A Simple Budget-balanced Mechanism *

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Abstract

In the private values single object auction model, we construct a *satisfactory* mechanism - a symmetric, dominant strategy incentive compatible, budget-balanced, and ex-post individually rational mechanism. Our mechanism allocates the object with positive probability to *only* those agents who have the highest value. This probability is at least $(1 - \frac{2}{n})$, where n is the number of agents. Hence, our mechanism converges to efficiency at a linear rate as the number of agents grow. Our mechanism has a simple interpretation: a fixed allocation probability is allocated using a second-price Vickrey auction whose revenue is redistributed among all the agents in a simple way.

Keywords. budget-balanced mechanisms, Green-Laffont mechanism, Pareto optimal mechanism.

JEL Keywords. D82, D71, D02.

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1 Introduction

This paper considers the problem of allocating a unit of resource among a set of agents who have private valuation for it. Transfers are allowed but preferences over transfers are quasi-linear. However, transfers have to balance. Examples of such problems include: allocating a bequest among claimants, deciding on a venue of a public good (hospital) among various municipalities, sharing a unit of time on a supercomputer owned jointly by various firms etc.

Efficiency in this problem requires that the agent with the highest valuation must be given the entire resource. We follow a mechanism design approach to construct a new dominant strategy incentive compatible (DSIC), budget-balanced, and *nearly* efficient mechanism for this problem. The mechanism design literature on this topic centers around an impossibility result of Green and Laffont (1977): there is no efficient, DSIC, and budget-balanced mechanism. This paper presents a new avenue for escaping this impossibility result by *burning probabilities*.

Relax efficiency by burning probabilities. We describe a DSIC, budget-balanced, and individually rational mechanism that *only* allocates probabilities to the highest-valued agent(s) and burns (wastes) the remaining probabilities. With $n \geq 3$ agents and at a generic valuation profile $v_1 > v_2 > \ldots > v_n$, our mechanism allocates the object to agent 1 (highest valuation agent) with probability $(1 - \frac{2}{n}) + \frac{2}{n}\frac{v_3}{v_2}$. Our mechanism can be simply stated as: a second-price auction of this probability followed by a redistribution of the revenue of the second-price auction among all the agents, where agents 1 and 2 receive an amount $\frac{v_3}{n}$ each and every other agent receives an amount $\frac{v_2}{n}$. Such a redistribution is crucial to maintain incentives. Notice that the mechanism converges to efficiency at a linear rate. Our mechanism can be thought to be an answer to the following question:

> *What allocation probability can be auctioned using a second-price auction whose revenue can be redistributed among all the agents?*

By the Green-Laffont impossibility result, this allocation probability is strictly less than 1, and our mechanism shows that it is larger than $(1 - \frac{2}{n})$. We show that in the class of all mechanisms that allocate the object to only the highest valued agent, our mechanism is *welfare undominated*, i.e., every mechanism in this class gives less welfare at *some* valuation profile.

We now discuss some of the other attempts to escape the Green-Laffont impossibility theorem and argue how they compare to burning probabilities.

Relax solution concept. Cramton et al. (1987) show that there is an efficient, Bayesian incentive compatible, budget-balanced, and individually rational mechanism for this problem. [1] Hence, the Green-Laffont impossibility can be completely overcome by relaxing the solution concept to Bayesian incentive compatibility. We also point out that d'Aspremont and Gérard-Varet (1979); Arrow (1979) construct mechanisms (now called the dAGV mechanisms), which are efficient, Bayesian incentive compatible, and budget-balanced. But the dAGV mechanisms are not individually rational.

The advantage of a DSIC mechanism is that it is prior-free and more robust to strategic manipulation. This is probably the reason that a long literature exists investigating the possibility and impossibility boundaries of DSIC, efficient, and budget-balanced mechanisms - see Hurwicz and Walker (1990); Laffont and Maskin (1980); Green and Laffont (1979); Walker (1980). Our mechanism adds to this literature and provides a new reason to look at DSIC mechanisms.

Relax budget-balance by burning money. Another way of overcoming the Green-Laffont impossibility result is to relax the budget-balance constraint. Recent papers follow this approach by relaxing budget-balance to a *no-deficit* condition (i.e., the designer can only earn revenue). Their objective is to redistribute as much revenue as possible from an efficient and DSIC mechanism - Cavallo (2006); Guo and Conitzer (2009); Moulin (2009, 2010) are notable contributions. By well-known revenue equivalence results, the only class of efficient and DSIC mechanisms are Groves mechanisms (Holmström, 1979). In Guo and Conitzer (2009); Moulin (2009), they propose Groves mechanisms that redistribute a large fraction of revenue as number of agents grow - unlike the mechanism in Cavallo (2006), these mechanisms are complicated to describe. The main difference from this literature to ours is that budget-balance is a necessary constraint for us, and we are interested in exploring the limitations of imposing DSIC and budget-balance as constraints.

