Monotonicity and Incentive Compatibility *

Debasis Mishra†

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Abstract

Understanding multi-dimensional mechanism design.

1 The Model

Let $M = \{1, \ldots, m\}$ be a finite set of agents. Every agent has private information, which can be multi-dimensional. This is called his **type**. The space from which an agent draws his type is called his type space. Let T_i denote the type space of agent $i \in M$. We assume $T_i \subseteq \mathbb{R}^n$ for some integer $n \geq 1$. Let $T^m = \times_{i=1}^m T_i$. Also, denote $T^m_{-i} = \times_{j \neq i: j \in M} T_j$. Let $t_{-i} = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_m\}$.

The set of outcomes or alternatives is denoted by A. The valuation of agent i is a mapping $v_i : A \times T_i \rightarrow \mathbb{R}$. An allocation rule is a mapping $f : T^m \rightarrow A$. A payment rule is a mapping $p : T^m \rightarrow \mathbb{R}^m$. An allocation rule along with a payment rule is called a **mechanism**.

A classical example of multi-dimensional mechanism design problem is the design of auctions for selling multiple objects. If there are k objects, every agent (buyer) has a 2^k dimensional type space, denoting his value for every possible bundle of objects. An outcome says which objects are assigned to which agents.

Definition 1 *An allocation rule f is* **dominant strategy incentive compatible** *(DSIC) if there exists a payment rule p such that for every agent $i \in M$ and for every $t_{-i} \in T^m_{-i}$, we have*

$$v_i(f(t_i, t_{-i}), t_i) - p(t_i, t_{-i}) \geq v_i(f(s_i, t_{-i}), t_i) - p(s_i, t_{-i}) \qquad \forall\ s_i, t_i \in T_i. \tag{1}$$

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†Indian Statistical Institute, 7 Shaheed Jit Singh Marg, New Delhi - 110016, India

Notice that the set of inequalities 1, are for a given agent $i \in M$ and for a given type profile $t_{-i} \in T^m_{-i}$ of other agents. Our objective is to investigate when these inequalities have a solution for every $i \in M$ and for every $t_{-i} \in T^m_{-i}$. To simplify notation, we can therefore fix an agent $i \in M$ and a type profile of other agents $t_{-i} \in T^m_{-i}$, and investigate what allocation rules are DSIC. Hence, without loss of generality, we can analyse an economy with a single agent.

1.1 Examples of Allocation Rule

Here, we give some examples of allocation rules.

- **Constant allocation:** The constant allocation rule f^c allocates some $a \in A$ for every $\mathbf{t} \in T^m$. In particular, there exists a $a \in A$ such that for every $\mathbf{t} \in T^m$ we have $f^c(\mathbf{t}) = a$.
- **Dictator allocation:** The dictator allocation rule f^d allocates the *best* outcome of some **dictator** agent $i \in M$. In partcular, let $i \in M$ be the dictator agent. Then, for every $t_i \in T_i$ and every $t_{-i} \in T^m_{-i}$, $f^d(t_i, t_{-i}) \in \arg\max_{a \in A} v_i(a, t_i)$.
- **Efficient allocation:** The efficient allocation rule f^e is the one which maximizes the sum of values of agents. In particular, for every $\mathbf{t} \in T^m$, $f^e(\mathbf{t}) \in \arg\max_{a \in A} \sum_{i \in M} v_i(a, t_i)$.

Later, we will show that f^c, f^d, f^e are all DSIC.

1.2 A Single Agent Model

From this section onwards, we restrict attention to an economy with a single agent. From the discussion in the previous section, restricting to an economy with a single agent is without loss of generality. The type space of the agent is $n-$dimensional and is denoted by $T \subseteq \mathbb{R}^n$. The set of outcomes or alternatives is denoted by A. The valuation of agent is a mapping $v : A \times T \to \mathbb{R}$. We assume $v(a, t)$ is finite for every $a \in A$ and for every $t \in T$ [1]. An allocation rule is a mapping $f : T \to A$. A payment rule is a mapping $p : T \to \mathbb{R}$.

Definition 2 *An allocation rule f is* **dominant strategy incentive compatible** *(DSIC) if there exists a payment rule p such that*

$$v(f(t), t) - p(t) \geq v(f(s), t) - p(s) \qquad \forall\ s, t \in T. \tag{2}$$

[1] This is a mild restriction.

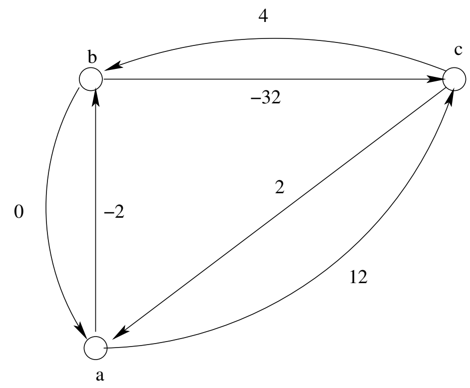

Figure 1: A directed graph

Allocation rule f along with a payment rule p is called a mechanism. The question we are interested in is: *What kind of allocation rules are DSIC?* In the sequel, we fix an allocation rule f and verify when it can be DSIC. We assume that the allocation rule is onto. If it is not onto, set A to be the range of f. Hence, the assumption that f is onto is not restrictive. So for every $a_k \in A$, there exists a $t \in T$ such that $f(t) = a_k$.

2 Graphs, Cycles, and Cycle Monotonicity

2.1 Basic Graph Definitions

Since some concepts of graphs will be used extensively, we define them in this section. A **directed graph** is tuple (T, E), where T is called the set of nodes and E is called the set of edges. An edge is an ordered pair of nodes. The set T can be finite or infinite. A **complete** directed graph is a directed graph (T, E) is one in which for every $i, j \in T$ $(i \neq j)$ [2], there is an edge from i to j. In this note, we will only be concerned with complete directed graphs and refer to them as graphs. Also, we will associate with a graph (T, E) a length function $l : E \to \mathbb{R}$.

A (finite) **path** in a graph (T, E) is a sequence of distinct nodes $(t_1, \dots, t_k)$. A (finite) **cycle** in a graph (T, E) is a sequence of nodes $(t_1, \dots, t_k, t_1)$ where $(t_1, \dots, t_k)$ is a path. The length of a path $P = (t_1, \dots, t_k)$ is the sum of lengths of edges in that path P, i.e., $l(P) = l(t_1, t_2) + \ldots + l(t_{k-1}, t_k)$. Similarly, the length of a cycle $C = (t_1, \dots, t_k, t_1)$ is the sum of lengths of edges in the cycle, i.e., $l(C) = l(t_1, t_2) + \ldots + l(t_{k-1}, t_k) + l(t_k, t_1)$.

Figure 1 gives an example of a graph. A cycle in this graph is (a, b, c, a) with length -32. A path in this graph is (c, b, a) with length 4.

[2]We do not allow edges from a node to itself.

2.2 Cycle Monotonicity

In this section, we give a characterization of DSIC. We make no assumptions on T, A, and $v(\cdot,\cdot)$. Denote $l(s,t) = v(f(t),t) - v(f(s),t)$. DSIC condition can then be restated as follows. Allocation rule f is DSIC if there exists a payment rule p such that

$$p(t) - p(s) \leq l(s,t) \qquad \forall\ s,t \in T. \tag{3}$$

Equations 3 has a graph theoretic interpretation. Every type in T is a node in the graph - hence, it may be an infinite graph. There is a directed edge from every $s \in T$ to every $t \in T \setminus \{s\}$. The weight of edge from s to t, denoted as (s,t), is $l(s,t)$. We call this graph the **type graph** and denote it as T_f. A well known result in graph theory states that inequalities in 3 have a solution if the corresponding graph has no cycles of negative length.

Definition **3 (Cycle Monotonicity)** *The allocation rule f satisfies* **cycle monotonicity** *if for every finite and distinct sequence of types $(t_1, t_2, \ldots, t_k)$ $(k \geq 2)$ we have*

$$l(t_1,t_2) + l(t_2,t_3) + \ldots + l(t_{k-1},t_k) + l(t_k,t_1) \geq 0. \tag{4}$$

The distinct sequence of nodes $(t_1, \ldots, t_k)$ and the edges $(t_1,t_2),(t_2,t_3),\ldots,(t_{k-1},t_k),(t_k,t_1)$ is called a cycle in type graph T_f. If we do not include the edge (t_k,t_1) and consider just the distinct sequence of nodes $(t_1,\ldots,t_k)$ and the edges $(t_1,t_2),(t_2,t_3),\ldots,(t_{k-1},t_k)$, then it is called a path in the graph T_f. For any path P in graph T_f, we denote its length by $l(P)$, which is equal to the sum of length of edges in path P. Similarly, for any cycle C in graph T_f, we denote its length by $l(C)$.

Theorem **1 (Rockafellar (1970), Rochet (1987))** *The allocation rule f is DSIC if and only if it satisfies cycle monotonicity.*

Proof: Suppose f is DSIC. Consider a finite and distinct sequence of points $(t_1, t_2, \ldots, t_k)$ with $k \geq 2$. Since f is DSIC, there exists a payment rule p such that

$$\begin{aligned}
p(t_2) - p(t_1) &\leq l(t_1,t_2)\\
p(t_3) - p(t_2) &\leq l(t_2,t_3)\\
\ldots &\leq \ldots\\
\ldots &\leq \ldots\\
p(t_k) - p(t_{k-1}) &\leq l(t_{k-1},t_k)\\
p(t_1) - p(t_k) &\leq l(t_k,t_1).
\end{aligned}$$

Adding these inequalities, we obtain that $l(t_1,t_2) + l(t_2,t_3) + \ldots + l(t_{k-1},t_k) + l(t_k,t_1) \geq 0$.

Suppose f satisfies cycle monotonicity. For any two types $s, t \in T$, let $P(s,t)$ denote the set of all (finite) paths from s to t. The set $P(s,t)$ is non-empty because the direct edge from s to t always exists. Define the **shortest path** length from s to t ($s \neq t$) as follows.

$$dist_{T_f}(s,t) = \inf_{P \in P(s,t)} l(P).$$

Let $dist_{T_f}(s,s) = 0$ for all $s \in T$. First, we show that $dist_{T_f}(s,t)$ is finite. Consider any path $P \in P(s,t)$. By cycle monotonicity, $l(P) \geq -l(t,s)$. Hence, $dist_{T_f}(s,t) \geq -l(t,s)$. Since $v(a,r)$ is bounded for every $a \in A$ and for every $r \in T$, $l(t,s)$ is bounded. Hence, $dist_{T_f}(s,t)$ is finite.

Now, fix a type $r \in T$. Consider two types $s, t \in T$. We first prove a well-known lemma.

LEMMA **1** *Suppose f satisfies cycle monotonicity. For any $r, s, t \in T$ with $s \neq t$, we have $dist_{T_f}(r,t) \leq dist_{T_f}(r,s) + l(s,t)$.*

Proof: If $r = t$, $dist_{T_f}(r,t) = dist_{T_f}(r,r) = 0$. By cycle monotonicity, $dist_{T_f}(t,s) \geq -l(s,t)$ or $dist_{T_f}(t,s) + l(s,t) \geq 0 = dist_{T_f}(r,r) = dist_{T_f}(r,t)$. This gives $dist_{T_f}(r,s) + l(s,t) \geq dist_{T_f}(r,t)$. By cycle monotonicity, $dist_{T_f}(t,s) + l(s,t) \geq 0$. If $r = s$, then $dist_{T_f}(r,t) \leq l(r,t) = dist_{T_f}(r,s) + l(s,t)$. If $r \neq s \neq t$, consider any path P from r to s. We distinguish between two possible cases.

CASE 1: Path P contains t. In that case, let Q_1 be the path from r to t in P and Q_2 be the path from t to s. Hence, $l(P) = l(Q_1) + l(Q_2)$. Adding $l(s,t)$ on both sides, we get $l(P) + l(s,t) = l(Q_1) + l(Q_2) + l(s,t)$. Using cycle monotonicity, we get $l(P) + l(s,t) \geq l(Q_1) \geq dist_{T_f}(r,t)$. Hence, $l(P) + l(s,t) \geq dist_{T_f}(r,t)$.

CASE 2: Path P does not contain t. In that case, by definition $dist_{T_f}(r,t) \leq l(P) + l(s,t)$, i.e., $l(P) + l(s,t) \geq dist_{T_f}(r,t)$.

Hence, in both cases, we see $l(P) + l(s,t) \geq dist_{T_f}(r,t)$. Since this holds for every path from r to s, we have $dist_{T_f}(r,s) + l(s,t) \geq dist_{T_f}(r,t)$. ■

Now, define the following payment rule: let $p(s) = dist_{T_f}(r,s)$ for all $s \in T$. Take any $s, t \in T$. We have $p(t) - p(s) = dist_{T_f}(r,t) - dist_{T_f}(r,s) \leq l(s,t)$ from Lemma 1. Hence, f is DSIC. ■

REMARK: The characterization in Theorem 1 also holds for the case when T is finite. In that case, number of paths in T_f is finite, and hence, for every $s, t \in T$, $dist_{T_f}(s,t) = \min_{P \in P(s,t)} l(P)$.

2.3 Allocation Graph

To prove Theorem 1, we constructed the graph T_f - called the **type graph** of allocation rule f. Here, we introduce another graph, called the **allocation graph**, and show its relation to the type graph. To define the graph, we need some extra notation. Let $T_a = \{t \in T : f(t) = a\}$ for every $a \in A$. The allocation graph A_f has a node for every alternative in A and a directed edge from every $a \in A$ to every $b \in A \setminus \{a\}$. The length of edge from a to b is $d(a,b) = \inf_{t \in T_b}[v(b,t) - v(a,t)]$ (this exists as long as it is not infinite, which we can assume without loss of generality).

Theorem **2** (**Rochet** (**1987**), **Rockafellar** (**1970**)) *The following are equivalent for the allocation rule f.*

1. *The allocation rule f is DSIC.*
2. *The type graph T_f has no cycle of negative length.*
3. *The allocation graph A_f has no cycle of negative length.*

Proof: The equivalence of 1 and 2 was established in Theorem 1. We establish an equivalence between 1 and 3. Suppose f is DSIC. Hence there exists a payment rule p such that Equations 1 holds. Consider $s, t \in T$ such that $f(s) = f(t) = a$. Hence, $v(f(s), t) = v(f(t), t)$. Since f is DSIC, $p(s) = p(t)$. Hence, without loss of generality, $p : A \to \mathbb{R}$. Consider a cycle in A_f with nodes $(a_1, \ldots, a_k)$. From DSIC, we can write

$$\begin{aligned} p(a_2) - p(a_1) &\leq v(a_2, t) - v(a_1, t) \qquad \forall\, t \in T_{a_2} \\ \textit{or},\ p(a_2) - p(a_1) &\leq \inf_{t \in T_{a_2}} [v(a_2, t) - v(a_1, t)] = d(a_1, a_2). \end{aligned}$$

Hence, we can write

$$\begin{aligned} p(a_2) - p(a_1) &\leq d(a_1, a_2) \\ p(a_3) - p(a_2) &\leq d(a_2, a_3) \\ \ldots &\leq \ldots \\ p(a_k) - p(a_{k-1}) &\leq d(a_{k-1}, a_k) \\ p(a_1) - p(a_k) &\leq d(a_k, a_1). \end{aligned}$$

Adding the inequalities, we get $d(a_1, a_2) + \ldots + d(a_{k-1}, a_k) + d(a_k, a_1) \geq 0$.

Suppose A_f has no cycles of negative length. As in the proof of Theorem 1, fix a node $a \in A_f$ and take shortest paths from a to every other node $b \in A_f$, denoted as $dist_{A_f}(a, b)$. Define $dist_{A_f}(a, a) = 0$. Define for all $b \in A$ and all $t \in T_b$, $p(t) = dist_{A_f}(a, b)$. Now consider $s, t \in T$. If $f(s) = f(t)$, then $p(s) - p(t) = 0 = v(f(t), t) - v(f(s), t)$. Suppose $f(s) = b$ and

$f(t) = c \neq b$. Then, $p(t) - p(s) = dist_{A_f}(a, c) - dist_{A_f}(a, b) \leq d(b, c) \leq v(f(t), t) - v(f(s), t)$, where the last inequality follows from the definition of $d(b, c)$ and the first inequality follows by Lemma 1. ■

Hence, DSIC is equivalent to requiring no negative cycles in the type graph or in the allocation graph.

