Programming
with Applications to Economics (M.S.Q.E.)
or
Optimization Techniques (M.Stat.)

Course webpage: http://www.isid.ac.in/∼dmishra/mp.html

Lectures - 17,18,19

1 Application: Envy-Free Allocation in Combinatorial Auctions

A **combinatorial auction** problem is described by a set of goods $N = \{1, \ldots, n\}$, a set of buyers $M = \{1, \ldots, m\}$, and the valuation function of each buyer $i \in M$, denoted by $v_i : 2^{|N|} \rightarrow \mathbb{R}_+$. The buyers have combinatorial values, i.e., for every **bundle** of goods S, every buyer $i \in M$ assigns a value $v_i(S)$.

A **partition** X of goods in N is $(X_1, \ldots, X_m)$ such that $X_i \subseteq N$ for all $i \in M$ and $X_i \cap X_j = \emptyset$ for all $i \neq j$. An **allocation** is a partition X and an assignment μ of every bundle in the partition to a unique player. So, $\mu_i(X)$ denotes the bundle assigned to buyer i.

Here is an example with $N = \{1, 2\}$ and $M = \{1, 2\}$. The valuations are given in Table 1 below. The possible partitions in the example in Table 1 are: $(\{1\}, \{2\})$, $(\{1, 2\}, \emptyset)$. Let

	$\{1\}$	$\{2\}$	$\{1,2\}$
$v_1(\cdot)$	8	6	12
$v_2(\cdot)$	5	9	15

Table 1: Example 1

$X = (\{1\}, \{2\})$. Then, an assignment μ can be $\mu_1(X) = \{1\}$ and $\mu_2(X) = \{2\}$. Another assignment μ' can be $\mu'_1(X) = \{2\}$ and $\mu'_2(X) = \{1\}$.

The goods are assigned to the buyers, and buyers make payments. The payment function of buyers is denoted by $p : M \to \mathbb{R}_+$, where $p(i)$ denotes the payment of buyer $i \in M$. If a buyer $i \in M$ gets a bundle S and pays an amound $p(i)$, then his payoff is $v_i(S) - p(i)$.

Definition 1 *An allocation (X, μ) is* **envy-free** *if there exists a payment function $p : M \to \mathbb{R}_+$ such that for every $i \in M$*

$$v_i(\mu_i(X)) - p(i) \geq v_i(\mu_j(X)) - p(j) \qquad \forall\, j \in M. \tag{\textbf{ENVY}}$$

The definition of envy-free allocation says that no buyer should be envious, in terms of payoff, of any other buyer, i.e., the payoff a buyer gets from his allocation and payment should be higher than the payoff he would have got from the allocation and payment of any other buyer.

For the example above, let us verify if (X, μ) (as defined earlier, $X = (\{1\}, \{2\})$ and $\mu_1(X) = \{1\}$ and $\mu_2(X) = \{2\}$) is envy-free. We will need to find $p(1)$ and $p(2)$ that satisfy

$$8 - p(1) \geq 6 - p(2)$$
$$9 - p(2) \geq 5 - p(1).$$

In other words $-4 \leq p(1) - p(2) \leq 2$. One possible solution is $p(1) = 2, p(2) = 0$.

Let Γ denote the set of all assignments.

Definition 2 *An allocation (X, μ) is* **p-efficient** *if for every assignment $\mu' \in \Gamma$, we have*

$$\sum_{i \in M} v_i(\mu_i(X)) \geq \sum_{i \in M} v_i(\mu'_i(X)) \tag{1}$$

In the example, allocation (X, μ) gives a total value of 17 to buyers, whereas allocation (X, μ') gives a total value of 11. Hence (X, μ) is p-efficient.