Relax efficiency by giving to others. If we burn money, we need not relax efficiency, and we can restrict attention to the Groves class of mechanisms. On the other hand, we may relax efficiency and search within the class of all DSIC and budget-balanced mechanisms. In Mishra and Sharma (2016), we describe a class of such mechanisms that we call ranking mechanisms. We further showed that it includes a mechanism which asymptotically converges to efficiency at an exponential rate as the number of agents grow. Long (2016) independently discovers

[1] They consider a more general problem with property rights. In our problem, there are no property rights. We can assign equal property rights to all the agents and apply their result.

the same set of mechanisms.

Ranking mechanisms include a simple mechanism proposed by Green and Laffont (1977), called the *residual claimant* mechanism. In that mechanism, an agent is uniformly randomly picked to be a residual claimant, and a Vickrey auction is held among the remaining agents. The revenue from the Vickrey auction is given to the residual claimant. This mechanism is DSIC and budget-balanced. It allocates the object to the highest valued agent with probability $(1 - \frac{1}{n})$, where n is the number of agents.

Relaxing efficiency takes one out of the comfortable class of Groves mechanisms - this means, one needs to worry about both the allocation rule and payment rule. This is the reason we see less work on non-efficient, DSIC, and budget-balanced mechanisms. Besides Mishra and Sharma (2016), papers by Hashimoto (2015) and Guo et al. (2011) discuss variants of the Green-Laffont mechanism and its properties. These mechanisms are very close to the Green-Laffont mechanism and differ from our mechanism significantly. Sprumont (2013) characterizes the class of DSIC, individually rational, deficit-free, and envy-free mechanisms. But he does not impose budget-balance. Drexl and Kleiner (2015) investigate expected welfare maximizing DSIC and budget-balanced mechanisms but only consider the case of two agents. Nath and Sandholm (2016) look at a more general mechanism design problem than ours but restrict attention to mostly deterministic mechanisms. Their main result says that deterministic mechanisms are like Green-Laffont mechanisms but without randomization. With randomization, they give some *approximation* guarantees using Green-Laffont type mechanisms.

In the papers described above, efficiency is relaxed by allocating the object with positive probability to agents who do not have the highest value - the Green-Laffont mechanism allocates the object to the second highest valued agent with $\frac{1}{n}$ probability and the mechanism in Mishra and Sharma (2016) allocate the object to almost $\frac{n}{2}$ agents with positive probability.

From a practical standpoint, this may lead to unpleasant situations sometimes. Consider a scenario where the highest valued agent has valuation 1 million and the second highest valued agent has valuation close to zero. Both the GL mechanism and the mechanisms in Mishra and Sharma (2016) allocates the object with positive probability to the second highest valued agent. Giving the object with positive probability to really low-valued agents when a high-valued agent is present may be problematic in certain practical settings.

This motivates us to explore a new direction for overcoming the Green-Laffont impossibility result. Compared to the mechanism in Mishra and Sharma (2016), our mechanism does not converge to efficiency at an *exponential* rate. However, unlike their mechanism, this mechanism is simpler to describe and *only* allocates the object to the highest valued agents.

2 The Model

We consider the standard single object independent private values model with $N = \{1, \ldots, n\}$ as the set of agents. Throughout, we assume that $n \geq 3$. Each agent $i \in N$ has a valuation v_i for the object. If he is given $\alpha_i \in [0, 1]$ of the object, or given the object with probability α_i, and he pays p_i for it, then his net utility is $\alpha_i v_i - p_i$. The set of all valuations for any agent is given by $V \equiv [0, \beta]$, where $\beta \in \mathbb{R}$. A valuation profile will be denoted by $\mathbf{v} \equiv (v_1, \ldots, v_n)$.

An **allocation rule** is a map $f : V^n \to [0, 1]^n$, where we denote by $f_i(\mathbf{v})$ the probability of agent i getting allocated the object at valuation profile $\mathbf{v}$. We assume that at all $\mathbf{v} \in V^n$, $\sum_{i \in N} f_i(\mathbf{v}) \leq 1$.