2.4 DSIC Allocation Rules

We revisit the constant (f^c), dictatorial (f^d), and efficient (f^e) allocation rules, and examine if they are DSIC.

- f^c: In the constant allocation rule, let $f^c(t) = a$ for all $t \in T$. In that case, for any $s, t \in T$, $v(f^c(t), t) = v(a, t) = v(f^c(s), t)$. Hence, $l(s, t) = 0$ for all $s, t \in T$. Thus, length of any finite cycle is zero. Since $l(s, t) = 0$ for all $s, t \in T$, a payment rule which makes f^c DSIC is $p(r) = 0$ for all $r \in T$. So, the constant allocation rule is DSIC *without money.*

- f^d: In the dictatorial allocation rule f^d, we consider two cases.

  Case 1: The agent under consideration is not the dictator. In that case, $f^d(s) = f^d(t)$ for all $s, t \in T$. Hence, $l(s, t) = 0$ for all $s, t \in T$, and length of any finite cycle is again zero. Such an agent requires no payment as in the constant allocation rule.

  Case 2: The agent under consider is the dictator. In that case, $f^d(s) = \arg\max_{a \in A} v(a, s)$. Hence, for any $s, t \in T$ we have $l(s, t) = v(f^d(t), t) - v(f^d(s), t) \geq 0$. So, any finite cycle has non-negative length. Note that $p(r) = 0$ for all $r \in T$ is a payment rule which makes f DSIC.

- f^e: For the efficient allocation rule f^e, we resort to our earlier notation with m agents. We consider a class of payment rules which makes f^e DSIC. For agent $i \in M$, it is given by:

  $$p^e(t_i, t_{-i}) = h_i(t_{-i}) - \sum_{j \neq i} v_j(f^e(t_i, t_{-i}), t_j) \qquad \forall\ t_i \in T_i,\ \forall\ t_{-i} \in T^m_{-i},$$

  where $h_i : T^m_{-i} \to \mathbb{R}$ for all i are a family of arbitrary functions. To see that p^e makes

f^e DSIC, consider $s_i, t_i \in T_i$. Then, we have

$$\begin{aligned} v_i(f^e(t_i,t_{-i}),t_i) - p^e(t_i,t_{-i}) &= \sum_{j \in M} v_j(f^e(t_i,t_{-i}),t_j) - h_i(t_{-i}) \\ &\geq v_i(f^e(s_i,t_{-i}),t_i) + \sum_{j \neq i} v_j(f^e(s_i,t_{-i}),t_j) - h_i(t_{-i}) \\ &= v_i(f^e(s_i,t_{-i}),t_i) - p^e(t_i,t_{-i}). \end{aligned}$$

3 Revenue Equivalence

Consider an allocation rule f which is DSIC. Let p be a payment rule which makes f DSIC. Let $\alpha \in \mathbb{R}$. Define $q(t) = p(t) + \alpha$ for all $t \in T$. Since $q(t) - q(s) = p(t) - p(s) \leq l(s,t)$, we see that q is also a payment that makes f DSIC. Is it possible that all payments that make f DSIC can be obtained by adding a suitable constant $\alpha \in \mathbb{R}$ to p? This property of an allocation rule is called **revenue equivalence**. Not all allocation rules satisfy revenue equivalence. Myerson (1981) showed that in the standard auction of single object (one-dimensional type space) every allocation rule satisfies revenue equivalence. The objective of this section is to identify allocation rules that satisfy revenue equivalence in more general settings.

Definition 4 *An allocation rule f satisfies* **revenue equivalence** *if for any two payment rules p and $\hat{p}$ that make f DSIC, there exists a constant $\alpha \in \mathbb{R}$ [3] such that*

$$p(t) = \hat{p}(t) + \alpha \qquad \forall\, t \in T. \tag{5}$$

The first characterization of revenue equivalence involves no assumptions on T, A, and $v(\cdot,\cdot)$.

Theorem 3 (**Heydenreich et al. (2009)**) *Suppose f is DSIC. Then the following are equivalent.*

1. *The allocation rule f satisfies revenue equivalence.*
2. *For all $s,t \in T$, we have $dist_{T_f}(s,t) + dist_{T_f}(t,s) = 0$.*
3. *For all $a,b \in A$, we have $dist_{A_f}(a,b) + dist_{A_f}(b,a) = 0$.*

[3]In a model with more than one agent α can be a (agent-specific) mapping from type profile of other players to real numbers.

Proof: We establish the equivalence of 1 and 2 first. Suppose f satisfies revenue equivalence. Consider any $s, t \in T$. Since f is DSIC, by Theorem 1, the following two payment rules makes f DSIC:

$$p^s(r) = dist_{T_f}(s, r) \qquad \forall r \in T$$
$$p^t(r) = dist_{T_f}(t, r) \qquad \forall r \in T.$$

Since revenue equivalence holds, $p^s(s) - p^t(s) = p^s(t) - p^t(t)$. But $p^s(s) = p^t(t) = 0$. Hence, $p^s(t) + p^t(s) = 0$, which implies that $dist_{T_f}(s, t) + dist_{T_f}(t, s) = 0$.

Now, suppose $dist_{T_f}(s, t) + dist_{T_f}(t, s) = 0$ for all $s, t \in T$. Consider any payment rule p that makes f DSIC. Take any path $P = (s, t_1, \ldots, t_k, t)$ from s to t. Now, $l(P) = l(s, t_1) + l(t_1, t_2) + \ldots + l(t_{k-1}, t_k) + l(t_k, t) \geq [p(t_1) - p(s)] + [p(t_2) - p(t_1)] + \ldots + [p(t_k) - p(t_{k-1})] + [p(t) - p(t_k)] = p(t) - p(s)$. Hence, $p(t) - p(s) \leq l(P)$ for any path P from s to t. Hence, $p(t) - p(s) \leq dist_{T_f}(s, t)$. Similarly, $p(s) - p(t) \leq dist_{T_f}(t, s)$. Hence, $0 = dist_{T_f}(s, t) + dist_{T_f}(t, s) \geq [p(s) - p(t)] + [p(t) - p(s)] = 0$. Hence, $p(s) - p(t) = dist_{T_f}(t, s)$, which is independent of $p(\cdot)$. Hence, revenue equivalence holds.

The equivalence of 1 and 3 can be established in a similar fashion. We have already seen that for DSIC f, we can consider payment rules to be a mapping from A to $\mathbb{R}$. Now, we can mimic the above proof for allocation graph A_f. ■

3.1 Restricted Versions of Revenue Equivalence

If T is convex, then revenue equivalence is proved in Rockafellar (1970).

Theorem **4 (Rockafellar (1970), Krishna and Maenner (2001))** *Suppose T is convex and $v : A \times \mathbb{R}^n \to \mathbb{R}$* [4] *is linear in type. If f is DSIC, then f satisfies revenue equivalence.*

Proof: We prove the following lemma.

Lemma **2 (Path Contraction)** *Suppose T is convex and $v : A \times \mathbb{R}^n \to \mathbb{R}$ is linear in type. For every $s, t \in T$ and every $\epsilon > 0$, there exists a sequence of points $r_1, \ldots, r_M$ such that $l(s, r_1) + l(r_1, r_2) + \ldots + l(r_M, t) + l(t, r_M) + l(r_M, r_{M-1}) + \ldots + l(r_2, r_1) + l(r_1, s) < \epsilon$.*

Figure 3.1 explains the lemma. For every $s, t \in T$, there exists a series of two-cycles (cycles involving two nodes) between s and t which add up to arbitrarily close to zero.
Proof: Let $\delta = t - s$. Define for every $j \in \{1, \ldots, k\}$, $r_j = s + \frac{j}{k+1}\delta$. By convexity of T, $r_j \in T$ for all $j \in \{1, \ldots, k\}$. Set $r_0 = s$ and $r_{k+1} = t$. Now, due to linearity of $v(\cdot, \cdot)$, we can

[4] Notice that we require $v : A \times \mathbb{R}^n \to \mathbb{R}$ instead of $v : A \times T \to \mathbb{R}$. We need this to make use of linearity of v in t. Since $T \subseteq \mathbb{R}^n$, this is without loss of generality.

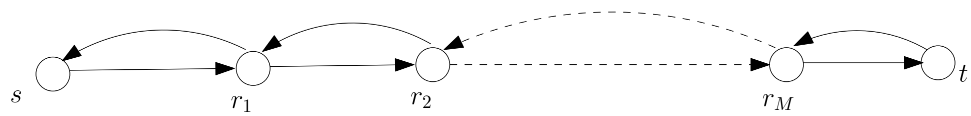

Figure 2: For any $s, t \in T$, we find a series of cycles between s and t with arbitrarily short length

write

$$\begin{aligned} l(r_j, r_{j+1}) &= v(f(r_{j+1}), r_{j+1}) - v(f(r_j), r_{j+1}) \\ &= \big[v(f(r_{j+1}), s) - v(f(r_j), s)\big] + \frac{j+1}{k+1}\big[v(f(r_{j+1}), \delta) - v(f(r_j), \delta)\big] \\ l(r_{j+1}, r_j) &= v(f(r_j), r_j) - v(f(r_{j+1}), r_j) \\ &= \big[v(f(r_j), s) - v(f(r_{j+1}), s)\big] + \frac{j}{k+1}\big[v(f(r_j), \delta) - v(f(r_{j+1}), \delta)\big]. \end{aligned}$$

Hence, we can write

$$l(r_j, r_{j+1}) + l(r_{j+1}, r_j) = \frac{1}{k+1}\big[v(f(r_{j+1}), \delta) - v(f(r_j), \delta)\big].$$

Hence, we can write

$$\sum_{j=0}^{k} \big[l(r_j, r_{j+1}) + l(r_{j+1}, r_j)\big] = \frac{1}{k+1}\big[v(f(t), \delta) - v(f(s), \delta)\big].$$

Since $v(\cdot, \cdot)$ is finite, $v(f(t), \delta) - v(f(s), \delta)$ is finite. Hence, we can choose k sufficiently large so that

$$\sum_{j=0}^{k} \big[l(r_j, r_{j+1}) + l(r_{j+1}, r_j)\big] < \epsilon.$$

■

Note that by cycle monotonicity, $dist_{T_f}(s, t) + dist_{T_f}(t, s) \geq 0$. Assume for contradiction $dist_{T_f}(s, t) + dist_{T_f}(t, s) = \epsilon > 0$. From Lemma 2, there exists paths from s to t and t to s such that sum of their lengths is less than ϵ. This is a contradiction. Hence, $dist_{T_f}(s, t) + dist_{T_f}(t, s) = 0$. By Theorem 3, f satisfies revenue equivalence. ■

3.2 Connected Type Space

We now identify another domain of types where revenue equivalence holds. It follows from the results in Chung and Olszewski (2007). We use the proof in Heydenreich et al. (2009).

The main assumption in this section is T is a connected subset of topological space $\mathbb{R}^n$. We remind the following fact about connected sets.

FACT **1** *For a subset T of topological space $\mathbb{R}^n$, the following conditions are equivalent:*

1. *Set T is connected.*
2. *Set T cannot be written as union of two non-empty* **separated sets**, *where two sets A and B are separated if $A \cap cl(B) = \emptyset$ and $B \cap cl(A) = \emptyset$.*
3. *Set T cannot be partitioned into two non-empty open sets.*

In next two theorems, we make the assumption that T is a connected set in $\mathbb{R}^n$.

THEOREM **5 (Chung and Olszewski (2007))** *Suppose $T \subseteq \mathbb{R}^n$ is a connected set, A is finite, and v is continuous in type. If f is DSIC, then f satisfies revenue equivalence.*

Proof: We do the proof in three steps.

STEP 1 - TWO CYCLE CONNECTED: Let $A = A_1 \cup A_2$ be partition of A. Let $T_1 = \{t \in T : f(t) \in A_1\}$ and $T_2 = \{t \in T : f(t) \in A_2\}$. Hence, T_1 and T_2 define a partition of T. Since T is a connected space, by Fact 1, $cl(T_1) \cap cl(T_2) \neq \emptyset$. Let $t \in cl(T_1) \cap cl(T_2)$. Hence, there are sequences $\{s_n\}_{n\geq 1} \in T_1$ and $\{t_n\}_{n\geq 1} \in T_2$ such that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = t$. Since A is finite, there exists $a_1 \in A_1$ and $a_2 \in A_2$ such that there are subsequences $\{s_{n^k}\}$ with $f(s_{n^k}) = a_1$ for all k and $\{t_{n^r}\}$ with $f(t_{n^r}) = a_2$ for all r. Since $v(\cdot, \cdot)$ is continuous with respect to type, we can write

$$\begin{aligned} 0 &= v(a_2, t) - v(a_1, t) + v(a_1, t) - v(a_2, t) \\ &= \lim_{n^r \to 0} [v(a_2, t_{n^r}) - v(a_1, t_{n^r})] + \lim_{n^k \to 0} [v(a_1, s_{n^k}) - v(a_2, s_{n^k})] \\ &\geq d(a_1, a_2) + d(a_2, a_1) \geq 0, \end{aligned}$$

where the last two inequalities follow from the definition of $d(\cdot, \cdot)$ and the fact that every cycle has non-negative length (since f is DSIC). Hence, $d(a_1, a_2) + d(a_2, a_1) = 0$. Thus, we have shown that for any partition $A_1 \cup A_2 = A$ of A, there exists $a_1 \in A_1$ and $a_2 \in A_2$ such that $d(a_1, a_2) + d(a_2, a_1) = 0$. [5]

STEP 2 - PATH CONTRACTION: Next, we show that for any $a, b \in A$, there exists $a_1, a_2, \ldots, a_k \in A$ such that $a_1 = a, a_k = b$ and $d(a_i, a_{i+1}) + d(a_{i+1}, a_i) = 0$ for all $i \in \{1, \ldots, k-1\}$. Call such a path a **zero path** from a to b. Fix $a \in A$ and let A_1 be the set of outcomes such

[5] This property is referred to as **two-cycle connected** property by Heydenreich et al. (2009).

that for every $b \in A_1$, there is a zero path from a to b. Let $A_2 = A \setminus A_1$. Assume for contradiction $A_2 \neq \emptyset$. By our earlier claim in Step 1, there exists $a_1 \in A_1$ and $a_2 \in A_2$ such that $d(a_1, a_2) + d(a_2, a_1) = 0$. By definition of A_1, $d(a, a_1) + d(a_1, a) = 0$. Hence, (a, a_1, a_2) is a zero path from a to a_2, and $a_2 \in A_1$. This is a contradiction.

Step 3 - Revenue Equivalence: Finally, for any $a, b \in A$, $dist_{A_f}(a, b) + dist_{A_f}(b, a) \leq d(P^{ab}) + d(P^{ba})$, where $d(P^{ab})$ is the length of any path P^{ab} from a to b in A_f and $d(P^{ba})$ is the length of any path P^{ba} from b to a in A_f. Since there is a zero path from a to b, we get $dist_{A_f}(a, b) + dist_{A_f}(b, a) \leq 0$. Since f is DSIC, cycle monotonicity implies that $dist_{A_f}(a, b) + dist_{A_f}(b, a) \geq 0$. Hence, $dist_{A_f}(a, b) + dist_{A_f}(b, a) = 0$. The result then follows from Theorem 3. ■

The next theorem generalizes, in some way, Theorem 5. We alter two conditions in Theorem 5: we make A countable but also make v *equicontinuous*.

For completeness, let us revisit the definitions of continuity and uniform continuity.

Definition **5** *A function $f : T \to \mathbb{R}$, where $T \subseteq \mathbb{R}^n$ is continuous at $x \in T$ if for every $\epsilon > 0$ there exists $\delta(x) > 0$ such that if $y \in T$ and $||y - x|| < \delta(x)$ then $|f(y) - f(x)| < \epsilon$. Function f is* **continuous** *if it is continuous at every $x \in T$.*

The important point is that we can choose different $\delta(x)$ at different x in a continuous function.