Theorem 1 *An allocation (X, μ) is envy-free if and only if it is p-efficient.*

Proof: For every $i, j \in M$ and every allocation (X, μ), denote $\delta_i(j) = v_i(\mu_i(X)) - v_i(\mu_j(X))$. We can write the inequality (**ENVY**) as

$$p(i) - p(j) \leq \delta_i(j) \qquad \forall\, i, j \in M. \tag{2}$$

The constraint graph corresponding to the difference constraints in (2) is as follows: (a) assign a vertex to every buyer, an edge between every pair of buyers in both directions - this results in a complete directed graph; (b) cost of edge from j to i is $\delta_i(j)$. Call this graph the **envy graph**. From our result relating difference constraints and constraint graph cycles, an allocation X is envy-free if and only if the corresponding envy graph has no negative cycles.

Note that the $p_j \geq 0$ for all $j \in M$ constraints are redundant since any envy-free solution can be scaled to get a non-negative solution.

Now, suppose an allocation (X, μ) is envy-free. Then the envy graph has no negative cycles. Suppose (X, μ) is not p-efficient. Then there exists some assignment μ' such that

$$\sum_{i \in M} [v_i(\mu_i'(X)) - v_i(\mu_i(X))] > 0. \tag{3}$$

Let $S = \{i \in M : \mu_i(X) \neq \mu_i'(X)\}$. Hence the previous inequality can be written as

$$\sum_{i \in S} [v_i(\mu_i(X)) - v_i(\mu_i'(X))] < 0. \tag{4}$$

But for every $i \in S$, $\mu_i'(X) = \mu_j(X)$ for some $j \neq i$. Without loss of generality let $S = \{1, \ldots, k\}$, $\mu_i(X) = X_i$, and $\mu_i'(X) = X_{i+1}$ if $i < k$ and $\mu_k'(X) = X_1$. So, inequality (4) can be rewritten as

$$\sum_{i=1}^{k-1} [v_i(X_i) - v_i(X_{i+1})] + [v_k(X_k) - v_k(X_1)] < 0 \tag{5}$$

$$\Rightarrow \sum_{i=1}^{k-1} \delta_i(i+1) + \delta_k(1) < 0. \tag{6}$$

Left hand side of inequality (6) is the length of cycle $1 \rightarrow k \rightarrow (k-1) \rightarrow \ldots \rightarrow 2 \rightarrow 1$. Hence, the envy graph has a negative cycle, which is a contradiction.

Now, suppose (X, μ) is p-efficient but not envy-free. Since (X, μ) is not envy-free, there is a negative cycle in the corresponding envy graph. Without loss of generality, let this cycle be $1 \rightarrow k \rightarrow (k-1) \rightarrow \ldots \rightarrow 2 \rightarrow 1$ and $\mu_i = X_i$. Then, define $\mu_i'(X) = X_{i+1}$ for $i < k$, $\mu_k'(X) = X_1$, and $\mu_i(X) = \mu_i'(X)$ for all other i. Because of negative cycle, we can write

$$\sum_{i=1}^{k-1} \delta_i(i+1) + \delta_k(1) < 0$$

$$\Rightarrow \sum_{i=1}^{k-1} [v_i(X_i) - v_i(X_{i+1})] + [v_k(X_k) - v_k(X_1)] < 0$$

$$\Rightarrow \sum_{i=1}^{k} v_i(\mu_i(X)) < \sum_{i=1}^{k} v_i(\mu_i'(X))$$

$$\Rightarrow \sum_{i \in M} v_i(\mu(X)) < \sum_{i \in M} v_i(\mu_i'(X)).$$

Hence, (X, μ) is not p-efficient, which is a contradiction. ■

We modify the previous example to illustrate Theorem 1. The new example has one more buyer (see Table 2). Let $X = (\{1\}, \{2\}, \emptyset)$ and $\mu_1(X) = \{2\}$, $\mu_2(X) = \emptyset$, and $\mu_3(X) = \{1\}$.