A **payment rule** of agent i is a map $p_i : V^n \to \mathbb{R}$. A collection of payment rules of all the agents will be denoted by $\mathbf{p} \equiv (p_1, \ldots, p_n)$. A **mechanism** is a pair $(f, \mathbf{p})$. We require our mechanism to satisfy the following three properties:

- A mechanism $(f, \mathbf{p})$ is **dominant strategy incentive compatible (DSIC)** if for every $i \in N$, for every $v_{-i} \in V^{n-1}$, and for every $v_i, v_i' \in V$, we have

$$v_i f_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq v_i f_i(v_i', v_{-i}) - p_i(v_i', v_{-i}).$$

- A mechanism $(f, \mathbf{p})$ is **budget-balanced (BB)** if for every $\mathbf{v} \in V^n$, we have

$$\sum_{i \in N} p_i(\mathbf{v}) = 0.$$

- A mechanism $(f, \mathbf{p})$ is **symmetric** if for every $\mathbf{v} \in V^n$ and for every $i, j \in N$ with $v_i = v_j$, we have

$$f_i(\mathbf{v}) = f_j(\mathbf{v}), \quad p_i(\mathbf{v}) = p_j(\mathbf{v}).$$

We call a mechanism **satisfactory** if it is DSIC, BB, and symmetric. Symmetry allows us to consider a mild notion of fairness in our mechanism. It also explicitly rules out *dictatorial* mechanisms, where a dictator agent is given the object for free at all valuation profiles. We are interested in finding satisfactory mechanisms that are almost efficient in the following sense.

At any valuation profile $\mathbf{v}$, denote by $\mathbf{v}[k]$ the set of agents who have the k-th highest valuation at $\mathbf{v}$. More formally,

$$\mathbf{v}[1] := \{i \in N : v_i \geq v_j \ \forall \ j \in N\}.$$

Having defined $\mathbf{v}[k-1]$, we recursively define $\mathbf{v}[k]$ as

$$\mathbf{v}[k] := \{i \in N \setminus (\cup_{k'=1}^{k-1} \mathbf{v}[k']) : v_i \geq v_j \ \forall \ j \in N \setminus (\cup_{k'=1}^{k-1} \mathbf{v}[k'])\}.$$

Definition 1 *An allocation rule f is **efficient** at $\mathbf{v}$ if*

$$\sum_{i \in \mathbf{v}[1]} f_i(\mathbf{v}) = 1.$$

An allocation rule f is efficient if it is efficient at all $\mathbf{v} \in V^n$. A mechanism $(f, \mathbf{p})$ is efficient if f is efficient.

The efficiency of a BB mechanism is equivalent to maximizing the total welfare of agents at every profile of valuations. To see this, note that the total welfare of agents at a valuation profile $\mathbf{v}$ from a mechanism $(f, \mathbf{p})$ is

$$\sum_{i \in N} \Big[v_i f_i(\mathbf{v}) - p_i(\mathbf{v}) \Big] = \sum_{i \in N} v_i f_i(\mathbf{v}),$$

where the second equality followed from BB. This is clearly maximized by assigning the object to the highest valued agents.

3 A Top-only Satisfactory Mechanism

We now define our mechanism. Informally, the mechanism can be described in very simple terms as follows.

1. Agents are asked to report their values, and suppose the reported values are $v_1 > v_2 > \ldots > v_n$ - we consider reported values to be strictly ordered for simplicity.
2. Probability $\pi(v_2, v_3) = (1 - \frac{2}{n}) + \frac{2}{n}\frac{v_3}{v_2}$ is auctioned using a second-price auction. In particular,
   (a) Agent 1 wins the probability $\pi(v_2, v_3)$.
   (b) Agent 1 pays $v_2 \pi(v_2, v_3) \equiv (1 - \frac{2}{n})v_2 + \frac{2}{n}v_3$.
3. To maintain budget-balance, the generated revenue from the second-price auction, $v_2\pi(v_2, v_3)$, is redistributed among agents as follows:
   (a) Agents 1 and 2 receive an amount $\frac{v_3}{n}$ each.
   (b) Each agent j, where $j > 2$, receives an amount $\frac{v_2}{n}$.

Before formally defining the mechanism, we comment on some obvious properties of the mechanism. The probability auctioned in the mechanism depends on the (reported) values of second and third highest valued agents. Loosely, this cannot distort the incentives in the auction because all the allocation probabilities only go to the highest valued agent. Further, the redistribution amount of each agent does not depend on his own reported valuation, and hence, maintains incentive compatibility. This makes the overall mechanism DSIC. It is clearly budget-balanced. By breaking the ties carefully, we make it symmetric. Finally, each agent gets non-negative payoff in the mechanism, ensuring ex-post individual rationality. Also, by definition, only the highest valued agent gets the object with positive probability.