Definition **6** *A function $f : T \to \mathbb{R}$, where $T \subseteq \mathbb{R}^n$ is* **uniformly continuous** *if for every $x \in T$ and for every $\epsilon > 0$ there exists $\delta > 0$ such that if $y \in T$ and $||y - x|| < \delta$ then $|f(y) - f(x)| < \epsilon$.*

In the definition of uniform continuous functions, the δ is same for every $x \in T$. Hence, a uniform continuous function is continuous but the converse is not true. For example, the function $f : \mathbb{R} \to \mathbb{R}_+$ defined as $f(x) = x^2$ for all $x \in \mathbb{R}$ is continuous but not uniformly continuous. If T is compact, then continuity and uniform continuity is equivalent (this is known as Heine-Cantor Theorem).

We require something stronger than uniform continuity.

Definition **7** *The valuation function $v : A \times T \to \mathbb{R}$ is* **equicontinuous** *if for every $\epsilon > 0$ and every $t \in T$, there exists a $\delta > 0$ such that for all $a \in A$ and for all $s \in T$ with $||t - s|| < \delta$, we have $|v(a, s) - v(a, t)| < \epsilon$.*

Note that δ may not depend on a, implying that equicontinuity of v is stronger than just uniform continuity in type.

Theorem 6 (Heydenreich et al. (2009)) *Suppose $T \subseteq \mathbb{R}^n$ is a connected set, A is countable, and $v(\cdot,\cdot)$ is equicontinuous. If f is DSIC, then it satisfies revenue equivalence.*

Proof: We do the proof in three steps.
Step 1 - Partition of T: Assume for contradiction f is DSIC but does not satisfy revenue equivalence. By Theorem 3, there exists $a, b \in A$ such that $dist_{A_f}(a,b) + dist_{A_f}(b,a) > 0$. As A is countable, the set $\{dist_{A_f}(x,a) + dist_{A_f}(a,x) : x \in A\}$ is countable. Hence, there exists a $z > 0$ such that $A_1 = \{x \in A : dist_{A_f}(x,a) + dist_{A_f}(a,x) < z\}$ and $A_2 = \{x \in A : dist_{A_f}(x,a) + dist_{A_f}(a,x) > z\}$ with A_1, A_2 non-empty, disjoint, and defining a partition of A. Let $T_1 = \{t \in T : f(t) \in A_1\}$ and $T_2 = \{t \in T : f(t) \in A_2\}$. Since f is onto, T_1 and T_2 are non-empty and define a partition of T.

Step 2 - T_1 is open: Let $t \in T_1$ and $f(t) = x \in A_1$. Then, $dist_{A_f}(x,a) + dist_{A_f}(a,x) = z - \epsilon$ for some $\epsilon > 0$. As v is equicontinuous, there is a $\delta > 0$ such that $|v(a',t) - v(a',s)| < \frac{\epsilon}{2}$ for all $a' \in A$ and for all $s \in T$ with $||s - t|| < \delta$. Let $s \in T$ such that $||s - t|| < \delta$ and let $y = f(s)$. Now,

$$
\begin{aligned}
dist_{A_f}(a,y) + dist_{A_f}(y,a) &\leq dist_{A_f}(a,x) + d(x,y) + d(y,x) + dist_{A_f}(x,a) \\
&= z - \epsilon + d(x,y) + d(y,x) \\
&\leq z - \epsilon + v(y,s) - v(x,s) + v(x,t) - v(y,t) \\
&\leq z - \epsilon + |v(y,s) - v(y,t)| + |v(x,t) - v(x,s)| \\
&< z - \epsilon + \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= z.
\end{aligned}
$$

Hence, $dist_{A_f}(a,y) + dist_{A_f}(y,a) < z$, implying $y \in A_1$. Thus, $s \in T_1$ for any s in the δ ball around t. Hence, T_1 is open.

Step 3 - Revenue Equivalence: Since T is connected, either $cl(T_1) \cap T_2 \neq \emptyset$ or $cl(T_2) \cap T_1 \neq \emptyset$. Since T is connected and T_1 is open, T_2 cannot be open. Hence, T_2 is closed. In that case, $cl(T_2) = T_2$. By definition $T_2 \cap T_1 = \emptyset$, and hence, $cl(T_2) \cap T_1 = \emptyset$. Then, we must have $cl(T_1) \cap T_2 \neq \emptyset$. Let $t \in cl(T_1) \cap T_2$. Clearly, $t \notin T_1$ and $t \in T_2$. Since $t \in cl(T_1)$, there exists a sequence of types in T_1, say $\{t_n\}_{n \geq 1}$, that converge to $t \in T_2$. Let $x = f(t) \in A_2$. Then, $dist_{A_f}(x,a) + dist_{A_f}(a,x) = z + \epsilon$ for some $\epsilon > 0$. Since v is equicontinuous, there is a δ such that $|v(a',t) - v(a',s)| < \frac{\epsilon}{2}$ for all $a' \in A$ and for all $s, t \in T$ with $||s - t|| < \delta$. Choose k such that $|t_k - t| < \delta$. Note that $t_k \in t_1$ and $f(t_k) \in A_1$. Let $f(t_k) = y$. So,

$dist_{A_f}(a, y) + dist_{A_f}(y, a) < z$. Now,

$$\begin{aligned} z + \epsilon &= dist_{A_f}(x, a) + dist_{A_f}(a, x) \\ &\leq d(x, y) + dist_{A_f}(y, a) + dist_{A_f}(a, y) + d(y, x) \\ &< z + d(x, y) + d(y, x) \\ &\leq z + v(y, s) - v(x, s) + v(x, t) - v(y, t) \\ &\leq z + |v(y, s) - v(y, t)| + |v(x, s) - v(x, t)| \\ &< z + \epsilon. \end{aligned}$$

This is a contradiction. Hence f must satisfy revenue equivalence. ■

3.2.1 Examples

The continuity assumption in Theorem 5 is crucial. For instance, consider the following example. Let $A = \{a, b\}$ and $T = [0, 1]$. Let the valuation function be $v(a, t) = 1$ if $t < 0.5$ and $v(a, t) = 0$ for $t \geq 0.5$; $v(b, t) = 0.5$ for all t. So, $v(a, \cdot)$ is discontinuous at $t = 0.5$. Let the allocation rule f be as follows: $f(t) = a$ for $t < 0.5$ and $f(t) = b$ for $t \geq 0.5$. The inequalities for DSIC are:

$$\begin{aligned} 1 - p(a) &\geq 0.5 - p(b) \\ 0.5 - p(b) &\geq 0 - p(a). \end{aligned}$$

Hence, $-0.5 \leq p(a) - p(b) \leq 0.5$. Two payment rules which make f DSIC are: $p(a) = 0, p(b) = 0.5$ and $p'(a) = p'(b) = 0$.

Similarly, if T is finite, revenue equivalence need not hold. Consider $T = \{s, t\}$ and $A = \{a, b\}$. Let $v(a, s) = 1$ and $v(b, s) = 0.5$; $v(a, t) = 0$ and $v(b, t) = 0.5$. Let $f(s) = a$ and $f(t) = b$. It can be verified that the DSIC inequalities are same as for the previous example. Hence, revenue equivalence does not hold. In both the examples, $dist_{A_f}(a, b) = dist_{A_f}(b, a) = 0.5$. Hence, by Theorem 3 revenue equivalence does not hold.

The following example from Holmstrom (1979) is insightful. He considers an economy with two agents. Let $A = [0, 1]$ and $T_1 = T_2 = [0, 1]$. The valuation functions are defined as follows. For every $a \in A$ and every $t_1 \in T_1$, $t_2 \in T_2$, we have

$$\begin{aligned} v_1(a, t_1) &= 0 && \text{if } a \leq t_1 \\ &= t_1 - a && \text{if } a > t_1. \\ v_2(a, t_2) &= t_2 + \frac{a}{2}. \end{aligned}$$

The allocation rule is the efficient rule. So, it is defined as, for every $t_1 \in T_1$ and $t_2 \in T_2$,

$$f(t_1, t_2) \in \arg\max_{a \in A} \left[v_1(a, t_1) + v_2(a, t_2) \right]$$

Claim 1 *For every $t_1, t_2 \in [0,1] = T_1 = T_2$, we have $f(t_1,t_2) = t_1$.*

Proof: Define $V(a,t_1,t_2) = v_1(a,t_1) + v_2(a,t_2)$. If $f(t_1,t_2) = t_1$, then $V(f(t_1,t_2),t_1,t_2) = v_1(t_1,t_1) + v_2(t_1,t_2) = t_2 + \frac{t_1}{2}$. If $f(t_1,t_2) = a < t_1$, then $V(f(t_1,t_2),t_1,t_2) = v_1(a,t_1) + v_2(a,t_2) = t_2 + \frac{a}{2} < t_2 + \frac{t_1}{2} = V(t_1,t_1,t_2)$. If $f(t_1,t_2) = a > t_1$, then $V(f(t_1,t_2),t_1,t_2) = v_1(a,t_1) + v_2(a,t_2) = t_1 - a + t_2 + \frac{a}{2} = t_2 + t_1 - \frac{a}{2} < t_2 + \frac{t_1}{2} = V(t_1,t_1,t_2)$. Hence, $f(t_1,t_2) = t_1$. ■

Now fix the type of agent 2 to $t_2 \in T_2$ and consider agent 1. Consider $s,t \in T_1$. If $s < t$, then $l(s,t) = v_1(f(t),t) - v_1(f(s),t) = v_1(t,t) - v_1(s,t) = 0$. If $s > t$, then $l(s,t) = v_1(t,t) - v_1(s,t) = 0 - (t-s) = s - t > 0$. Hence, every edge in the type graph has non-negative length. Now, consider $s,t \in T$ such that $s < t$. Since every edge in type graph has non-negative length, the length of every finite path from s to t has to be non-negative. But $l(s,t) = 0$. Hence, $dist_{T_f}(s,t) = 0$. Now, consider $s,t \in T$ such that $s > t$. Consider any finite path $(s, s_1, \ldots, s_k, t)$. Since length of edge (u,v) such that $u < v$ is zero, we can, without loss of generality, assume $s > s_1 > \ldots > s_k > t$. Length of this path is $l(s,s_1) + l(s_1,s_2) + \ldots + l(s_{k-1},s_k) + l(s_k,t) \geq s - t$. Hence, $dist_{T_f}(s,t) = l(s,t) = (s-t)$. Hence, for any $s,t \in T_1$, we have $dist_{T_f}(s,t) + dist_{T_f}(t,s) = |s-t| > 0$. By Theorem 3, f does not satisfy revenue equivalence. Holmstrom (1979) construct specific payment functions for this example which does not satisfy revenue equivalence. It is clear that for agent 1, zero payment in all types is one payment rule and any rule in the Groves class of payment rules is also feasible. These two do not differ by a constant.

This example illustrates that when A is not countable, even if T is convex, there are allocation rules which do not satisfy revenue equivalence when v is piecewise linear in type. So, linearity of v is crucial in Theorem 4. Also, Theorem 5 fails if A is not countable.

3.3 A Characterization for Finite Set of Alternatives

The following characterization for finite A is due to Chung and Olszewski (2007). Let A be a finite set and B_1, B_2 be non-empty partition of A and $r : A \to \mathbb{R}$. For every $\epsilon > 0$, define

$$V_1(\epsilon) = \cup_{b_1 \in B_1}\{t \in T : v(b_1,t) - v(b_2,t) > r(b_1) - r(b_2) + \epsilon \ \forall \ b_2 \in B_2\}$$
$$V_2(\epsilon) = \cup_{b_2 \in B_2}\{t \in T : v(b_2,t) - v(b_1,t) > r(b_2) - r(b_1) + \epsilon \ \forall \ b_1 \in B_1\}.$$

Type space T is called **splittable** if there exists such $B_1, B_2 \subsetneq A$ with $B_1 \cap B_2 = \emptyset$, and $B_1 \cup B_2 = A$, and $r : A \to \mathbb{R}$, and $\epsilon > 0$ such that $T = V_1(\epsilon) \cup V_2(\epsilon)$. Note that if $t \in V_1(\epsilon) \cap V_2(\epsilon)$, then for some $b_1 \in B_1$ and for some $b_2 \in B_2$, we have $v(b_1,t) - v(b_2,t) > r(b_1) - r(b_2)$ and $v(b_2,t) - v(b_1,t) > r(b_2) - r(b_1)$, which is not possible. Hence, $V_1(\epsilon) \cap V_2(\epsilon) = \emptyset$.

Theorem **7** (**Chung and Olszewski** (**2007**)) *Suppose A is finite. Every DSIC allocation rule satisfies revenue equivalence if and only if T is not splittable.*

Proof: Suppose every DSIC allocation rule satisfies revenue equivalence. Assume for contradiction T is splittable, and the concerned disjoint sets are B_1 and B_2, and the function is r. Consider the following allocation rule f:

$$f(t) \in \arg\max_{a \in A}[v(a,t) - r(a)].$$

One payment rule which makes f DSIC is p defined as $p(t) = r(f(t))$ for all $t \in T$ - this follows from the definition of f. Here is another payment rule which makes f DSIC. Define p' as:

$$p'(t) = r(f(t)) + \frac{\epsilon}{2} \qquad \forall\ t \in V_1(\epsilon)$$
$$p'(t) = r(f(t)) - \frac{\epsilon}{2} \qquad \forall\ t \in V_2(\epsilon).$$

Note that (a) p' is a well defined payment rule since $T = V_1(\epsilon) \cup V_2(\epsilon)$ and $V_1(\epsilon) \cap V_2(\epsilon) = \emptyset$ and (b) $p(t) - p'(t)$ is not a constant for all $t \in T$. Now, consider $s \in V_1(\epsilon)$. By definition of $s \in V_1(\epsilon)$, there exists $b \in B_1$ such that for all $a \in A_1$ we have

$$v(b,s) - r(b) > v(a,s) - r(a) + \epsilon.$$

Hence, $f(s) \in B_1$. Let $f(s) = b \in B_1$. Then, again by definition of $V_1(\epsilon)$, $v(b,s) - [r(b) + \frac{\epsilon}{2}] > v(a,s) - [r(a) - \frac{\epsilon}{2}]$ for all $a \in B_2$. Hence, the agent has no incentive to deviate to a type t such that $f(t) \in B_2$. By the choice of f and p', he has no incentive to deviate to a type t such that $f(t) \in B_1$. Thus, we have a payment rule p' which makes f DSIC but does not differ from f by a constant. Hence, f does not satisfy revenue equivalence.

Suppose T is not splittable. Consider an allocation rule f which is DSIC. Assume for contradiction that f does not satisfy revenue equivalence. By Theorem 3 we have $dist_{A_f}(a,b) + dist_{A_f}(b,a) > 0$ for some $a, b \in A$. Define the following sets:

$$B_1 = \{c \in A : dist_{A_f}(a,c) + dist_{A_f}(c,a) = 0\}$$
$$B_1 = \{c \in A : dist_{A_f}(a,c) + dist_{A_f}(c,a) > 0\}.$$

Clearly, B_1 and B_2 are partitions of A with $a \in B_1$ and $b \in B_2$. From the definition of B_2 for every $b_1 \in B_1$ and $b_2 \in B_2$ we get $dist_{A_f}(a,b_2) + d(b_2,b_1) + dist_{A_f}(b_1,a) > 0$ (otherwise, we will have $dist_{A_f}(b_1,b_2) + dist_{A_f}(b_2,b_1) = 0$). Since $a, b_1 \in B_1$, we have $dist_{A_f}(a,b_1) + dist_{A_f}(b_1,a) = 0$, which implies that $dist_{A_f}(a,b_2) + d(b_2,b_1) - dist_{A_f}(a,b_1) > 0$. Hence, we get for every $b_1 \in B_1$ and $b_2 \in B_2$ we have

$$d(b_2,b_1) > dist_{A_f}(a,b_1) - dist_{A_f}(a,b_2).$$

Since B_1 and B_2 are finite, there exists $\delta > 0$ such that

$$d(b_2,b_1) > dist_{A_f}(a,b_1) - dist_{A_f}(a,b_2) + \delta.$$

We use this to define the r function as follows.

$$r(c) = dist_{A_f}(a, c) + \delta \qquad \forall\, c \in B_1$$
$$r(c) = dist_{A_f}(a, c) \qquad \forall\, c \in B_2.$$

So, for all $b_1 \in B_1$ and $b_2 \in B_2$, we have $d(b_2, b_1) > dist_{A_f}(a, b_1) + \delta - dist_{A_f}(a, b_2) = r(b_1) - r(b_2)$. Also $dist_{A_f}(a, b_1) + d(b_1, b_2) \geq dist_{A_f}(a, b_2)$. Hence, $d(b_1, b_2) \geq r(b_2) - r(b_1) + \delta$. This implies that $d(b_1, b_2) > r(b_2) - r(b_1)$. Since B_1 and B_2 are finite, there exists $\epsilon > 0$ such that for all $b_1 \in B_1$ and for all $b_2 \in B_2$ we have

$$d(b_2, b_1) > r(b_1) - r(b_2) + \epsilon$$
$$d(b_1, b_2) > r(b_2) - r(b_1) + \epsilon.$$

By definition of $d(b_2, b_1)$ for all $t \in T_{b_1}$ we have $v(b_1, t) - v(b_2, t) \geq d(b_2, b_1)$ and, similarly, for all $t \in T_{b_2}$ we have $v(b_2, t) - v(b_1, t) \geq d(b_1, b_2)$. This implies that $T_{b_1} \subseteq V_1(\epsilon)$ and $T_{b_2} \subseteq V_2(\epsilon)$. Hence, $T \subseteq V_1(\epsilon) \cup V_2(\epsilon)$. Thus, T is splittable, a contradiction. ■

Chung and Olszewski (2007) use the characterization in Theorem 7 to deduce results in Theorems 5 and 6. The proofs we give for Theorems 5 and 6 are based on Heydenreich et al. (2009), which uses Theorem 3.