	$\{1\}$	$\{2\}$	$\{1,2\}$
$v_1(\cdot)$	8	6	12
$v_2(\cdot)$	5	9	15
$v_3(\cdot)$	7	7	14

Table 2: Example 2

The envy graph for this example is shown in Figure 1. Two cycles in this graph can be easily detected: $1 \to 2 \to 1$ and $1 \to 3 \to 1$. Hence, (X, μ) is not an envy-free allocation. This can also be verified from the fact that it is not a p-efficient allocation. The p-efficient allocation is (X, μ') where $\mu'_1(X) = \{1\}$, $\mu'_2(X) = \{2\}$, and $\mu'_3(X) = \emptyset$.

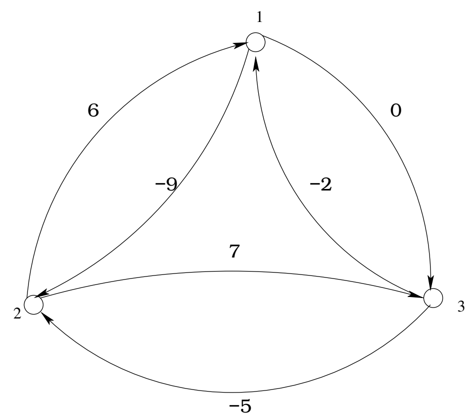

Figure 1: Envy graph for the example in Table 2

2 Application: Efficient Combinatorial Auctions

In this section, we continue to study the combinatorial auctions problem. We will make some subtle changes in notations. As before $N = \{1, \ldots, n\}$ is the set of goods and $M = \{1, \ldots, m\}$ is the set of buyers. Let $\Omega = \{S : S \subseteq N\}$ be the set of all bundles of goods. The valuation function of buyer $i \in M$ is $v_i : \Omega \to \mathbb{R}_+$. For buyer $i \in M$, $v_i(S)$ denotes the value on bundle $S \in \Omega$. We assume $v_i(\emptyset) = 0$ for all buyers $i \in M$.

An **allocation** $X = (X_1, \ldots, X_m)$ is a m dimensional vector such that $X_i \in \Omega$ for all $i \in M$, $X_i \cap X_j = \emptyset$ for all $i \neq j$, and $\cup_{i \in M} X_i \subseteq N$. In an allocation X, X_i denotes the bundle of goods assigned to buyer $i \in M$. So, the two differences from our earlier notation are (a) an allocation X indicates a partition and an assignment of goods and (b) not all goods need to be assigned in an allocation. Let $\mathbb{X}$ be the set of all allocation. An allocation

$X \in \mathbb{X}$ is **efficient** if

$$\sum_{i \in M} v_i(X_i) \geq \sum_{i \in M} v_i(Y_i) \qquad \forall\, Y \in \mathbb{X}.$$

As an example, we look at the example in Table 2. The efficient allocation for this example is $(\{1\}, \{2\}, \emptyset)$, meaning buyer 1 gets good 1, buyer 2 gets good 2, and buyer 3 gets nothing. This gives a total value of 17, which is higher than the total value obtained in any other allocation.

2.1 Formulation as an Integer Program

Our objective is to formulate the problem of finding an efficient allocation. The decision variable is: $x_i(S) \in \{0, 1\}$ for all buyers $i \in M$ and for all $S \in \Omega$. $x_i(S)$ should be 1 if buyer $i \in M$ is assigned bundle $S \in \Omega$, and zero otherwise. We should have two sets of constraints: (1) to ensure that every buyer gets some bundle of goods (may be the empty set) and (2) to ensure that every good is assigned to at most one buyer. The objective function maximizes the total value of buyers.

$$V(M, N; v) = \max \sum_{i \in M} \sum_{S \in \Omega} v_i(S) x_i(S)$$

$$\text{s.t.} \tag{CA-IP}$$

$$\sum_{S \in \Omega} x_i(S) = 1 \qquad \forall\, i \in M \tag{7}$$

$$\sum_{i \in M} \sum_{S \in \Omega : j \in S} x_i(S) \leq 1 \qquad \forall\, j \in N \tag{8}$$

$$x_i(S) \in \{0, 1\} \qquad \forall\, i \in M,\ \forall\, S \in \Omega. \tag{9}$$