We now define the mechanism carefully to handle ties in reported values.

Definition 2 *Our mechanism $\mathcal{M}^* \equiv (f^*, \mathbf{p}^*)$ is defined as follows. The allocation rule f^* is defined as: for every $\mathbf{v}$ with $v_1 \geq v_2 \geq v_3 \geq \ldots \geq v_n$, we have*

$$f_i^*(\mathbf{v}) := \begin{cases} \frac{1}{|\mathbf{v}[1]|}\Big[\big(1 - \frac{2}{n}\big) + \big(\frac{2}{n}\big)\frac{v_3}{v_2}\Big] & \text{if } i \in \mathbf{v}[1] \\ 0 & \text{otherwise} \end{cases}$$

where $\frac{0}{0}$ is assumed to be 1. The payment of each agent $i \in N$ is given by

$$p_i^*(\mathbf{v}) := p_i^*(0, v_{-i}) + v_i f_i(\mathbf{v}) - \int_0^{v_i} f_i(x_i, v_{-i}) dx_i,$$

where $p_i^*(0, v_{-i})$ is defined as

$$p_i^*(0, v_{-i}) = \begin{cases} -\frac{v_3}{n} & \text{if } i \in \{1, 2\} \\ -\frac{v_2}{n} & \text{otherwise} \end{cases}$$

Though the formal definition involves defining payments using a Myersonian formula, it coincides with our informal description for the generic case when $v_1 > v_2 > v_3 > \ldots > v_n$. To see this, note that in this case, $f_1(\mathbf{v}) = \pi(v_2, v_3) = (1 - \frac{2}{n}) + \frac{2}{n}\frac{v_3}{v_2}$ and $f_i(\mathbf{v}) = 0$ for all $i > 1$. Further, $f_1(x_1, v_{-1}) = \pi(v_2, v_3)$ for all $x_1 \in (v_2, v_1]$ and $f_1(x_1, v_{-1}) = 0$ for all $x_1 < v_2$. Finally, $f_i(x_i, v_{-i}) = 0$ for all $x_i \leq v_i$ for all $i \neq 1$. These observations imply that the payment defined using the Myersonian formula in the above description coincides with the payments in the informal description.

Tie-Breaking. Tie-breaking in our mechanism is done in a symmetric way. We illustrate this with an example. Suppose $N = \{1, 2, 3\}$. There are three possible ties that can happen, and we describe our mechanism in each of the cases.

1. Suppose $v_1 = v_2 = v_3$. Then the object is given to each agent with equal probability and no probability is burnt:

$$f_1^*(v_1, v_2, v_3) = f_2^*(v_1, v_2, v_3) = f_3^*(v_1, v_2, v_3) = \frac{1}{3}.$$

   Notice that for every i,

$$v_i f_i^*(v_1, v_2, v_3) - \int_0^{v_i} f_i^*(x_i, v_{-i}) dx_i = \frac{1}{3} v_i.$$

   Further for every i, $p_i^*(0, v_{-i}) = -\frac{v_1}{3} = -\frac{v_2}{3} = -\frac{v_3}{3}$. Hence, we get that for every i,

$$p_i(v_1, v_2, v_3) = \frac{1}{3}(v_i - v_3) = 0.$$

   So, agents are distributed equal share of the object for free.

2. Suppose $v_1 = v_2 > v_3$. Then, the object is given with equal probability to agents 1 and 2, but some probability is burnt:

$$f_1^*(v_1, v_2, v_3) = f_2^*(v_1, v_2, v_3) = \frac{1}{2}\Big[\frac{1}{3} + \frac{2}{3}\frac{v_3}{v_2}\Big],$$

$$f_3^*(v_1, v_2, v_3) = 0.$$

   In this case, agent 3 receives a payment of $\frac{v_2}{3}$:

$$p_3^*(v_1, v_2, v_3) = p^*(0, v_1, v_2) = -\frac{v_2}{3}.$$

   For every $i \in \{1, 2\}$, we see that

$$v_i f_i^*(v_1, v_2, v_3) - \int_0^{v_i} f_i(x_i, v_{-i}) dx_i = v_i f_i^*(v_1, v_2, v_3),$$

   and $p_i^*(0, v_{-i}) = -\frac{v_3}{3}$. Hence, for every $i \in \{1, 2\}$, we get

$$p_i^*(v_1, v_2, v_3) = v_i f_i^*(v_1, v_2, v_3) - \frac{v_3}{3} = \frac{v_i}{6}.$$