Chung and Olszewski (2007) also give sufficient conditions for every DSIC allcation rule to satisfy revenue equivalence when A is not countable and $v(a, s) = a \cdot s$. The sufficient condition is called *bounded gridwise connected set*. It is similar to the characterization condition in Theorem 3. Smoothly connected sets, ala Holmstrom (1979), are bounded gridwise connected. In summary, the sufficient condition in Chung and Olszewski (2007) for uncountable A point to the following two type spaces where every DSIC allocation rule satisfies revenue equivalence: (1) for every pair $s, t \in T$, s and t can be connected by an arc of finite length; (2) for every pair $s, t \in T$, s and t can be connected by a smooth arc.

4 (2-Cycle) Monotonicity

Though Theorem 2 characterizes DSIC allocation rules, it requires verification of lengths of large number of cycles. The natural question is under what domains of types can we restrict attention to verification of lengths of smaller number of cycles. The following definition is a step towards that direction.

Definition 8 *An allocation rule f satisfies* **2-cycle monotonicity** *or simply* **monotonicity** *if for all $s, t \in T$*

$$l(s, t) + l(t, s) \geq 0.$$

Note that the definition of monotonicity can be stated in terms of allocations also. By virtue of Theorem 2, the allocation rule f satisfies monotonicity if for all $a, b \in A$, $d(a,b) + d(b,a) = 0$. Also, note that since DSIC is equivalent to cycle monotonicity and cycle monotonicity implies monotonicity, monotonicity is necessary for an allocation rule to be DSIC. In the rest of article, we discuss when monotonicity is sufficient for an allocation rule to be DSIC, i.e., when monotonicity implies cycle monotonicity.

The **domain** of a mechanism specifies a type space T, the allocation set A, and value function v. A domain (T, A, v) is called a **monotonicity domain** if every f that satisfies monotonicity also satisfies cycle monotonicity in this domain.

Note that monotonicity is called **weak monotonicity** in Bikhchandani et al. (2006), who characterize DSIC rules for combinatorial auction domains using this, i.e., A is finite, $T = \mathbb{R}^{|A|}_{+}$, and $v(a,t) =< a, t >$. Their characterization result is later subsumed in two subsequent papers - Saks and Yu (2005) and Monderer (2008), where they relax T to be a convex set (A is finite and $v(a,t) =< a, t >$). The monotonicity condition is also used for characterizing Bayes-Nash incentive compatible rules in Muller et al. (2007).

4.1 The Single Object Auction Case

It is a good exercise to look at the simple but insightful case of single object auction. In the single object auction case, the type set of an agent is one dimensional, i.e., $T \subseteq \mathbb{R}^1$. An allocation gives a probability of winning the object. Hence, $A \subseteq [0,1]$. This section examines this case. The analysis is taken from Vohra (2008) [6].

We say the valuation function v satisfies **strict increasing differences** [7] if for all $a, b \in A$ with $a > b$ and for all $s, t \in T$ with $s > t$ we have

$$v(a,s) - v(b,s) > v(a,t) - v(b,t).$$

It is standard to have $v(a,s) = a \times s$ for all $a \in A$ and $s \in T$. Note that such a form of v function satisfies strict increasing differences.

In the one dimensional case, the monotonicity condition has nice interpretation if we assume $v(a,s) = a \times s$. Consider $s, t \in T$ and $s > t$. Then $l(s,t) + l(t,s) = v(f(t),t) - v(f(s),t) + v(f(s),s) - v(f(t),s) = [f(t) - f(s)] \times (t - s)$. Hence, $l(s,t) + l(t,s) \geq 0$ is equivalent to saying $f(s) \geq f(t)$ for all $s > t$, i.e., f is **non-decreasing**. With strict increasing differences also, monotonicity is equivalent to non-decreasing f. This is shown in the next theorem.

Theorem **8** (**Vohra** (**2008**)) *Suppose $T, A \subseteq \mathbb{R}^1$ and v satisfies strict increasing differences. An allocation rule f is DSIC if and only if it is non-decreasing.*

[6] Vohra (2008) is also an excellent account of all the topics we discuss in this note.

[7] This condition is also known as the **single crossing property**.

Proof: Suppose f is DSIC. Then, consider any $s, t \in T$ with $s > t$. By cycle monotonicity, $l(s,t) + l(t,s) \geq 0$. Hence,

$$v(f(t),t) - v(f(s),t) + v(f(s),s) - v(f(t),s) \geq 0.$$

Rearranging, we get

$$v(f(t),s) - v(f(s),s) \leq v(f(t),t) - v(f(s),t) \tag{6}$$

Assume for contradiction, $f(s) < f(t)$. Then, Equation 6 contradicts strict increasing differences.

Suppose f is non-decreasing. To show that f is DSIC, we need to show f satisfies cycle monotonicity, i.e., length of any cycle having finite number of nodes (types) is non-negative. We use induction on number of nodes involved in a cycle.

First note that for any $s, t \in T$, $s > t$ implies that $f(s) \geq f(t)$, and strict increasing differences implies that $l(s,t) = v(f(t),t) - v(f(s),t) \geq v(f(t),s) - v(f(s),s) = -l(t,s)$. Hence, $l(s,t) + l(t,s) \geq 0$. So, any cycle involving two nodes has non-negative length.

Now consider a cycle with $(k+1)$ nodes, and assume that any cycle involving less than $(k+1)$ nodes has non-negative length. Let the cycle be $(t_1, t_2, \ldots, t_{k+1})$, and let, without loss of generality, $t_{k+1} > t_j$ for all $j \in \{1, \ldots, k\}$. We first show that $l(t_k, t_{k+1}) + l(t_{k+1}, t_1) \geq l(t_k, t_1)$.

Assume for contradiction $l(t_k, t_{k+1}) + l(t_{k+1}, t_1) < l(t_k, t_1)$. Then, $v(f(t_{k+1}), t_{k+1}) - v(f(t_k), t_{k+1}) + v(f(t_1), t_1) - v(f(t_{k+1}), t_1) < v(f(t_1), t_1) - v(f(t_k), t_1)$. Hence,

$$v(f(t_{k+1}), t_{k+1}) - v(f(t_k), t_{k+1}) < v(f(t_{k+1}), t_1) - v(f(t_k), t_1). \tag{7}$$

Since $t_{k+1} > t_1$ and $t_{k+1} > t_k$ implies $f(t_{k+1}) \geq f(t_k)$, Equation 7 contradicts strict increasing differences. Hence, $l(t_k, t_{k+1}) + l(t_{k+1}, t_1) \geq l(t_k, t_1)$.

Now, the length of the cycle $(t_1, t_2, \ldots, t_{k+1}, t_1)$ is $l(t_1, t_2) + \ldots + l(t_k, t_{k+1}) + l(t_{k+1}, t_1) \geq l(t_1, t_2) + \ldots + l(t_{k-1}, t_k) + l(t_k, t_1)$. But the term in the right is the length of the cycle $(t_1, t_2, \ldots, t_k, t_1)$, which has k nodes. By induction hypothesis, the length of this cycle is non-negative. Hence, $l(t_1, t_2) + \ldots + l(t_k, t_{k+1}) + l(t_{k+1}, t_1) \geq 0$. ■

A relatively easy extension of Theorem 8 to multidimensional case is possible. Suppose there is a linear ordering $\succ_A$ on the set of alternatives A and a linear ordering $\succ_T$ on the set of types T, where $T \subseteq \mathbb{R}^n$. In this case, we will say f is **non-decreasing** if for every $s \succ_T t$ we have either $f(s) = f(t)$ or $f(s) \succ_A f(t)$. We say valuation function satisfies **strict increasing differences** if for all $a \succ_A b$ and for all $s \succ_T t$ we have

$$v(a,s) - v(b,s) > v(a,t) - v(b,t).$$

It is easily seen that the proof of Theorem 8 does not use the fact that T and A are one-dimensional. Hence, the following result holds.

Theorem 9 *Suppose there is a linear ordering $\succ_A$ on the set of alternatives A and a linear ordering $\succ_T$ on the set of alternatives $T \subseteq \mathbb{R}^n$, and v satisfies strict increasing differences. An allocation rule f is DSIC if and only if it is non-decreasing.*

Proof: Mimic the steps in the proof of Theorem 8. ∎

As long as the assumptions of Theorem 9 holds, the result in Theorem 9 holds even when T is finite. Mualem and Schapira (2008) call a *domain* satisfying assumptions in Theorem 9 a **Monge** domain when T is finite. The result in Mualem and Schapira (2008) is a corollary to Theorem 9 (or Theorem 8).

4.2 Difference Set

The objective of this section is to uncover some structure of the problem. We make a series of claims. To illustrate the claims, we assume the following:

- **Assumption 1:** The allocation set A is finite. Moreover $\#A = n$. Hence, every element $t \in T$ can be represented as a vector over allocations. So, $v(a,t) = t_a$ represents, the value attached to allocation $a \in A$ in type t.

An example of a multidimensional setting is the combinatorial auction example. Let $M = \{1, \ldots, m\}$ be the set of objects. A buyer (agent) has value for every bundle of objects. The set of bundles is denoted by Ω. An allocation denotes which bundle is allocated to which buyer. So, here $A = \Omega$. The type of an agent is his value for various bundles - hence, $v(a,t) = t_a$ is the value of bundle $a \in A = \Omega$ in type t. Sometimes, it is convenient to think of A as a vector of 0s and 1s, where 1 indicates the bundle is allocated, and $v(a,t) =< a, t >$ gives the value of an allocation.

For every $a \in A$, define the **difference set** of a as

$$D_a = \big\{t \in T : v(a,t) - v(b,t) \geq d(b,a) = \inf_{s \in T_a} [v(a,s) - v(b,s)] \ \forall\, b \in A \setminus \{a\}\big\}.$$

Note that D_a is the interesection of T with a polyhedron $\{t \in \mathbb{R}^{|A|} : t_a - t_b \geq d(b,a) \ \forall\, b \in A \setminus \{a\}\}$. Hence if t is a point in the interior of D_a, we must have $t_a - t_b > d(b,a)$ for all $b \in A \setminus \{a\}$.

Define for every $B \subseteq A$, $D_B = \cap_{b \in B} D_b$. Note that $v(a,t) - v(b,t)$ is the increase in value of the agent at type t from alternative b to alternative a, whereas $d(b,a)$ is the infimum increase in the value of the agent from alternative b to alternative a, where the infimum is over all types s with $f(s) = a$. Since f is onto, $T_a \neq \emptyset$ and $D_a \neq \emptyset$.

Claim 2 *For every $a \in A$, we have $T_a \subseteq D_a$.*

Proof: For all $s \in T_a$, $v(a,s) - v(b,s) \geq d(b,a)$ for all $b \in A \setminus \{a\}$. Hence $T_a \subseteq D_a$ for all $a \in A$. ■

For any set S, let S^o denote the interior of S.

Claim **3** *Suppose f satisfies monotonicity. Then, for every $a, b \in A$, $D_a^o \cap D_b = \emptyset$.*

Proof: Assume for contradiction that there exists $t \in D_a^o \cap D_b$. But $v(a,t) - v(b,t) > d(b,a)$ since $t \in D_a^o$ and $v(b,t) - v(a,t) \geq d(a,b)$ since $t \in D_b$. This gives us $d(a,b) + d(b,a) < 0$, a contradiction to monotonicity. ■

Claim **4** *Suppose f satisfies monotonicity. Then, for every $a \in A$, we have $D_a^o \subseteq T_a$.*

Proof: Assume for contradiction there exists a $t \in D_a^o$ such that $t \notin T_a$. Since f is onto, $t \in T_b$ for some $b \neq a$. By Claim 2, $t \in D_b$. Hence, $t \in D_b \cap D_a^o$, a contradiction to Claim 3. ■

Claim **5** *If T is closed and convex, then D_a is closed and convex for every $a \in A$.*

Proof: For every $a \in A$, D_a is the intersection of a polyhedron with T. Since T is closed and convex, D_a is also closed and convex. ■

These claims point to a *phase diagram* of the type space. The entire type space can be divided into polyhedra (D_as). In the interior of these polyhedra, we get a fixed outcome, e.g., in the interior of D_a the outcome is a. The boundary between polyhedra can have any of the outcomes (at any point on the boundary) that define the boundary. An illustration is given in Figure 3.

Lemma **3 (Intersection Lemma, Monderer (2008))** *If f satisfies monotonicity and $\cap_{a \in A} D_a \neq \emptyset$* [8]*, then, f satisfies cycle monotonicity.*

Proof: Let $t \in \cap_{a \in A} D_a$. Consider any $a, b \in A$. Note that $v(a,t) - v(b,t) \geq d(b,a)$ since $t \in D_a$. Similarly, $v(b,t) - v(a,t) \geq d(a,b)$ since $t \in D_b$. Hence, $0 \geq d(a,b) + d(b,a) \geq 0$, where the last inequality comes from the fact that f satisfies monotonicity. Hence, $v(a,t) - v(b,t) = d(b,a)$ for all $a, b \in A$.

Now, consider a cycle $(a_1, \ldots, a_k)$ in A_f. The length of this cycle is $d(a_1,a_2) + d(a_2,a_3) + \ldots + d(a_{k-1},a_k) + d(a_k,a_1)$, and by substituting the value of $d(\cdot,\cdot)$, we get the length of this cycle to be 0. ■

Remark: Notice that the lemma is independent of the structure of the type space.

[8] If A is not finite, then this is an infinite intersection.

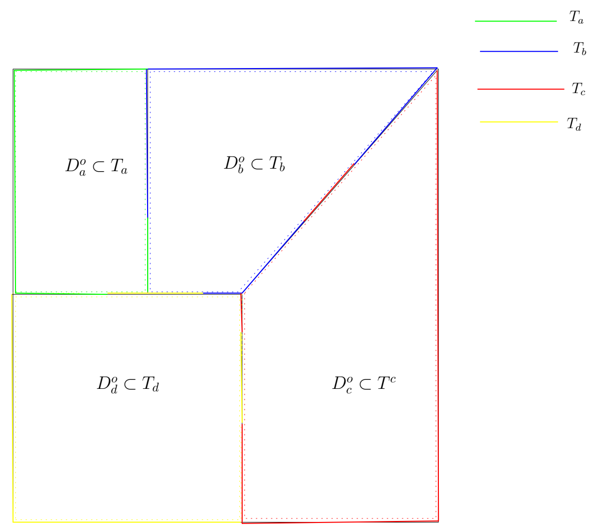

Figure 3: Phase Diagram of Type Space

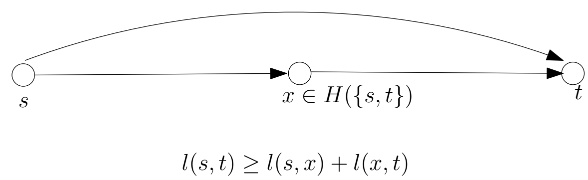

Figure 4: Decomposing an direct path into two smaller paths

4.3 Compact and Convex Type Space

In this section, we will assume that T is compact and convex. We show that T is a monotonicity domain in that case. For any set $S \subset T$, we write $H(S)$ to denote the convex hull of S.