The LP relaxation of formulation (**CA-IP**) does not always give integral solutions. However, if we restrict Ω to be only singleton bundles. i.e., buyers can be assigned at most one good (this is the assignment problem model we studied earlier), then the resulting constraint matrix becomes totally unimodular, and the LP relaxation always gives integral solution. Besides the assignment problem setting, there are other general settings where the LP relaxation of (**CA-IP**) gives integral solutions. The exact nature of these settings will be taught in an advanced mathematical economics course. Interested students can read Bikhchandani et al. (2002). These settings arise in specific types of valuation functions, and do not result in a TU constraint matrix.

We assume that the valuation functions are such that LP relaxation of (**CA-IP**) gives an optimal solution of formulation (**CA-IP**). Then, the efficient allocation problem can be reformulated as:

$$V(M,N;v) = \max \sum_{i \in M} \sum_{S \in \Omega} v_i(S) x_i(S)$$

$$\text{s.t.} \qquad \textbf{(CA-LP)}$$

$$\sum_{S \in \Omega} x_i(S) = 1 \qquad \forall\, i \in M \tag{10}$$

$$\sum_{i \in M} \sum_{S \in \Omega : j \in S} x_i(S) \leq 1 \qquad \forall\, j \in N \tag{11}$$

$$x_i(S) \geq 0 \qquad \forall\, i \in M,\ \forall\, S \in \Omega. \tag{12}$$

The dual of this formulation is:

$$V(M,N;v) = \min \sum_{i \in M} \pi_i + \sum_{j \in N} p_j$$

$$\text{s.t.} \qquad \textbf{(CA-DLP)}$$

$$\pi_i + \sum_{j \in S} p_j \geq v_i(S) \qquad \forall\, i \in M,\ \forall\, S \in (\Omega \setminus \emptyset) \tag{13}$$

$$\pi_i \geq 0 \qquad \forall\, i \in M \tag{14}$$

$$p_j \geq 0 \qquad \forall\, j \in N. \tag{15}$$

The dual variables have nice economic interpretations. We can think p_j to be the price of good j. Assume that if a buyer $i \in M$ gets a bundle of goods S, then he pays $\sum_{j \in S} p_j$, and the payoff he gets is $\pi_i(S) := v_i(S) - \sum_{j \in S} p_j$. Define $\pi_i(\emptyset) = 0$ for all $i \in M$. Hence, constraint (13) can be written as $\pi_i \geq \pi_i(S)$ for all $i \in M$ and for all $S \in \Omega$. Now, let us write the complementary slackness conditions. Let x be a feasible solution of (**CA-LP**) and (p, π) be a feasible solution of (**CA-DLP**). They are optimal if and only if

$$x_i(S)\Big[\pi_i - \pi_i(S)\Big] = 0 \qquad \forall\, i \in M,\ \forall\, S \in (\Omega \setminus \emptyset) \tag{16}$$

$$x_i(\emptyset)\pi_i = 0 \qquad \forall\, i \in M \tag{17}$$

$$p_j\Big[1 - \sum_{i \in M} \sum_{S \in \Omega : j \in S} x_i(S)\Big] = 0 \qquad \forall\, j \in N. \tag{18}$$

Equation (17) says that if $x_i(\emptyset) = 1$, then $\pi_i = 0 = \pi_i(\emptyset)$. Also, Equation (16) says that if $x_i(S) = 1$, then $\pi_i = \pi_i(S)$. Due to dual feasibility, we know that $\pi_i \geq \pi_i(S)$. Hence, at optimality $\pi_i = \max_{S \in \Omega} \pi_i(S)$ for every buyer $i \in M$ - this denotes the maximum payoff of buyer i at a given price vector p. Hence, an optimal solution of (**CA-DLP**) can be described by just $p \in \mathbb{R}^n_+$.