   These amounts correspond to a uniform randomization over two asymmetric Vickrey auction. In the first auction, the tie is broken in favor of agent 1, and the other, it is broken in favor of agent 2. In each auction, a probability of $\frac{1}{3} + \frac{2}{3}\frac{v_3}{v_2}$ is auctioned - in one auction, the winner is agent 1 and the other the winner is agent 2. In either case, the winner pays an amount equal to $\frac{v_2}{3} + \frac{2v_3}{3}$. This amount is shared between the agents as follows: agents 1 and 2 get $\frac{v_3}{3}$ by each and agent 3 gets $\frac{v_2}{3}$. Uniform randomization over these two auctions exactly give us our mechanism, and generates a symmetric mechanism.

3. Suppose $v_1 > v_2 = v_3$. Then, the object is given with probability 1 to agent 1:

$$f_1^*(v_1, v_2, v_3) = 1, f_2^*(v_1, v_2, v_3) = f_3^*(v_1, v_2, v_3) = 0.$$

In this case, agent 2 receives a payment equal to $\frac{v_3}{3}$ and agent 3 receives a payment equal to $\frac{v_2}{3} = \frac{v_3}{3}$.

For agent 1, notice that

$$v_1 f_1^*(v_1, v_2, v_3) - \int_0^{v_1} f_1^*(x_1, v_2, v_3) = v_2.$$

Hence, payment of agent 1 is

$$p_1^*(v_1, v_2, v_3) = p_1^*(0, v_2, v_3) + v_2 = \frac{2v_2}{3} = \frac{2v_3}{3}.$$

This amount exactly corresponds to the fact that a Vickrey auction of the entire object is conducted. This generates a revenue of v_2. This is distributed equally among all the agents, including agent 1 (the winner).

3.1 The Result

In this section, we state the main result of the paper. Before describing the main result, we introduce some notation. For satisfactory mechanism $\mathcal{M} \equiv (f, \mathbf{p})$, let $\mathcal{W}(\mathbf{v}; \mathcal{M})$ be the welfare generated at a valuation profile $\mathbf{v}$ by this mechanism:

$$\mathcal{W}(\mathbf{v}; \mathcal{M}) := \sum_{i \in N} \big[v_i f_i(\mathbf{v}) - p_i(\mathbf{v})\big] = \sum_{i \in N} v_i f_i(\mathbf{v}),$$

where the second equality follows from budget-balance.

Definition 3 *An allocation rule f is* **top-only** *if at every valuation profile $\mathbf{v}$, $f_i(\mathbf{v}) = 0$ if $i \notin \mathbf{v}[1]$. A mechanism $\mathcal{M} \equiv (f, \mathbf{p})$ is a top-only mechanism if f is a top-only allocation rule.*

The next definition is about the participation constraint of a mechanism.

Definition 4 *A mechanism $\mathcal{M} \equiv (f, \mathbf{p})$ satisfies* **ex-post individual rationality** *if for every $\mathbf{v}$ and every $i \in N$, we have*

$$v_i f_i(\mathbf{v}) - p_i(\mathbf{v}) \geq 0.$$

We are now ready to state the main result of the paper.

Theorem **1** *The mechanism $\mathcal{M}^* \equiv (f^*, \mathbf{p}^*)$ is a top-only satisfactory mechanism satisfying ex-post individual rationality. Further, there exists no other top-only satisfactory mechanism $\mathcal{M}$, such that*

$$\mathcal{W}(\mathbf{v}; \mathcal{M}) \geq \mathcal{W}(\mathbf{v}; \mathcal{M}^*) \qquad \forall\ \mathbf{v},$$

with strict inequality holding for some $\mathbf{v}$.

Theorem 1 establishes welfare-optimality of our mechanism in the class of top-only satisfactory mechanisms. We now give the proof of this theorem in the next section.