Lemma 4 **(Decomposition Lemma)** *Suppose T is convex, $v(\cdot,\cdot)$ is linear in type, and f satisfies monotonicity. Then, for every $s,t \in T$ and $x \in H(\{s,t\})$, we have $l(s,t) \geq l(s,x) + l(x,t)$ (see Figure 4).*

Proof: By convexity, $x \in T$. Let $t = x + \alpha(x - s)$ for some $\alpha > 0$. Then,

$$\begin{aligned} l(s,x) + l(x,t) &= v(f(x),x) - v(f(s),x) + v(f(t),t) - v(f(x),t) \\ &= v(f(x),x) - v(f(s),x) + v(f(s),t) - v(f(x),t) + l(s,t) \\ &= v(f(x),x) - v(f(s),x) + (1+\alpha)v(f(s),x) - \alpha v(f(s),s) \\ &- (1+\alpha)v(f(x),x) + \alpha v(f(x),s) + l(s,t) \\ &= l(s,t) - \alpha\big[[v(f(s),s) - v(f(x),s)] - [v(f(x),x) - v(f(s),x)]\big] \\ &= l(s,t) - \alpha\big[l(s,x) + l(x,s)\big] \\ &\leq l(s,t), \end{aligned}$$

where the inequality follows from monotonicity and the fact that $\alpha > 0$. ■

We will say f satisfies cycle monotonicity on $K \subset T$ if every cycle in the type graph corresponding to types in K has non-negative length. Similarly, we say f satisfies monotonicity on $K \subset T$ if every 2-cycle in the type graph corresponding to types in K has non-negative length.

Theorem **10 (Union Theorem, Monderer (2008))** *Suppose $v(\cdot,\cdot)$ is linear in type and f satisfies monotonicity. Let $T = T_1 \cup T_2$ be a closed and convex set, where T_1 and T_2 are non-empty closed and convex sets. If f satisfies cycle monotonicity on T_1 and T_2, then f satisfies cycle monotonicity.*

Proof: Since f satisfies cycle monotonicity on T_1 and T_2, we define two payment rules on T_1 and T_2. Let $s \in T_1 \cap T_2$ (such a s exists since T is connected). Define $p_1 : T_1 \to \mathbb{R}$ and $p_2 : T_2 \to \mathbb{R}$ as follows.

$$\begin{aligned} p_1(t_1) &= dist_{T_{1_f}}(s, t_1) \qquad \forall\ t_1 \in T_1 \\ p_2(t_2) &= dist_{T_{2_f}}(s, t_2) \qquad \forall\ t_2 \in T_2. \end{aligned}$$

By Lemma 1, p_1 and p_2 define payment rules that make f DSIC in T_1 and T_2 respectively. Clearly, p_1 and p_2 restricted to $T_1 \cap T_2$ define payment rules that makes f DSIC in $T_1 \cap T_2$. Since $T_1 \cap T_2$ is convex, we use revenue equivalence (Theorem 3) in $T_1 \cap T_2$ to conclude $p_1(t) - p_2(t) = \kappa$ for all $t \in T_1 \cap T_2$ for some constant κ [9]. Clearly, $p_2(t) + \kappa$ for all $t \in T_2$ define a payment rule that makes f DSIC in T_2. Now, we define a payment rule $\pi : T \to \mathbb{R}$ as follows: $\pi(t) = p_1(t)$ for all $t \in T_1$ and $\pi(t) = p_2(t) + \kappa$ for all $t \in T_2$. We will argue that π makes f DSIC in T.

Consider $t_1, t_2 \in T$. If $t_1, t_2 \in T_1$ or $t_1, t_2 \in T_2$, then we know that $\pi(t_2) - \pi(t_1) \leq l(t_1, t_2)$. Now, consider $t_1 \in T_1 \setminus T_2$ and $t_2 \in T_2 \setminus T_1$. Since T is convex, there exists $t \in T_1 \cap T_2$

[9] Recall that Theorem 3 did not require A to be finite.

such that $t \in H(\{t_1, t_2\})$, where $H(\{t_1, t_2\})$ is the convex hull of t_1 and t_2 [10]. Note that $\pi(t_2) - \pi(t) \leq l(t, t_2)$ and $\pi(t) - \pi(t_1) \leq l(t_1, t)$. Then, $\pi(t_2) - \pi(t_1) = [\pi(t_2) - \pi(t)] + [\pi(t) - \pi(t_1)] \leq l(t, t_2) + l(t_1, t) \leq l(t_1, t_2)$, where the last inequality follows from decomposition lemma (Lemma 4). ■

The main theorem uses the union theorem and the intersection lemma.

THEOREM **11** (**Saks and Yu** (**2005**), **Monderer** (**2008**)) *Suppose A is finite and v is linear in type. If T is a compact and convex set, then T is a monotonicity domain.*

Proof: We first prove that D_a for all $a \in A$ is compact and convex if T is compact and convex and v is linear in type. Since T is bounded, D_a is bounded for all $a \in A$. Since T is closed, any sequence $\{t_n\}_{n \geq 1}$ in D_a for any $a \in A$ will converge to a type $t \in T$. Since v is continuous in t, for any $b \in A \setminus \{a\}$, we have $v(a, t) - v(b, t) = \lim_{n \to \infty}[v(a, t_n) - v(b, t_n)] \geq d(b, a)$. Hence, $t \in D_a$. So, D_a is closed. Now, choose $s, t \in D_a$ for any $a \in A$. Let $x = \alpha s + (1 - \alpha)t$ for $\alpha \in (0, 1)$. Since T is convex, $x \in T$. So, for any $b \in T$, $v(a, x) - v(b, x) = \alpha[v(a, s) - v(b, s)] + (1 - \alpha)[v(a, t) - v(b, t)] \geq d(b, a)$. Hence, $x \in T$, implying that D_a is convex.

For the proof, we use double induction. The first induction is on $\#A$. If $\#A = 2$, then the allocation graph has only 2-cycles, which implies that f satisfies cycle monotonicity. Suppose the claim holds for any $\#A < k$ and consider A with $A = \{a_1, \ldots, a_k\}$.

For every T and f, define $r(T, f)$ to be the maximal number r $(1 \leq r \leq k)$ for which every set F of r distinct values from $\{1, \ldots, k\}$, the set $\cap_{j \in F} D_{a_j} \neq \emptyset$. The next induction is on $r(T, f)$.

Suppose $r(T, f) = 1$. So, there exists $a, b \in A$ such that $D_a \cap D_b = \emptyset$. The sets D_a and D_b are compact and convex. So, we can strictly separate them by a hyperplane. So, there exists, $0 \neq y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that

$$s \cdot y < \alpha < t \cdot y \qquad \forall\, s \in D_a,\ \forall\, t \in D_b.$$

Define $H_1 = \{t \in T : t \cdot y \leq \alpha\}$ and $H_2 = \{t \in T : t \cdot y \geq \alpha\}$. Note that H_1 and H_2 are compact and convex because T is compact and convex. Now, in each H_1 and H_2, f takes on at most $k - 1$ values, and therefore by our first induction hypothesis, f satisfies cycle monotonicity on each H_1 and H_2. By Theorem 10, f satisfies cycle monotonicity.

Suppose the claim holds for $1, \ldots, r - 1$ $(2 \leq r \leq k)$. We now prove it for $r(T, f) = r$. If $r = k$, it follows from Lemma 3. Hence, suppose $r < k$. Since $r < k$, there exists a set of $r + 1$ indices, say $\{1, \ldots, r + 1\}$ such that $\cap_{j=1}^{r+1} D_{a_j} = \emptyset$ and $\cap_{j=1}^{r} D_{a_j} \neq \emptyset$. Denote

[10] The complete argument is that the sets $T_1 \cap H(\{t_1, t_2\})$ and $T_2 \cap H(\{t_1, t_2\})$ are both closed and convex. Since $(T_1 \cap H(\{t_1, t_2\})) \cup (T_2 \cap H(\{t_1, t_2\})) = H(\{t_1, t_2\})$, by connectedness of convex sets, we conclude that $(T_1 \cap H(\{t_1, t_2\})) \cap (T_2 \cap H(\{t_1, t_2\})) \neq \emptyset$.

$R = \{a_1, \ldots, a_r\}$, $D_R = \cap_{j=1}^r D_{a_j}$ and $D_{r+1} = D_{a_{r+1}}$. The set D_R and D_{r+1} are compact and convex and may be strictly separated. So, there exists $0 \neq y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that

$$s \cdot y < \alpha < t \cdot y \qquad \forall\, s \in D_R,\ \forall\, t \in D_{r+1}.$$

Define $H_1 = \{t \in T : t \cdot y \leq \alpha\}$ and $H_2 = \{t \in T : t \cdot y \geq \alpha\}$. On H_1, the function f does not take on the value a_{r+1}. By our first induction hypothesis f satisfies cycle monotonicity on H_1. On H_2, if f takes on less than k values, then again by the first induction hypothesis, f satisfies cycle monotonicity on H_2. Else, f takes on k values on H_2. We show that $r(H_2, f) < r$.

Since $H_2 \subseteq T$, $d_{H_2}(b, a) \geq d_T(b, a)$ for all $a, b \in A$, where $d_S(b, a)$ denotes the length of edge in the allocation graph from b to a when we consider a set of types $S \subseteq T$. Define $H_{2_a} = \{t \in H_2 : v(a, t) - v(b, t) \geq d_{H_2}(b, a)\ \forall\, b \in A\}$ for every $a \in A$. Note that $H_{2_a} \subseteq D_a$ for all $a \in A$. Hence $\cap_{a \in R} H_{2_a} \subseteq (H_2 \cap D_R)$. By definition, $H_2 \cap D_R = \emptyset$. Hence $\cap_{a \in R} H_{2_a} = \emptyset$. So, $r(H_2, f) < r$. Thus, by our second induction hypothesis, f satisfies cycle monotonicity on H_2. Hence, by the union theorem (Theorem 10), f satisfies cycle monotonicity on T. ■

4.4 Convex Type Spaces

Theorem 11 can be easily generalized to the case where T is convex but need not be closed or bounded.

Theorem 12 (**Saks and Yu** (**2005**), **Monderer** (**2008**)) *Suppose A is finite and $v(\cdot, \cdot)$ is linear in type. If closure of T is convex, then T is a monotonicity domain.*

Proof: If T is compact, then we are done by Theorem 11. We prove the following lemma.

Lemma 5 *If $cl(T)$ is a monotonicity domain, then T is a monotonicity domain.*

Proof: Consider f on T which satisfies monotonicity. We will extend f to $cl(T)$. Define the boundary points of T as $T^b = cl(T) \setminus T$. For every $t \in T^b$, there exists an infinite sequence $\{t_k\}_{k \geq 1}$ such that $\lim_{k \to \infty} t_k = t$. Hence, there exists a $a \in A$ such that for infinite number of types t_j in $\{t_k\}_{k \geq 1}$ we have $f(t_j) = a$ (this is because A is finite). Hence, for every $t \in T^b$, there exists a sequence of $\{t_j\}_{j \geq 1}$ such that $\lim_{j \to \infty} t_j = t$ and $f(t_j) = a$ for every $j \geq 1$. Let $f(t) = a$.

We will show that the extension of f to $cl(T)$ satisfies monotonicity. By assumption, f satisfies monotonicity on T. Hence, consider two types $t_1, t_2 \in cl(T)$ such that at least one of them is in T_b. Let $t_1 \in T^b$ and $t_2 \in T$. Let $f(t_1) = a$ and $f(t_2) = b$. Let $\{t_j\}_{j \geq 1}$ be a sequence in T which converges to t_1 with $f(t_j) = a$ for all t_j in the sequence. Now, using continuity of v with respect to type,

$$\begin{aligned} l(t_1, t_2) + l(t_2, t_1) &= v(b, t_2) - v(a, t_2) + v(a, t_1) - v(b, t_1) \\ &= \lim_{j\to\infty} l(t_j, t_2) + \lim_{j\to\infty} [v(a, t_j) - v(b, t_j)] \\ &= \lim_{j\to\infty} [l(t_j, t_2) + l(t_2, t_j)] \\ &\geq 0, \end{aligned}$$

where the last inequality comes from the fact that t_2 and every type in sequence $\{t_j\}_{j\geq 1}$ is in T, where monotonicity holds. A similar argument works when $t_1, t_2 \in T_b$ (in this case, we need to consider the sequence in T which converges to t_1).

Hence, f satisfies monotonicity on $cl(T)$. Since $cl(T)$ is a monotonicity domain, f satisfies cycle monotonicity on $cl(T)$. Hence, f satisfies cycle monotonicity on T. ■

Consider a T. Suppose $cl(T)$ is bounded, then we are done by Theorem 11 and Lemma 5. If $cl(T)$ is not bounded, consider any finite cycle in $cl(T)$, and assume it has negative length. The convex hull of such a cycle, $H(C)$, will be compact and convex. Since $cl(T)$ is convex, $H(C) \subseteq cl(T)$. By Theorem 11 f satisfies cycle monotonicity on $H(C)$, contradicting the fact the C is a negative cycle. ■

5 Affine Maximizers as Mechanisms

In this section, we show a different kind of characterization of DSIC rules. This is due to Roberts (1979). The proof is from Lavi et al. (2009). We denote $M = \{1, \ldots, m\}$ to be the set of agents. Let A be finite and $|A| \geq 3$. Also, we assume the type set of every agent is $T = \mathbb{R}^{|A|}$. A type profile of agents will be denoted as $\mathbf{t}$ and the type vector of agent i in type profile $\mathbf{t}$ is t_i, where $t_i(a)$ denotes his value for allocation $a \in A$. Then the objective of the section is to prove the following theorem of Roberts (1979). The proof is borrowed from Lavi et al. (2009) and Vohra (2008). Specially, the graph theoretic interpretation of the proof is borrowed from Vohra (2008).

Theorem **13** (**Roberts (1979)**) *Suppose A is finite, $|A| \geq 3$, $T = \mathbb{R}^{|A|}$, and consider an onto allocation rule f. The allocation rule f is DSIC if and only if there exists non-zero $w \in \mathbb{R}^m_+$ and $\kappa \in \mathbb{R}^{|A|}$ such that*

$$f(\mathbf{t}) \in \arg\max_{a\in A} \big[\sum_{i\in M} w_i t_i(a) - \kappa_a\big] \qquad \forall\ \mathbf{t} \in T^m.$$

Proof: Suppose there exists $w \in \mathbb{R}^m_+$ and $\kappa \in \mathbb{R}^{|A|}$ such that

$$f(\mathbf{t}) \in \arg\max_{a \in A} \big[\sum_{i \in M} w_i t_i(a) - \kappa_a\big] \qquad \forall\ \mathbf{t} \in T^m.$$

We show that f satisfies cycle monotonicity, and hence DSIC. Fix types of agents other than agent i at $\mathbf{t_{-i}}$. Now, consider the type graph of agent i when type profile of other agents is $\mathbf{t_{-i}}$. Assume for contradiction that a cycle $(t_i^1, t_i^2, \ldots, t_i^k, t_i^1)$ has negative length in this graph, and let $f(t_i^j, \mathbf{t_{-i}}) = \mathbf{a_j}$ for all $j \in \{1, \ldots, k\}$. i.e.,

$$l(t_i^1, t_i^2) + l(t_i^2, t_i^3) + \ldots + l(t_i^{k-1}, t_i^k) + l(t_i^k, t_i^1) < 0$$
$$[t_i^2(a_2) - t_i^2(a_1)] + [t_i^3(a_3) - t_i^3(a_2)] + \ldots + [t_i^k(a_k) - t_i^k(a_{k-1})] + [t_i^1(a_1) - t_i^1(a_k)] < 0.$$

Since $w_i > 0$, we get

$$[w_i t_i^2(a_2) - w_i t_i^2(a_1)] + [w_i t_i^3(a_3) - w_i t_i^3(a_2)] + \ldots + [w_i t_i^k(a_k) - w_i t_i^k(a_{k-1})] + [w_i t_i^1(a_1) - w_i t_i^1(a_k)] < 0.$$

Now, denote $\mathbf{t^j} = (\mathbf{t_i^j}, \mathbf{t_{-i}})$ for all $j \in \{1, \ldots, k\}$. Adding and subtracting $\sum_{j=1}^k \sum_{l \in M: l \neq i} w_l t_l^j(a_j)$ to the above inequality, we get

$$\sum_{l \in M} w_l[t_l^2(a_2) - t_l^2(a_1)] + \sum_{l \in M} w_l[t_l^3(a_3) - t_l^3(a_2)] + \ldots + \sum_{l \in M} w_l[t_l^k(a_k) - t_l^k(a_{k-1})] + \sum_{l \in M} w_l[t_l^1(a_1) - t_i^l(a_k)] < 0. \quad (8)$$

But, by definition, for any $a, b \in A$ and any $\mathbf{t}$ with $f(\mathbf{t}) = \mathbf{a}$ we have

$$\sum_{l \in M} w_l[t_l(a) - t_l(b)] \geq \kappa_a - \kappa_b.$$

Hence, we can write Inequality (8) as

$$[\kappa_{a_2} - \kappa_{a_1}] + [\kappa_{a_3} - \kappa_{a_2}] + \ldots + [\kappa_{a_k} - \kappa_{a_{k-1}}] + [\kappa_{a_1} - \kappa_{a_k}] < 0.$$

Since the LHS of the above inequality is zero, we get a contradiction.