We will introduce some more notations. **Demand set** of a buyer $i \in M$ at a price vector $p \in \mathbb{R}^n_+$ is defined as $D_i(p) = \{S \in \Omega : \pi_i(S) \geq \pi_i(T)\ \forall\, T \in \Omega\}$. In the example above, consider a price vector $p = (4, 4)$ (i.e., price of good 1 is 4, good 2 is 4, and bundle 1,2 is 4+4=8). At this price vector, $D_1(p) = \{\{1\}, \{1, 2\}\}$, $D_2(p) = \{\{2\}\}$, and $D_3(p) = \{\{1\}, \{2\}\}$. Consider another price vector $p' = (7, 8)$. At this price vector, $D_1(p') = \{\{1\}\}$, $D_2(p') = \{\{2\}\}$, and $D_3(p') = \{\emptyset, \{1\}\}$.

Definition 3 *A price vector $p \in \mathbb{R}^n_+$ and an allocation X is called a* **Walrasian equilibrium** *if*

1. *$X_i \in D_i(p)$ for all $i \in M$ (every buyer gets a bundle with maximum payoff),*
2. *$p_j = 0$ for all $j \in N$ such that $j \notin \cup_{i \in M} X_i$ (unassigned goods have zero price).*

The price vector $p' = (7, 8)$ along with allocation $(\{1\}, \{2\}, \emptyset)$ is a Walrasian equilibrium of the previous example since $\{1\} \in D_1(p')$, $\{2\} \in D_2(p')$, and $\emptyset \in D_3(p')$.

Theorem 2 *(p, X) is a Walrasian equilibrium if and only if X corresponds to an optimal solution of (**CA-LP**) and p corresponds to an optimal solution of (**CA-DLP**).*

Proof: Suppose (p, X) is a Walrasian equilibrium. Then p generates a feasible solution of (**CA-DLP**) - this feasible solution is generated by setting $\pi_i = \max_{S \in \Omega}[v_i(S) - \sum_{j \in S} p_j]$ for all $i \in M$, and X corresponds to a feasible solution of (**CA-LP**) - this feasible solution is generated by setting $x_i(X_i) = 1$ for all $i \in M$ and setting zero all other x variables. Now, $X_i \in D_i(p)$ for all $i \in M$ implies that $\pi_i = \pi_i(X_i)$ for all $i \in M$, and this further implies that Equations (16) and (17) is satisfied. Similarly, $p_j = 0$ for all $j \in N$ that are not assigned in X. This means Equation (18) is satisfied. Since complementary slackness conditions are satisfied, these are also optimal solutions.

Now, suppose p is an optimal solution of (**CA-DLP**) and X corresponds to an optimal solution of (**CA-LP**). Then, the complementary slackness conditions imply that the conditions for Walrasian equilibrium is satisfied. Hence, (p, X) is a Walrasian equilibrium. ■

Another way to state Theorem 2 is that a Walrasian equilibrium exists if and only if an optimal solution of (**CA-LP**) gives an efficient allocation (an optimal solution of (**CA-IP**). This is because if a Walrasian equilibrium (p, X) exists, then X is an optimal solution of (**CA-LP**) that is integral. Hence it is an optimal solution of (**CA-IP**) or an efficient allocation. So, every allocation corresponding to a Walrasian equilibrium is an efficient allocation (this is termed as the **first welfare theorem** of economics).