3.2 Proof of Theorem 1

First, we show that $\mathcal{M}^*$ is a satisfactory mechanism - it is clearly a top-only mechanism. Fix a valuation profile $\mathbf{v}$ with $v_1 \geq v_2 \geq \ldots \geq v_n$, and observe the following using the definition of $p_i(\mathbf{v})$ for each i:

$$\begin{aligned}
\sum_{i\in N} p_i^*(\mathbf{v}) &= \sum_{i\in N} p_i^*(0, v_{-i}) + \sum_{i\in N} v_i f_i^*(\mathbf{v}) - \sum_{i\in N} \int_0^{v_i} f_i^*(x_i, v_{-i}) dx_i \\
&= \sum_{i\in N} p_i^*(0, v_{-i}) + v_1 f_1^*(\mathbf{v}) - \sum_{i\in N} \int_0^{v_i} f_i^*(x_i, v_{-i}) dx_i \quad \text{(by symmetry)} \\
&= \sum_{i\in N} p_i^*(0, v_{-i}) + v_1 f_1^*(\mathbf{v}) - (v_1 - v_2) f_1^*(\mathbf{v}) \quad \text{(by definition of } f^*) \\
&= \sum_{i\in N} p_i^*(0, v_{-i}) + v_2 f_1^*(\mathbf{v}) \\
&= \sum_{i\in N} p_i^*(0, v_{-i}) + v_2 (1 - \frac{2}{n}) + v_3 \frac{2}{n} \\
&= 0 \qquad \text{(by definition of } p_i^*(0, v_{-i}) \text{ for each } i)
\end{aligned}$$

This establishes that $\mathcal{M}^*$ is budget-balanced. For DSIC, we invoke the characterization of Myerson (1981), which states that an arbitrary mechanism $\mathcal{M} \equiv (f, \mathbf{p})$ is DSIC if and only if

1. Monotonicity. for all $i \in N$, for all v_{-i}, and for all v_i, v_i' with $v_i > v_i'$, we have

$$f_i(v_i, v_{-i}) \geq f_i(v_i', v_{-i}) \tag{1}$$

2. Revenue equivalence. for all $i \in N$, for all v_{-i}, and for all v_i, we have

$$p_i(v_i, v_{-i}) = p_i(0, v_{-i}) + v_i f_i(v_i, v_{-i}) - \int_0^{v_i} f_i(x_i, v_{-i}) dx_i. \tag{2}$$

Monotonicity is clearly satisfied by f^* and revenue equivalence is satisfied by $\mathbf{p}^*$ by definition. Hence, $\mathcal{M}^*$ is DSIC. Finally, since f^* is symmetric, $\mathbf{p}^*$ is also symmetric by construction. Hence, $\mathcal{M}^*$ is symmetric. This implies that $\mathcal{M}^*$ is a top-only satisfactory mechanism.

For ex-post individual rationality, note that for every $i \in N$ and for all $\mathbf{v}$, using revenue equivalence, we have

$$v_i f_i^*(\mathbf{v}) - p_i^*(\mathbf{v}) = \int_0^{v_i} f_i^*(x_i, v_{-i}) dx_i - p_i^*(0, v_{-i}) \geq 0,$$

where the inequality follows since $p_i^*(0, v_{-i}) \leq 0$ by definition.

Now, we move to the second part of the proof where we show that no other top-only satisfactory mechanism can welfare-dominate $\mathcal{M}^*$. To do this, we define some additional properties of an allocation rule, which is satisfied by f^*.

Definition 5 *An allocation rule f satisfies property*

P0. *if for every $\mathbf{v}$ with $|\mathbf{v}[1]| = 2$, we have $f_i(\mathbf{v}) = 0$ for all $i \notin \mathbf{v}[1]$.*

P1. *if for every $\mathbf{v}$ with $|\mathbf{v}[1]| > 2$, we have $\sum_{i \in \mathbf{v}[1]} f_i(\mathbf{v}) = 1$.*

P2. *if for every $\mathbf{v}$ with $\mathbf{v}[1] = \{k\}$ and $|\mathbf{v}[2]| > 1$, we have $f_k(\mathbf{v}) = 1$.*

Notice that f^* satisfies Properties P0, P1, and P2. Before completing the proof of the theorem, we state and prove an important proposition.

Proposition 1 *Suppose $(f, \mathbf{p})$ is a satisfactory mechanism and f satisfies Properties P0, P1, and P2. Then, for every $\mathbf{v}$ with $v_1 \geq v_2 \geq v_3 \geq \ldots \geq v_n$, we have*

$$\sum_{i \in N} p_i(0, v_{-i}) = -\frac{1}{n}\Big[(n-2)v_2 + 2v_3\Big].$$

Proof: We start off by establishing a property of payments.

Lemma 1 *Suppose $(f, \mathbf{p})$ is a satisfactory mechanism and f satisfies Properties P0, P1, and P2. For every $v_{-1} \equiv (v_2, v_3, \ldots, v_n)$ with $v_2 \geq v_3 \geq \ldots \geq v_n$, we have*

$$p_1(0, v_{-1}) = -\frac{v_3}{n}.$$

Proof: We do the proof in three steps.