Now, suppose f is DSIC and onto. Fix any non-zero and non-negative w vector. Note that for a given w, the κ vectors exist if for every $a, b \in A$ we have

$$\kappa_a - \kappa_b \leq \inf_{\mathbf{t}: f(\mathbf{t}) = a} \sum_{i \in M} w_i[t_i(a) - t_i(b)].$$

We already know that these inequalities have a solution if the underlying graph has no cycles of negative length. The underlying graph has a finite set of nodes, one for each allocation in

A, and is a complete directed graph. The length of edge from allocation b to allocation a is denoted as $d_w(b, a)$, and is given by

$$d_w(b, a) = \inf_{\mathbf{t}: f(\mathbf{t}) = a} \sum_{i \in M} w_i [t_i(a) - t_i(b)].$$

Denote this directed weighted complete graph as G^w. If we can find a non-zero and non-negative w such that G^w has no cycles of negative length, then the claim is proved.

Now, consider a cycle $C = (a_1, a_2, \ldots, a_k, a_1)$ in G^w. The length of this cycle is:

$$\begin{aligned} l(C) = &\inf_{\mathbf{t}: f(\mathbf{t}) = a_2} \sum_{i \in M} w_i [t_i(a_2) - t_i(a_1)] + \inf_{\mathbf{t}: f(\mathbf{t}) = a_3} \sum_{i \in M} w_i [t_i(a_3) - t_i(a_2)] + \ldots \\ &+ \inf_{\mathbf{t}: f(\mathbf{t}) = a_k} \sum_{i \in M} w_i [t_i(a_k) - t_i(a_{k-1})] + \inf_{\mathbf{t}: f(\mathbf{t}) = a_1} \sum_{i \in M} w_i [t_i(a_1) - t_i(a_k)]. \end{aligned}$$

Now for each $a_j \in \{a_1, \ldots, a_k\}$, choose $\mathbf{t}^j$ such that $f(\mathbf{t}^j) = a_j$. Then, $l(C) \geq 0$ if and only if for every such choice of $\mathbf{t}^1, \ldots, \mathbf{t}^k$ we have

$$\begin{aligned} &\sum_{i \in M} w_i [t_i^2(a_2) - t_i^2(a_1)] + \sum_{i \in M} w_i [t_i^3(a_3) - t_i^3(a_2)] + \ldots \\ &+ \sum_{i \in M} w_i [t_i^k(a_k) - t_i^k(a_{k-1})] + \sum_{i \in M} w_i [t_i^1(a_1) - t_i^1(a_k)] \geq 0. \end{aligned}$$

The RHS of the above inequality can be written as dot product of two vectors w and b, where $b \in \mathbb{R}^m$ is defined as:

$$b_i = [t_i^2(a_2) - t_i^2(a_1)] + [t_i^3(a_3) - t_i^3(a_2)] + \ldots + [t_i^k(a_k) - t_i^k(a_{k-1})] + [t_i^1(a_1) - t_i^1(a_k)].$$

Let K be the set of all such vectors. Hence, we need to prove that there exists a non-zero and non-negative $w \in \mathbb{R}_+^m$ such that

$$w \cdot b \geq 0 \qquad \forall\ b \in K.$$

The important steps in this proof will be

- If b is associated with some cycle $(a_1, \ldots, a_k, a_1)$, then it is associated with (a_1, a_k, a_1), i.e., attention can be restricted to 2-cycles.
- If b is associated with some cycle (a_1, a_k, a_1), then it is associated with every 2-cycle. Hence, attention can be restricted to only one 2-cycle.
- The set K is convex, and is disjoint from the negative orthant. Hence, separating hyperplane theorem immediately gives the desired result.

We do the proof by proving a series of technical lemmas.

Lemma **6** *Suppose f is DSIC. For some* $\mathbf{t}$*, let* $f(\mathbf{t}) = a$*. Let* $\mathbf{s}$ *be another type profile such that for all* $b \neq a$ [11]

$$s_i(a) - s_i(b) > t_i(a) - t_i(b) \qquad \forall\, i \in M.$$

Then, $f(\mathbf{s}) = a$*.*

Proof: By definition $t_i \neq s_i$ for all $i \in M$. Now, consider the following set of $m+1$ types. Define $\mathbf{t}^k$ for all $k \in \{0, 1, \ldots, m\}$ as

$$\begin{aligned} t_i^k &= t_i \qquad \forall\, i > k \\ t_i^k &= s_i \qquad \forall\, i \leq k. \end{aligned}$$

Note that $\mathbf{t}^0 = t$ and $\mathbf{t}^m = s$. Further for any $k \in \{0, 1, \ldots, m\}$, $\mathbf{t}^k$ and $\mathbf{t}^{k+1}$ differ in the type of agent $k+1$. Also, by definition, for every $k \in \{0, 1, \ldots, m\}$, we have

$$t_{k+1}^{k+1}(a) - t_{k+1}^{k+1}(b) > t_{k+1}^{k}(a) - t_{k+1}^{k}(b) \qquad \forall\, b \neq a. \tag{9}$$

We now prove the claim by induction arguments. Assume for contradiction $f(\mathbf{t}^1) = b \neq a$. Then, by the monotonicity for agent 1, we get

$$\begin{aligned} t_1^0(a) - t_1^0(b) + t_1^1(b) - t_1^1(a) &\geq 0 \\ t_1^1(a) - t_1^1(b) &\leq t_1^0(a) - t_1^0(b). \end{aligned}$$

This is a contradiction by Equation (9). Now, assume $f(\mathbf{t}^j) = a$ for all $j \leq k$ for some $k \geq 1$. We show that $f(\mathbf{t}^{k+1}) = a$. Assume for contradiction $f(\mathbf{t}^{k+1}) = b \neq a$. Then, by the monotonicity condition for agent $k+1$, we get

$$\begin{aligned} t_{k+1}^{k}(a) - t_{k+1}^{k}(b) + t_{k+1}^{k+1}(b) - t_{k+1}^{k+1}(a) &\geq 0 \\ t_{k+1}^{k+1}(a) - t_{k+1}^{k+1}(b) &\leq t_{k+1}^{k}(a) - t_{k+1}^{k}(a). \end{aligned}$$

This is a contradiction by Equation (9). ■

Lemma **7** *Suppose f is DSIC. For some* $\mathbf{t}$*, let* $f(\mathbf{t}) = a$*. Let* $\mathbf{s}$ *be another type profile such that for some* $b \neq a$

$$s_i(a) - s_i(b) > t_i(a) - t_i(b) \qquad \forall\, i \in M.$$

Then, $f(\mathbf{s}) \neq b$*.*

[11] The following condition is referred to as "Positive Association of Differences (PAD) in Roberts (1979). Essentially, the claim is every DSIC allocation rule satisfies PAD.

Proof: Assume for contradiction that $f(\mathbf{s}) = b$. Define a new type profile $\mathbf{r}$ as follows. For every $i \in M$, let $s_i(a) - s_i(b) = t_i(a) - t_i(b) + \epsilon_i$ for $\epsilon_i > 0$. Now, define $\mathbf{r}$ as

$$\begin{aligned} r_i(b) &= s_i(b) \qquad \forall\ i \in M \\ r_i(a) &= s_i(a) - \frac{\epsilon_i}{2} \qquad \forall\ i \in M \\ r_i(c) &= \min(s_i(c), t_i(c) - t_i(a) + s_i(a)) - \epsilon_i \qquad \forall\ i \in M. \end{aligned}$$

We do the proof in two steps.

Step 1: Consider $\mathbf{r}$ and $\mathbf{s}$, and note that $f(\mathbf{s}) = b$. Then, for every $i \in M$

$$r_i(b) - r_i(a) = s_i(b) - s_i(a) + \frac{\epsilon}{2} > s_i(b) - s_i(a).$$

For any $c \notin \{a, b\}$ and for every $i \in M$

$$r_i(b) - r_i(c) \geq s_i(b) - s_i(c) + \epsilon_i > s_i(b) - s_i(c)$$

Hence, by Lemma 6, $f(\mathbf{r}) = b$.

Step 2: Consider $\mathbf{r}$ and $\mathbf{t}$, and note that $f(\mathbf{t}) = a$. Then, for every $i \in M$

$$r_i(a) - r_i(b) = s_i(a) - \frac{\epsilon}{2} - s_i(b) = t_i(a) - t_i(b) + \frac{\epsilon}{2} > t_i(a) - t_i(b).$$

For any $c \notin \{a, b\}$ and for every $i \in M$

$$r_i(a) - r_i(c) \geq s_i(a) - \frac{\epsilon}{2} - t_i(c) + t_i(a) - s_i(a) + \epsilon > t_i(a) - t_i(c).$$

Hence, by Lemma 6, $f(\mathbf{r}) = a$. This is a contradiction with the result derived in Step 1. ■

We now define an important set for the proof. For every $a, b \in A$, let

$$P(b, a) = \{\delta \in \mathbb{R}^m : \exists\ \mathbf{t} \in \mathbb{R}^{|A|} \text{ such that } f(\mathbf{t}) = a, \text{ and } \delta_i = t_i(a) - t_i(b)\ \forall\ i \in M\}.$$

Since f is onto, the set $P(b, a)$ is non-empty for every $a, b \in A$. Notice that for any $w \in \mathbb{R}^m_{++}$,

$$d_w(b, a) = \inf_{\mathbf{t}: f(\mathbf{t}) = a} w \cdot [t(a) - t(b)] = \inf_{\delta \in P(b,a)} w \cdot \delta.$$

We now prove some properties of the "P sets".

Lemma **8** *Suppose f is DSIC. For every $a, b \in A$, the following are true:*

1. *If $\delta \in P(b, a)$, then for every $\epsilon \in \mathbb{R}^m_{++}$ we have $(\delta + \epsilon) \in P(b, a)$* [12].

[12] This means the set $P(b, a)$ is upper comprehensive for all $a, b \in A$.

2. *If* $(\delta - \epsilon) \in P(b,a)$ *for some* $\delta \in \mathbb{R}^m$ *and some* $\epsilon \in \mathbb{R}^m_{++}$, *then* $-\delta \notin P(a,b)$.

3. *If* $\delta \notin P(b,a)$, *then* $-\delta \in P(a,b)$.

Proof: Fix $a, b \in A$.

Proof of (1): Since $\delta \in P(b,a)$, there exists a $\mathbf{t}$ such that $f(\mathbf{t}) = a$ and $\delta = t(a) - t(b)$. Consider the following type profile:

$$\begin{aligned} s_i(a) &= t_i(a) \qquad \forall\, i \in M \\ s_i(c) &= t_i(c) - \epsilon_i \qquad \forall\, c \neq a,\ \forall\, i \in M. \end{aligned}$$

Note that for all $c \neq a$ and for all $i \in M$, $s_i(a) - s_i(c) = t_i(a) - t_i(c) + \epsilon_i > t_i(a) - t_i(c)$. Hence, by Lemma 6, $f(\mathbf{s}) = a$. Now, $s(a) - s(b) = t(a) - t(b) + \epsilon = \delta + \epsilon$. Hence, $\delta + \epsilon \in P(b,a)$.

Proof of (2): Since $(\delta - \epsilon) \in P(b,a)$, there exists a $\mathbf{t}$ such that $f(\mathbf{t}) = a$ and $t(a) - t(b) = \delta - \epsilon$. Assume for contradiction $-\delta \in P(a,b)$. Then, there exists $\mathbf{s}$ such that $f(\mathbf{s}) = b$ and $s(b) - s(a) = -\delta$ or $s(a) - s(b) = \delta > t(a) - t(b)$. But by Lemma 7, we must have $f(\mathbf{s}) \neq b$. This is a contradiction.

Proof of (3): Consider for every $c \notin \{a,b\}$, $\alpha^c \in P(c,b)$. Now, consider $\epsilon \in \mathbb{R}^m_{++}$. Define the following type profile:

$$\begin{aligned} s_i(a) &= 0 \qquad \forall\, i \in M \\ s_i(b) &= \delta_i \qquad \forall\, i \in M \\ s_i(c) &= \delta_i - \alpha^c_i + \epsilon_i \qquad \forall\, i \in M. \end{aligned}$$

Note that $s(b) - s(c) = \alpha^c + \epsilon > \alpha^c$. This implies that $f(\mathbf{s}) \in \{a,b\}$. But $s(b) - s(a) = \delta \notin P(b,a)$. Hence, $f(\mathbf{s}) \neq b$. Hence, $f(\mathbf{s}) = a$. But $s(a) - s(b) = -\delta$ implies that $-\delta \in P(a,b)$. ■

Define for every $a, b \in A$,

$$P(a,b) + P(b,a) = \{x \in \mathbb{R}^m : \exists\, x_1 \in P(a,b), x_2 \in P(b,a) \text{ with } x_1 + x_2 = x\}.$$

Let $\overline{P(a,b) + P(b,a)}$ denote the interior of the set $P(a,b) + P(b,a)$. As a consequence of Lemma 8, we get that $0 \notin \overline{P(a,b) + P(b,a)}$. To see this, assume for contradiction $0 \in \overline{P(a,b) + P(b,a)}$. Then, there exists $\epsilon \in \mathbb{R}^m_{++}$ such that $-\epsilon \in P(a,b) + P(b,a)$. This further implies that there exists $x - \epsilon \in P(a,b)$ and $-x \in P(b,a)$. This is a contradiction by Lemma 8.

Next, we define the following important notion. For every $a, b \in A$, define

$$h(b,a) = \inf_{\mathbf{t}: f(\mathbf{t})=a} \max_{i \in M} [t_i(a) - t_i(b)] = \inf_{\delta \in P(b,a)} \max_{i \in M} \delta_i.$$

Lemma **9** *If f is DSIC, then for every $a, b \in A$, $h(b,a)$ is a finite real number.*

Proof: Clearly, for every $a, b \in A$, $h(b,a)$ cannot be arbitrarily high. Assume for contradiction $h(b,a)$ can be made arbitrarily small. By Lemma 8, the set $P(b,a)$ is upper comprehensive, and this will imply that $P(b,a) = \mathbb{R}^m$. Now, pick $\delta \in P(a,b)$. This means, there exists $\mathbf{s}$ such that $f(\mathbf{s}) = b$ and $s_i(b) - s_i(a) = \delta_i$ for all $i \in M$. Since $P(b,a) = \mathbb{R}^m$, we can pick a type $\mathbf{t}$ such that $f(\mathbf{t}) = a$ and $t_i(a) - t_i(b) < -\delta_i$ for all $i \in M$. But this in turn implies that $t_i(b) - t_i(a) > s_i(b) - s_i(a)$ for all $i \in M$. By Lemma 7, $f(\mathbf{t}) \neq a$. This is a contradiction. ■

Denote a vector in $\mathbb{R}^m$, whose every component equals $\epsilon \in \mathbb{R}$ by $\mathbf{1}_\epsilon$.