There are combinatorial auction problems where a Walrasian equilibrium may not exist. Consider the example in Table 3. It can be verified that this example does not have a Walrasian equilibrium Suppose there is a Walrasian equilibrium (p, X). By Theorem 2 and the earlier discussion, X is efficient, i.e, $X = (\emptyset, \{1, 2\})$. Since $X_1 = \emptyset$, by definition of Walrasian equilibrium $\pi_1(\emptyset) = 0 \geq \pi_1(\{1\}) = 8 - p_1$, i.e., $p_1 \geq 8$. Similarly, $p_2 \geq 11$. This

	$\{1\}$	$\{2\}$	$\{1,2\}$
$v_1(\cdot)$	8	11	12
$v_2(\cdot)$	5	9	18

Table 3: An example where Walrasian equilibrium does not exist

means $p_1 + p_2 \geq 19$. But $X_2 = \{1,2\}$, and $\pi_2(\{1,2\}) = 18 - (p_1 + p_2) \leq -1 < 0 = \pi_2(\emptyset)$. Hence $\{1,2\} \notin D_2(p)$. This is a contradiction since (p, X) is a Walrasian equilibrium.

An feasible solution of (**CA-LP**), which is not integral and gives an objective function value higher than the optimal solution of (**CA-IP**) is: $x_1(\{1\}) = x_1(\{2\}) = 0.5$ and $x_2(\{1,2\}) = x_2(\emptyset) = 0.5$. The value of objective function of (**CA-LP**) from this feasible solution is $(8+11)*0.5+18*0.5 = 18.5 > 18 =$ objective function value of optimal solution of (**CA-IP**). Hence linear relaxation of (**CA-IP**) does not give an integral solution in this example.

3 Incentive Compatible Auctions

Auctions are a popular form of selling and buying resources. A rich field of economic theory is the field of auction theory. Informally, an auction is described by a set of rules, which takes as input *bids* of bidders, and gives us output a *payment* and an *allocation* as a function of the bids.

There is a set of resources (say indivisible goods) to be allocated to a set of bidders. Let $M = \{1, \dots, m\}$ be the set of m bidders. An outcome is an allocation indicating who gets which resources. Let Γ be the set of all outcomes, assumed to be finite. Each bidder is described by his **type**. A type is a multi-dimensional vector describing a bidder. Each dimension of the type can say something about the bidder. For example, if there are multiple goods, then the *value* on every good (or bundle of goods) can be a dimension in the type vector. Let T be the set of all types, assumed to be finite. We also assume that every type is k-dimensional. We denote $T = \{t_1, \dots, t_n\}$ - so, there are exactly n possible k-dimensional types. For every outcome $\alpha \in \Gamma$ a bidder with type $t \in T$ associates a utility $v(\alpha : t)$ for the outcome. The crucial assumption is types are **private** information - every bidder knows only his own type, and knows it exactly. The auctioneer is unaware of the types of bidders. The $v(\cdot)$ function is known to everyone, and is same across all the bidders. We do not distinguish bidders with the same type. This is called the **symmetric** setting.

An example is the sale of a single good. The possible outcomes are who gets the object, which can be described by a vector of zeros and ones with not more than one 1. The type of a bidder is single dimensional, describing the exact value of the object if he gets it. We assume that the set of all possible types T is a finite set, meaning that the possible value of the good to a bidder is finite. So, T can be $\{h, l\}$. This may indicate the value a bidder gets on the object - say with type h the value is 20 and with type l it is 15. If α is an outcome in which a bidder with type l gets the object, then $v(\alpha : l) = 15$. One can extend this example to more than two types also.

The above example can be extended for multiple goods, meaning multi-dimensional types. As before let $\{h, l\}$ be the set of possible types for every good, but assume that there are two goods. An outcome may decide which buyer gets which good (if at all). The type of a bidder is two dimensional - indicating type over both the goods. Possible types are $T = \{(h, h), (h, l), (l, h), (l, l)\}$.

An **allocation rule** is a function $f : T^m \rightarrow \Gamma$ that outputs an outcome for every profile of types of bidders. In the two good example above, if there are 3 bidders, then a profile of types is: $(h, h), (h, l), (h, h)$. This indicates an instance where 3 bidders have these three types. Given the profile of types, the allocation rule outputs an outcome - which bidder gets which objects.