Step 1. Pick v_{-1} such that $v_2 = v_3 = \theta \geq v_4 \geq \ldots \geq v_n$. Pick a type profile $\mathbf{v} \equiv (v_1, v_{-1})$ such that $v_1 = \theta$. If $\theta = 0$ this is the zero type profile, and by symmetry and budget-balance, the claim is true. Hence, suppose that $\theta > 0$. Let $K := |(0, v_{-1})[1]|$. Since $K \geq 2$, we have $|\mathbf{v}[1]| > 2$, and Property P1 implies that $\sum_{i \in \mathbf{v}[1]} f_i(\mathbf{v}) = 1$. Further, consider a type profile (x_1, v_{-1}), where $x_1 < \theta$. Such a type profile also satisfies $|(x_1, v_{-1})[1]| > 1$, and Property P0 and P1 imply that $f_1(x_1, v_{-1}) = 0$.

We now do the proof using induction on K. Using the observations in the previous paragraph along with symmetry and revenue equivalence formula, we get for all $i \in \mathbf{v}[1]$,

$$p_i(\mathbf{v}) = p_i(0, v_{-i}) + \frac{1}{K+1}\theta. \tag{3}$$

If $K = n - 1$, then $\mathbf{v}[1] = N$, and adding the above inequalities and using symmetry and BB, we get

$$p_1(0, v_{-1}) = -\frac{\theta}{n}.$$

Else, we assume that for all $K' > K$, the claim is true. Then, we have for all $i \notin \mathbf{v}[1]$, $|(0, v_{-i})[1]| = |\mathbf{v}[1]| = K + 1$, and induction hypothesis implies that

$$p_i(\mathbf{v}) = p_i(0, v_{-i}) = -\frac{\theta}{n}, \tag{4}$$

Adding Equations 3 and 4, and using BB and symmetry, we get

$$0 = (K+1)p_1(0, v_{-1}) + \theta - (n - K - 1)\frac{\theta}{n}.$$

Simplifying, we get,

$$p_1(0, v_{-1}) = -\frac{\theta}{n}.$$

This shows that if $|(0, v_{-1})[1]| > 1$, then the claim is true.

Step 2. Let $\mathbf{v}$ be a type profile such that for all $k > 2$ and for all $i \in \mathbf{v}[k]$, we have $v_i = 0$, and $|\mathbf{v}[1]| = 1$ and $|\mathbf{v}[2]| > 1$. In this step, we show that if $\theta = v_i > 0$ for every $i \in \mathbf{v}[2]$, then

$$p_i(0, v_{-i}) = -\frac{\theta}{n}.$$

Suppose $\mathbf{v}[1] = \{1\}$. By Step 1,

$$p_1(0, v_{-1}) = -\frac{\theta}{n}. \tag{5}$$

Further, by Property P2, $f_1(\mathbf{v}) = 1$. Further, for all $x_1 \in (\theta, v_1)$, we have $f_1(x_1, v_{-1}) = 1$ and for all $x_1 < \theta$, we have $f_1(x_1, v_{-1}) = 0$ - the latter observation follows from the fact that $|(x_1, v_{-1})[1]| > 1$ and Properties P0 and P1. Hence, using Equation 5 and Equation 2, we get

$$p_1(\mathbf{v}) = -\frac{\theta}{n} + v_1 - (v_1 - \theta) = (1 - 1/n)\theta. \tag{6}$$

Suppose $|\mathbf{v}[2]| = K$. By Property P2, $f_i(\mathbf{v}) = 0$ for all $i \in \mathbf{v}[2]$. Hence, for each $i \in \mathbf{v}[2]$, Equation 2 implies that

$$p_i(\mathbf{v}) = p_i(0, v_{-i}) \tag{7}$$

If $K = n - 1$, by adding Equations 6 and 7, and using BB and symmetry, we get for every $i \in \mathbf{v}[2]$,

$$0 = (n-1)p_i(0, v_{-i}) + (1 - 1/n)\theta.$$

This simplifies to $p_i(0, v_{-i}) = -\frac{\theta}{n}$.

Now, we use induction on K. Suppose the claim is true for all $K' > K$ and $K < n - 1$. By construction, for all $j > 2$ and for all $i \in \mathbf{v}[j]$, $v_i = 0$. We can construct another type profile $\mathbf{v}'$ such that $v'_i = \theta$ and $v'_j = v_j$ for all $j \neq i$. Note that $|\mathbf{v}'[2]| = K + 1$. Hence, induction hypothesis implies that

$$p_i(0, v'_{-i}) = p_i(0, v_{-i}) = p_i(\mathbf{v}) = -\frac{\theta}{n}. \tag{8}$$

Adding Equations 6, 7, and 8, and using BB and symmetry we get for every $i \in \mathbf{v}[2]$,

$$0 = Kp_i(0, v_{-i}) + (1 - 1/n)\theta - \frac{n - K - 1}{n}\theta.$$

This simplifies to $p_i(0, v_{-i}) = -\frac{\theta}{n}$, as desired.