Lemma **10** *For every $a, b \in A$,*

$$h(a,b) = \inf\{\epsilon \in \mathbb{R} : \mathbf{1}_\epsilon \in P(a,b)\}.$$

Proof: Choose a $\delta \in P(a,b)$. By Lemma 8, there exists an $\epsilon \in \mathbb{R}$ such that $\mathbf{1}_\epsilon \in P(a,b)$. By the definition of $h(a,b)$, the claim follows. ■

Lemma **11** *Suppose f is DSIC. If there exists a $\mathbf{t}$ such that for some $a, b \in A$ we have*

$$t_i(a) - t_i(b) < h(b,a) \qquad \forall\ i \in M.$$

Then, $f(\mathbf{t}) \neq a$. Further, if

$$t_i(a) - t_i(b) > h(b,a) \qquad \forall\ i \in M,$$

then $f(\mathbf{t}) \neq b$.

Proof: This follows from the definition of $h(b,a)$. Assume for contradiction there exists a $\mathbf{t}$ such that $f(\mathbf{t}) = a$ and $t_i(a) - t_i(b) < h(b,a)$ for all $i \in M$. Then, by the definition of $h(b,a)$, we have

$$\max_{i \in M} t_i(a) - t_i(b) \geq h(b,a).$$

This means for some $i \in M$, we have $t_i(a) - t_i(b) \geq h(b,a)$. This is in contradiction with the supposition in the lemma.

For the second part, assume $t_i(a) - t_i(b) > h(b,a)$ for all $i \in M$. By the definition of $h(b,a)$, there exists some $\mathbf{s}$ with $f(\mathbf{s}) = a$ such that

$$t_i(a) - t_i(b) > \max_{j \in M}[s_j(a) - s_j(b)] \qquad \forall\, i \in M.$$

Hence,

$$t_i(a) - t_i(b) > s_i(a) - s_i(b) \qquad \forall\, i \in M.$$

By Lemma 7, $f(\mathbf{t}) \neq b$. ■

LEMMA **12** *Suppose f is DSIC. For every $a, b \in A$, we have $h(a,b) + h(b,a) = 0$.*

Proof: We consider two possible cases.

CASE 1: Suppose that $h(a,b) + h(b,a) > 0$. Now choose a type $\mathbf{t}$ such that the following conditions are satisfied for every $i \in M$:

$$t_i(a) - t_i(b) < h(b,a) \tag{10}$$
$$t_i(b) - t_i(a) < h(a,b) \tag{11}$$
$$t_i(c) - t_i(a) < h(a,c) \qquad \forall\, c \notin \{a,b\} \tag{12}$$

Such a $\mathbf{t}$ exists and can be constructed as follows. Let $\epsilon = h(a,b) + h(b,a) > 0$. Then, choose $\mathbf{t}$ as:

$$t_i(b) = 0 \qquad \forall\, i \in M$$
$$t_i(a) = h(b,a) - \frac{\epsilon}{2} \qquad \forall\, i \in M$$
$$t_i(c) = h(b,a) + h(a,c) - \epsilon \qquad \forall\, c \notin \{a,b\},\ \forall\, i \in M.$$

Now, for any $i \in M$, $t_i(a) - t_i(b) = h(b,a) - \frac{\epsilon}{2} < h(b,a)$. Further, $t_i(b) - t_i(a) = -h(b,a) + \frac{\epsilon}{2} = h(a,b) - \frac{\epsilon}{2} < h(a,b)$ and for any $c \notin \{a,b\}$, we get $t_i(c) - t_i(a) = h(a,c) - \frac{\epsilon}{2} < h(a,c)$.

Now, Inequalities (10, 11,12) and Part 1 of Lemma 11 imply that $f(\mathbf{t}) \neq A$. This is a contradiction.

CASE 2: Suppose that $h(a,b) + h(b,a) < 0$. Let $\epsilon = -h(a,b) - h(b,a) > 0$. Now choose a type $\mathbf{t}$ such that the following conditions are satisfied for every $i \in M$:

$$t_i(b) - t_i(a) > h(a,b) \tag{13}$$
$$t_i(a) - t_i(b) > h(b,a) \tag{14}$$
$$t_i(c) - t_i(a) < h(a,c) \qquad \forall\, c \notin \{a,b\}. \tag{15}$$

Such a $\mathbf{t}$ exists and can be constructed as follows:

$$t_i(a) = 0 \qquad \forall\, i \in M$$
$$t_i(b) = h(a,b) + \frac{\epsilon}{2} \qquad \forall\, i \in M$$
$$t_i(c) = h(a,c) - \epsilon \qquad \forall\, c \notin \{a,b\},\ \forall\, i \in M.$$

Note that for all $i \in M$, $t_i(b) - t_i(a) = h(a,b) + \frac{\epsilon}{2} > h(a,b)$. Further, $t_i(a) - t_i(b) = -h(a,b) - \frac{\epsilon}{2} = h(b,a) + \frac{\epsilon}{2} > h(b,a)$ and for all $c \notin \{a,b\}$ we have $t_i(c) - t_i(a) = h(a,c) - \epsilon < h(a,c)$.

Now, Inequalities (15) and Part 1 of Lemma 11 imply that $f(\mathbf{t}) \in \{a,b\}$. Similarly, Inequalities (13,14) and Part 2 of Lemma 11 imply that $f(\mathbf{t}) \notin \{a,b\}$. This is a contradiction.

Cases 1 and 2 imply that $h(a,b) + h(b,a) = 0$. ■

Lemma **13** *Suppose f is DSIC. If $\delta^{c,b} \in P(c,b)$ and $\delta^{b,a} \in P(a,b)$, then for all $\epsilon > 0$*

$$\delta^{c,b} + \delta^{b,a} + \mathbf{1}_\epsilon \in P(c,a).$$

Proof: Consider $\mathbf{t}$ which satisfying the following inequalities.

$$t_i(a) - t_i(b) = \delta_i^{b,a} + \frac{\epsilon}{2} \qquad \forall\, i \in M \tag{16}$$
$$t_i(b) - t_i(c) = \delta_i^{c,b} + \frac{\epsilon}{2} \qquad \forall\, i \in M \tag{17}$$
$$t_i(x) - t_i(a) < h(x,a) \qquad \forall\, x \notin \{a,b,c\},\ \forall\, i \in M. \tag{18}$$

Such a $\mathbf{t}$ exists. For example, set $t_i(b) = 0\ \forall\, i \in M$, and find $t_i(a)$ and $t_i(c)$ from Equations 16 and 17 respectively. Finally, for all $x \notin \{a,b,c\}$ set $t_i(x) = t_i(a) + h(x,a) - \epsilon$. By Inequality 18 and Lemma 11, we get that $f(\mathbf{t}) \in \{a,b,c\}$.

Similarly, Equation 17 implies that $t_i(b) - t_i(c) > \delta_i^{c,b}$ for all $i \in M$. This means there exists $\mathbf{s}$ such that $f(\mathbf{s}) = b$ and $s_i(b) - s_i(c) = \delta_i^{c,b} < t_i(b) - t_i(c)$ for all $i \in M$. By Lemma 7 $f(\mathbf{t}) \neq c$. Using an analogous argument for Equation 16, we conclude that $f(\mathbf{t}) \neq b$. Hence, $f(\mathbf{t}) = a$. But $t_i(a) - t_i(c) = \delta_i^{b,a} + \delta_i^{c,b} + \epsilon$ for all $i \in M$. This implies that $\delta^{b,a} + \delta^{c,b} + \mathbf{1}_\epsilon \in P(c,a)$. ■

The next lemma uses the result in the last lemma to prove an important milestone in the proof.

Lemma **14** *Suppose f is DSIC. Let $w \in \mathbb{R}^m_{++}$ be such that every cycle involving two nodes in G^w has non-negative length, i.e., for every $a, b \in A$ we have $d_w(a,b) + d_w(b,a) \geq 0$. Then, every cycle in G^w has non-negative length.*

Proof: Since for every $a, b \in A$, $d_w(a,b) + d_w(b,a) \geq 0$, the edge lengths of G^w are finite. Now, consider $a, b, c \in A$. We show that $d_w(a,b) + d_w(b,c) \geq d_w(a,c)$. For this, note that by Lemma 13 for every $\delta^{a,b} \in P(a,b)$ and $\delta^{b,c} \in P(b,c)$ there exists $\delta^{a,c} \in P(a,c)$ such that $\delta^{a,c} = \delta^{a,b} + \delta^{b,c} + \mathbf{1}_\epsilon$, where ϵ is positive and can be made arbitrarily close to zero. Hence,

$$w \cdot [\delta^{a,b} + \delta^{b,c}] = w \cdot \delta^{a,c} - w \cdot \mathbf{1}_\epsilon.$$

Since ϵ can be made arbitrarily small and $w \in \mathbb{R}^m_{++}$, we conclude that

$$w \cdot [\delta^{a,b} + \delta^{b,c}] \geq \inf_{\delta \in P(a,c)} w \cdot \delta = d_w(a,c).$$

Hence,

$$\inf_{\delta \in P(a,b)} w \cdot \delta + \inf_{\delta \in P(b,c)} w \cdot \delta \geq d_w(a,c).$$

This shows that $d_w(a,b) + d_w(b,c) \geq d_w(a,c)$, i.e., edge lengths of G^w satisfy triangle inequality.

Now, consider any cycle $(a_1, a_2, \ldots, a_k, a_1)$. By triangle inequality, the length of this cycle is greater than or equal to the length of the cycle (a_1, a_k, a_1). By our assumption, this cycle has non-negative length. Hence, the cycle $(a_1, a_2, \ldots, a_k, a_1)$ has non-negative length. ■

Lemma **15** *Suppose f is DSIC. If $\delta^{c,a} \in P(c,a)$, $\delta^{a,b} \in P(a,b)$, and $\delta^{b,a} \in P(b,a)$, then for all $\epsilon > 0$*

$$\delta^{c,a} + \delta^{a,b} + \delta^{b,a} + \mathbf{1}_{2\epsilon} \in P(c,a).$$

Proof: By Lemma 13, $\delta^{c,a} + \delta^{a,b} + \mathbf{1}_\epsilon \in P(c,b)$. Applying Lemma 13 again, $\delta^{c,a} + \delta^{a,b} + \delta^{b,a} + \mathbf{1}_{2\epsilon} \in P(c,a)$. ■

Lemma **16** *Suppose f is DSIC. If $\delta_1, \delta_2 \in \overline{P(a,b) + P(b,a)}$, then $\delta_1 + \delta_2 \in \overline{P(a,b) + P(b,a)}$.*

Proof: Let $\delta_i = \delta_i^{a,b} + \delta_i^{b,a}$, where $\delta_i^{a,b} \in P(a,b)$ and $\delta_i^{b,a} \in P(b,a)$ for $i = 1, 2$. Without loss of generality, let $\delta_1^{a,b} \in \overline{P(a,b)}$. Then, for some $\epsilon > 0$, $\delta_1^{a,b} - \mathbf{1}_\epsilon \in P(a,b)$. Pick $\epsilon' > 0$ and arbitrarily small, i.e., $\epsilon > 4\epsilon'$. We know, $\mathbf{1}_{h(a,c)+\epsilon'} \in P(a,c)$ and $\mathbf{1}_{h(c,a)+\epsilon'} \in P(c,a)$. By Lemma 15,

$$\mathbf{1}_{h(c,a)+\epsilon'} + \delta_1^{a,b} - \mathbf{1}_\epsilon + \delta_1^{b,a} + \mathbf{1}_{2\epsilon'} \in P(c,a).$$

Again using Lemma 15,

$$\delta_2^{a,b} + \mathbf{1}_{h(a,c)+\epsilon'} + \mathbf{1}_{h(c,a)+\epsilon'} + \delta_1^{a,b} - \mathbf{1}_\epsilon + \delta_1^{b,a} + \mathbf{1}_{2\epsilon'} \in P(a,b).$$

Since $h(a,c) + h(c,a) = 0$, we get $\delta_2^{a,b} + \delta_1^{a,b} + \delta_1^{b,a} + \mathbf{1}_{4\epsilon'-\epsilon} \in P(a,b)$. Hence, $\delta_2^{a,b} + \delta_1^{a,b} + \delta_1^{b,a} \in \overline{P(a,b)}$. Since $\delta_2^{b,a} \in P(b,a)$, we get that $\sum_{i=1,2} [\delta_i^{a,b} + \delta_i^{b,a}] \in \overline{P(a,b) + P(b,a)}$. ■

Lemma **17** *Suppose f is DSIC. If $\delta \in \overline{P(a,b)+P(b,a)}$, then $\frac{\delta}{2} \in \overline{P(a,b)+P(b,a)}$.*

Proof: Assume for contradiction $\frac{\delta}{2} \notin \overline{P(a,b)+P(b,a)}$. Since $h(a,b)+h(b,a)=0$, this implies that $\frac{\delta}{2}+\mathbf{1}_{h(a,b)}+\mathbf{1}_{h(b,a)}-\mathbf{1}_{\epsilon} \notin P(a,b)+P(b,a)$ for some sufficiently small $\epsilon > 0$. Since $\mathbf{1}_{h(b,a)}+\mathbf{1}_{\epsilon} \in \overline{P(b,a)}$, $\frac{\delta}{2}+\mathbf{1}_{h(a,b)}-\mathbf{1}_{2\epsilon} \notin P(a,b)$. By Lemma 8, $-\frac{\delta}{2}-\mathbf{1}_{h(a,b)}+\mathbf{1}_{2\epsilon} \in P(b,a)$. But $\mathbf{1}_{h(a,b)}+\mathbf{1}_{\epsilon} \in \overline{P(a,b)}$. Hence, $-\frac{\delta}{2}+\mathbf{1}_{3\epsilon} \in \overline{P(a,b)+P(b,a)}$. But, $\delta \in \overline{P(a,b)+P(b,a)}$ implies that for sufficiently small $\epsilon > 0$, we will have $\delta - \mathbf{1}_{3\epsilon} \in \overline{P(a,b)+P(b,a)}$. Using Lemma 16, we get that $\frac{\delta}{2} \in \overline{P(a,b)+P(b,a)}$. This is a contradiction. ■

We now come to the next major milestone of the proof.

Lemma **18** *Suppose f is DSIC. For every $a,b \in A$, the set $\overline{P(a,b)+P(b,a)}$ is convex.*

Proof: Pick $\delta_1, \delta_2 \in \overline{P(a,b)+P(b,a)}$. By Lemma 16 and 17, we get that $\frac{\delta_1+\delta_2}{2} \in \overline{P(a,b)+P(b,a)}$. Since $\overline{P(a,b)+P(b,a)}$ is an open set, this shows that it is convex. ■

Lemma **19** *Suppose f is DSIC. For every distinct $a,b,x,y \in A$,*

$$\overline{P(a,b)+P(b,a)} = \overline{P(a,x)+P(x,a)}$$
$$\overline{P(a,b)+P(b,a)} = \overline{P(x,y)+P(y,x)}.$$

Proof: Suppose $\delta^{a,b}+\delta^{b,a} \in \overline{P(a,b)+P(b,a)}$ with $\delta^{a,b} \in P(a,b)$ and $\delta^{b,a} \in \overline{P(b,a)}$. Hence, there exists $\epsilon > 0$ such that $\delta^{b,a}-\mathbf{1}_{\epsilon} \in P(b,a)$. Pick any $x \neq a,b$. Since $\mathbf{1}_{h(x,a)+\epsilon'} \in P(x,a)$, where $\epsilon' > 0$ and $4\epsilon' = \epsilon$. By Lemma 15, $\delta^{b,a}+\delta^{a,b}+\mathbf{1}_{h(x,a)+3\epsilon'-\epsilon} \in P(x,a)$. But $\mathbf{1}_{h(a,x)+\epsilon'} \in \overline{P(a,x)}$. Using $h(a,x)+h(x,a)=0$, we get that $\delta^{a,b}+\delta^{b,a} \in \overline{P(a,x)+P(x,a)}$. Hence, $\overline{P(a,b)+P(b,a)} \subseteq \overline{P(a,x)+P(x,a)}$. Interchanging the role of b and x in the above argument establishes, $\overline{P(a,b)+P(b,a)} = \overline{P(a,x)+P(x,a)}$. Now, repeat the above argument for $\overline{P(a,x)+P(x,a)}$ with $y \neq a,x$. This will give, $\overline{P(a,x)+P(x,a)} = \overline{P(x,y)+P(y,x)}$. This gives $\overline{P(a,b)+P(b,a)} = \overline{P(x,y)+P(y,x)}$. ■

Lemma **20** *Fix some $w \in \mathbb{R}^m_{++}$ and $a,b \in A$. If $d_w(a,b)+d_w(b,a) < 0$, then for any $x,y \in A$, we have $d_w(x,y)+d_w(y,x) < 0$.*

Proof: If $d_w(a,b)+d_w(b,a) < 0$, then for some $\delta^{a,b} \in P(a,b)$ and $\delta^{b,a} \in P(b,a)$, we must have $w \cdot [\delta^{a,b}+\delta^{b,a}] < 0$. This implies that for some $\delta \in \overline{P(a,b)+P(b,a)}$, $w \cdot \delta < 0$. By Lemma 19, the claim follows. ■

By Lemma 20, it now suffices to show that there exists a $w \in \mathbb{R}^m_{++}$ such that $w \cdot \delta \geq 0$ for all $\delta \in \overline{P(a,b)+P(b,a)}$ for some $a,b \in A$. Since $0 \notin \overline{P(a,b)+P(b,a)}$ and $\overline{P(a,b)+P(b,a)}$

is convex by Lemma 18, there exists a $w \in \mathbb{R}^m$ such that $w \cdot \delta \geq 0$ for all $\delta \in \overline{P(a,b) + P(b,a)}$ (this follows from the separating hyperplane theorem).