A **payment rule** is a function $p : T^m \rightarrow \mathbb{R}^m$ that specifies the payment of every bidder for every profile of types of bidders. In this discussion, we will focus on specific types of auctions in which bidders will be asked to reveal their types. These are called **direct revelation mechanisms**. So, an auction is specified by an allocation rule f and a payment rule p. There are other types of auctions also - but it is enough to focus on direct revelation mechanisms for theoretical purposes.

Two examples of specific auction types are: (a) first price auction (b) second price auction. Both the auctions are used in the setting of selling a single good. Bidders are asked to report their types or values. Given the reported values, the allocation rule is to give the good to the bidder with the highest value in both the auctions. In the first price auction, the price is set equal to the highest value, whereas in the second price auction, the price is set equal to the second highest value.

Obviously, there are many kinds of auctions possible. The question is what are the auctions in which bidders will reveal their **true types**. Investigating such auctions is a central research topic in auction theory (more generally *mechanism design*) literature. We write a profile of types as $\mathbf{t}$ and write $\mathbf{t} = (t, \mathbf{t}^{-\mathbf{i}})$. The crucial assumption we make here is that of **quasi-linearity**. Quasi-linearity says that if a bidder i has type t, and reveals his type to be s whereas other bidders report types as $\mathbf{t}^{-i}$, then his payoff in an auction with allocation rule f and payment rule p is:

$$v(f(s, \mathbf{t}^{-i}) : t) - p^i(s, \mathbf{t}^{-i}).$$

Given the quasi-linearity assumption, we define an incentive compatible allocation rule.

Definition 4 *An allocation rule $f : T^m \rightarrow \Gamma$ is* **incentive compatible** *if there exists a payment rule $p : T^m \rightarrow \mathbb{R}^m$ such that for all bidders $i \in M$, for all types $s, t \in T$*

$$v(f(\mathbf{t}) : t) - p^i(\mathbf{t}) \geq v(f(s, \mathbf{t}^{-i}) : t) - p^i(s, \mathbf{t}^{-i}) \qquad \forall\ \mathbf{t}^{-i} \in T^{m-1}. \qquad \textbf{(IC)}$$

An **incentive compatible auction** has an allocation rule f and a payment rule p that satisfies constraint (**IC**).

3.1 Negative Cycles and Incentive Compatibility

In this section, we fix a bidder $i \in M$, and fix the profile of bidders other than i as $\mathbf{t}^{-i}$. So, we suppress the dependence on these in notations, and simplify inequality (**IC**) as

$$v(f(t):t) - p(t) \geq v(f(s):t) - p(s) \qquad \forall\ s,t \in T.$$

$$p(t) - p(s) \leq \Big[v(f(t):t) - v(f(s):t)\Big] \qquad \forall\ s,t \in T. \tag{IC-G}$$

Constraints (**IC-G**) are difference constraints. We construct an **allocation graph** for every bidder $i \in M$ and every profile of types $\mathbf{t}^{-i} \in T^{m-1}$ as follows: a node for every type $t \in T$, an edge for every order pair of types (s,t) with $s \neq t$ and $s,t \in T$, weight of edge from s to t is $\Big[v(f(t):t) - v(f(s):t)\Big]$.

Theorem 3 *An allocation rule f is incentive compatible if and only if for every bidder $i \in M$ and every profile of types $\mathbf{t}^{-i} \in T^{m-1}$, the corresponding allocation graph has no negative cycles.*

The proof of the theorem follows from what we have already learnt about difference constraints. There are many settings where Theorem 3 can be made stronger. The length of a cycle between a pair of nodes s and t is denoted by $l^2(s,t) = v(f(t):t) - v(f(s):t) + v(f(s):s) - v(f(t):s)$. From Theorem 3, it is clear that every incentive compatible allocation rule must satisfy $l^2(s,t) \geq 0$ for all $s,t \in T$, and for every agent and every profile of types of other agents. This condition is also sufficient in a variety of settings. We will refer to this condition as **2-cycle condition**.

References

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