To conclude, we show that $w \in \mathbb{R}^m_{++}$. Assume for contradiction $w_1 < 0$. By the choice of w, the graph G^w has no cycles of negative length. This implies that there exists constants $\kappa \in \mathbb{R}^{|A|}$ such that f is an affine maximizer. Consider $\mathbf{t}$ such that $f(\mathbf{t}) = a$. Consider the type profile $\mathbf{s}$ such that $s_i = t_i$ for all $i \neq 1$ and $s_1(a) = t_1(a) + \epsilon$ and $s_1(b) = t_1(b) - \epsilon$ for all $b \neq a$ for some $\epsilon > 0$. By monotonicity, $f(\mathbf{s}) = a$. Now, by affine maximization, we get for every $b \neq a$,

$$\sum_{i \neq 1} w_i[t_i(b) - t_i(a)] + \kappa_b - \kappa_a \leq w_1[t_1(a) - t_1(b)]$$
$$\sum_{i \neq 1} w_i[s_i(b) - s_i(a)] + \kappa_b - \kappa_a \leq w_1[s_1(a) - s_1(b)] = w_1[t_1(a) - t_1(b)] + 2w_1\epsilon.$$

Using the fact,

$$\sum_{i \neq 1} w_i[s_i(b) - s_i(a)] = \sum_{i \neq 1} w_i[t_i(b) - t_i(a)],$$

we get that

$$\sum_{i \in M} w_i[t_i(b) - t_i(a)] + \kappa_b - \kappa_a \leq 2w_1\epsilon.$$

Since $w_1 < 0$ and $\epsilon > 0$ can be chosen arbitrarily large, we get a contradiction [13]. ■

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Appendix: Characterizing Monotonicity Domains

In this section, we try to characterize monotonicity domains. The assumption we make are the following:

A1 The set of allocations A is a finite set. Moreover, every $a \in A$ is written as a vector $a = (a_1, \ldots, a_n)$.

A2 The valuation function $v : A \times T \to \mathbb{R}$ is of the following form: for every $a \in A$, for every $t \in T$, $v(a, t) = a \cdot t$.

Throughout the rest of this section, we asssume **A1** and **A2**.

An example where these assumptions hold is the following. In combinatorial auctions, n can be the set of bundles of goods and every $a \in A$ is a vector of 0s and 1s with exactly one 1, representing the bundle of good allocated. A type $t \in T$ represents the value for various bundles of goods. Hence, $v(a,t) = a \cdot t$ gives the value from allocation a when type is t by a simple dot product.

Since this valuation function is linear in type, our earlier results hold. Namely, every T whose closure is convex is a monotonicity domain. We try to prove: If T is a monotonicity domain, then its closure is convex.

5.1 Two Dimensional Type Space

We prove the following theorem in this section.

Theorem **14** (**Monderer** (**2008**)) *If $T \subseteq \mathbb{R}^2$ ($n = 2$) is a monotonicity domain, then $cl(T)$ is convex.*

We do the proof by proving a series of lemmas. For any set $T \subseteq \mathbb{R}^n$, let $x_1, \ldots, x_k \in T$ and $\lambda_1 \ldots \lambda_k \in \mathbb{R}$. Then, $\sum_{j=1}^{k} \lambda_j x_j$ is called

- a **linear combination** of $x_1, \ldots, x_k$,
- a **affine combination** of $x_1, \ldots, x_k$ if $\sum_{j=1}^{k} \lambda_j = 1$, and
- a **convex combination** of $x_1, \ldots, x_k$ if $\sum_{j=1}^{k} \lambda_j = 1$ and $\lambda_j \geq 0$ for all $j \in \{1, \ldots, k\}$,

The **linear space** spanned by T is defined as

$$L_T = \{t \in \mathbb{R}^n : t \text{ is a linear combination of points in } T\}.$$

The **affine space** spanned by T is defined as

$$A_T = \{t \in \mathbb{R}^n : t \text{ is an affine combination of points in } T\}.$$

For any set T, denote by M_T the linear space spanned by all difference vectors in T. A set of points $x_1, \ldots, x_{k+1}$ are affinely (linearly) independent, if none of them is an affine (linear) combination of others.

The convex hull of affine $(k+1)$ affine independent points in dimension k is called a **simplex**. If $L = \{t_0, \ldots, t_k\}$ is a set of $k+1$ affinely independent points in $\mathbb{R}^k$, then $\Delta(L)$ will denote the simplex generated by L and its relative interior is denoted by $\Delta^0(L)$.

Lemma **21** *Let $T \subseteq \mathbb{R}^n$, where $n \geq 2$ and $L = \{t_0, \ldots, t_n\}$ be $n+1$ affinely independent types. Then, there exists $z_0, \ldots, z_n \in M_L$ such that*

$$\sum_{i=0}^{n} z_i = 0,$$

and for all $i, j, s \in \{0, \ldots, n\}$ with $i \neq j$ and $j \neq s$

$$(t_j - t_i) \cdot z_s = 0$$
$$(t_j - t_s) \cdot z_s < 0.$$

Proof: Fix s and $j \neq s$. Since the dimension of M_L is n, there exists a $z \neq 0$ such that $z \in M_L$ and z is perpendicular to $n-1$ types $t_j - t_i$ with $i \neq j$ and $i \neq s$. So, for all $i \neq j$ and $i \neq s$ we have

$$(t_j - t_i) \cdot z = 0. \tag{19}$$

But z cannot be perpendicular to $t_j - t_s$ because of dimension of M_L. Without loss of generality, we can assume

$$(t_j - t_s) \cdot z < 0. \tag{20}$$

Note that if z satisfies Equations (19) and (20), then αz with $\alpha > 0$ and $\alpha \in \mathbb{R}$ also satisfies Equations (19) and (20).

Choose $z_0, \ldots, z_n \in M_L$ as above. Note that there exists $\alpha_0, \ldots, \alpha_n$, not all equal to zero such that

$$\sum_{i=0}^{n} \alpha_i z_i = 0.$$

Without loss of generality, let $\alpha_0 > 0$. For any $s > 0$ note that

$$\begin{aligned} 0 &= (t_0 - t_s) \cdot (\sum_{i=0}^{n} \alpha_i z_i) \\ &= \sum_{i=0}^{n} \alpha_i (t_0 - t_s) \cdot z_i \\ &= \alpha_0 (t_0 - t_s) \cdot z_0 + \alpha_s (t_0 - t_s) \cdot z_s \qquad \text{(Using Equation (19)}. \end{aligned}$$

This implies that

$$\alpha_s = -\alpha_0 \frac{(t_0 - t_s) \cdot z_0}{(t_0 - t_s) \cdot z_s}.$$

By Equation (20), $\alpha_s > 0$. Thus, replacing z_s by $\alpha_s z_s$ for all $s \in \{0, \ldots, n\}$ gives us the desired result. ■

LEMMA **22** *Every finite affine independent set $L = \{t_0, \ldots, t_n\} \subseteq T \subseteq \mathbb{R}^n$, where $n \geq 2$ is not a monotonicity domain.*

Moreover, there exists a monotone function $f : L \to A$, where $A = \{y_0, \ldots, y_n\}$ which is not cyclically monotone and satisfies the following:

1. *$y_0, \ldots, y_n \in M_L$.*

2. *$f(t_j) = y_j$ for all $j \in \{0, \ldots, n\}$.*

3. *$d(y_i, y_j) = t_j \cdot (y_j - y_i)$ for all $j \neq i$ and $i, j \in \{0, \ldots, n\}$.*

4. *$d(y_0, y_1) + d(y_1, y_2) + \ldots + d(y_{n-1}, y_n) + d(y_n, y_0) < 0$.*

Proof: Let $z_0, \ldots, z_n$ be as in Lemma 21. Define $y_0, \ldots, y_n$ as follows:

$$y_0 = 0,\ y_1 = -z_n, y_n = z_{n-1},\ y_i = y_{i-1} - z_{i-2} \qquad \forall\ i \geq 2.$$

For monotonicity, let $0 < i < j$. We get

$$\begin{aligned} d(y_i, y_j) + d(y_j, y_i) &= (t_j - t_i) \cdot (y_j - y_i) \\ &= (t_j - t_i) \cdot (-\sum_{k=i-1}^{j-2} z_k) \\ &= -(t_j - t_i) \cdot z_i \qquad \text{(By Lemma 21)} \\ &> 0 \qquad \text{(By Lemma 21).} \end{aligned}$$

For $i = 0$ and $j > 1$, we get

$$\begin{aligned} d(y_0, y_j) + d(y_j, y_0) &= (t_0 - t_j) \cdot (y_0 - y_j) \\ &= (t_j - t_0) \cdot y_j \\ &= (t_j - t_0) \cdot (-z_n - \sum_{k=0}^{j-2} z_k) \\ &= -(t_j - t_0) \cdot z_0 \qquad \text{(By Lemma 21)} \\ &> 0 \qquad \text{(By Lemma 21).} \end{aligned}$$

Finally, for $i = 0$ and $j = 1$, we get

$$\begin{aligned} d(y_0, y_1) + d(y_1, y_0) &= (t_0 - t_1) \cdot (y_0 - y_1) \\ &= (t_1 - t_0) \cdot y_1 \\ &= (t_1 - t_0) \cdot z_n \\ &= 0. \qquad \text{(By Lemma 21)} \end{aligned}$$

Now, for cycle monotonicity, observe the following.

$$\begin{aligned}
d(y_n, y_0) + \sum_{i=0}^{n-1} d(y_i, y_{i+1}) &= (y_1 - y_0) \cdot t_1 + \ldots + (y_n - y_{n-1}) \cdot t_n + (y_0 - y_n) \cdot t_0 \\
&= (t_0 - t_1) \cdot y_0 + \sum_{i=1}^{n} (t_{i-1} - t_i) \cdot y_i \\
&= (t_1 - t_0) \cdot (-z_n) + \sum_{i=1}^{n} (t_i - t_{i-1}) \cdot (-z_n + \sum_{k=0}^{i-2} z_k) \\
&= (t_n - t_{n-1}) \cdot (-z_n) \\
&< 0 \qquad \text{(By Lemma 21)}.
\end{aligned}$$

Hence, cycle monotonicity is violated. ∎

5.2 Results for Dimension 2

For $L = \{t_0, t_1, t_2\}$, define $f(t_0) = y_0 = 0$, $f(t_1) = y_1 = z_2$, and $f(t_2) = y_2 = -z_1$. Then, $d(y_0, y_1) + d(y_1, y_0) = (t_1 - t_0) \cdot (y_1 - y_0) = -(t_1 - t_0) \cdot z_2 = 0$. Also, $d(y_0, y_2) + d(y_2, y_0) = -(t_2 - t_0) \cdot z_1 = 0$, and $d(y_1, y_2) + d(y_2, y_1) = (t_1 - t_2) \cdot (z_2 + z_1) = -(t_1 - t_2) \cdot z_0 = 0$. Finally, $d(y_0, y_1) + d(y_1, y_2) + d(y_2, y_0) = t_1 \cdot (y_1 - y_0) + t_2 \cdot (y_2 - y_1) + t_0 \cdot (y_0 - y_2) = (t_0 - t_1) \cdot y_0 + (t_1 - t_2) \cdot y_1 + (t_2 - t_0) \cdot y_2 = (t_1 - t_2) \cdot z_2 + (t_0 - t_2) \cdot z_1 < 0$.

We extend the affinely independent set in dimension 2 to a non-convex set as follows. Define for every $i, j \in \{1, 2, 3\}$ and $i \neq j$,

$$d_L(y_i, y_j) = t_j \cdot (y_j - y_i).$$

$$Q_j = \{t \in A_L \setminus \Delta^0(L) : t \cdot (y_j - y_i) \geq d_L(y_i, y_j) \ \forall \ i \neq j\}.$$

Now, set $f(t) = y_j$ if $t \in Q_j$ - in case t belongs to $Q_0 \cap Q_1$ set $f(t) = y_1$, if t belongs to $Q_1 \cap Q_2$ set $f(t) = y_2$, and if t belongs to $Q_2 \cap Q_0$ set $f(t) = y_0$. This extends f from L to $\cup_{j=1}^{3} Q_j$.

Note two important things:

1. For every $t_j \in L$ we have $t_j \in Q_j$.

2. For any $i, j \in \{1, 2, 3\}$ with $i \neq j$,

$$d(y_i, y_j) = \inf_{t \in Q_j} t \cdot (y_j - y_i) = d_L(y_i, y_j).$$

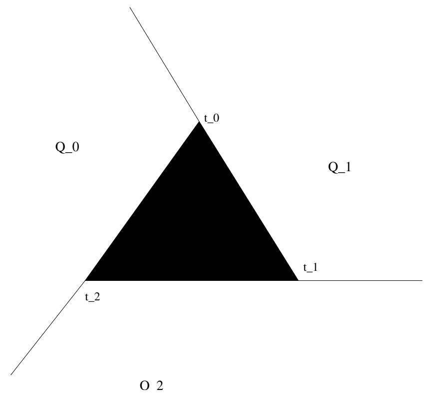

Figure 5: A non-convex domain in 2 dimension that is not a monotonicity domain

From the second fact and Lemma 22, it follows that f satisfies monotonicity on $\cup_{j=1}^{3} Q_j$ but fails cycle monotonicity on L.

Note that for any $i, j \in \{1, 2, 3\}$ with $i \neq j$ the interior of Q_i and the set Q_j do not intersect. Because if a t exists such that it belongs to interior of Q_i and in Q_j, then $0 = t \cdot (y_i - y_j) + t \cdot (y_j - y_i) > d(y_i, y_j) + d(y_j, y_i) \geq 0$, where the strict inequality comes from the fact that t is in the interior of Q_i and weak inequality comes from monotonicity of f.

Now, consider the two hyperplanes that define Q_0:

$$t \cdot (y_0 - y_1) = d_L(y_1, y_0)$$
$$t \cdot (y_0 - y_2) = d_L(y_2, y_0).$$

In the hyperplane $t \cdot (y_0 - y_1) = d_L(y_1, y_0)$ the points t_0 and t_1 lie. This is because $t_0 \cdot (y_0 - y_1) = d_L(y_1, y_0)$ and $t_1 \cdot (y_0 - y_1) = -t_1 \cdot (y_1 - y_0) = -d_L(y_0, y_1) = d_L(y_1, y_0)$. Similarly, t_0 and t_2 lie in the second hyperplane. Hence, the polyhedra Q_0 is defined by the line joining t_0 and t_1 and the line joining t_0 and t_2. Similarly, Q_1 is defined by the line joining t_1 and t_0 and the line joining t_1 and t_2, and finally Q_2 is defined by the line joining t_2 and t_0 and t_2 and t_1. See Figure 5.