Step 3. Now, we complete the proof. Pick a $\mathbf{v}$ with $\mathbf{v}[1] = \{1\}$ and $|\mathbf{v}[2]| > 1$. Suppose $v_k = \theta > 0$ for all $k \in \mathbf{v}[2]$. Note that by Step 1, the claim is proved if we show that for all $i \notin \mathbf{v}[1]$, we have $p_i(0, v_{-i}) = -\frac{\theta}{n}$ - in this case $(0, v_{-i})$ is a type profile such that $|(0, v_{-i})[1]| = 1$.

Suppose $K = |\mathbf{v}[2]|$. We use induction on K. If $K = n - 1$, the claim follow from Step 2. Suppose the claim is true for all $K' > K$. Pick $i \in \mathbf{v}[k]$, where $k > 2$. We can construct a type profile $\mathbf{v}'$ with $v'_i = \theta$ and $v_j = v'_j$ for all $j \neq i$. Since $|\mathbf{v}'[2]| = K + 1$, induction hypothesis implies that

$$p_i(0, v'_{-i}) = p_i(0, v_{-i}) = -\frac{\theta}{n}. \tag{9}$$

Now, at type profile $\mathbf{v}$, we know that $\mathbf{v}[1] = \{1\}$ and $|\mathbf{v}[2]| > 1$. By Property P2, $f_1(\mathbf{v}) = 1$ and for all $x_1 \in (\theta, v_1)$, we have $f_1(x_1, v_{-1}) = 1$. Further, by Property P0 and P1, $f_1(x_1, v_{-1}) = 0$ for all $x_1 < \theta$. Using these observations and Equation 2, we get

$$p_1(\mathbf{v}) = p_1(0, v_{-1}) + v_1 - (v_1 - \theta) = -\frac{\theta}{n} + \theta = (1 - 1/n)\theta, \tag{10}$$

where the second equality follows from Step 1. Since $f_i(\mathbf{v}) = 0$ for all $i \neq 1$, we can argue the following. For every $i \in \mathbf{v}[2]$, we have

$$p_i(\mathbf{v}) = p_i(0, v_{-i}). \tag{11}$$

For every $i \in \mathbf{v}[k]$, where $k > 2$, using Equation 9,

$$p_i(\mathbf{v}) = p_i(0, v_{-i}) = -\frac{\theta}{n}. \tag{12}$$

Adding Equations 10, 11, and 12, and using symmetry we get for every $i \in \mathbf{v}[2]$,

$$0 = K p_i(0, v_{-i}) + (1 - 1/n)\theta - (n - K - 1)\frac{\theta}{n}.$$

Simplifying, we get $p_i(0, v_{-i}) = -\frac{\theta}{n}$, as desired. ■

Now, we complete the proof of Proposition 1. Suppose $(f, \mathbf{p})$ is a satisfactory mechanism and f satisfies Properties P0, P1, and P2. Using Lemma 1, we immediately get that $p_i(0, v_{-i}) = -\frac{v_3}{n}$ if $i \in \{1, 2\}$ and $p_i(0, v_{-i}) = -\frac{v_2}{n}$ if $i \notin \{1, 2\}$. Using these equations, we get $\sum_{i \in N} p_i(0, v_{-i}) = -\frac{1}{n}\big[(n-2)v_2 + 2v_3\big]$. ■

Now, we complete the remaining part of Proof of Theorem 1. Assume for contradiction that mechanism $\tilde{\mathcal{M}} \equiv (\tilde{f}, \tilde{\mathbf{p}})$ is a top-only satisfactory mechanism such that for all $\mathbf{v}$, we have

$$\mathcal{W}(\mathbf{v}; \tilde{\mathcal{M}}) \geq \mathcal{W}(\mathbf{v}, \mathcal{M}^*), \tag{13}$$

with strict inequality holding for some $\mathbf{v}$.

Every top-only allocation rule satisfies Property P0. Since f^* satisfies Properties P1 and P2, Equation 13 implies that $\tilde{f}$ satisfies Properties P1 and P2 - this is because an implication of Equation 13 is that $\tilde{f}$ is efficient at all valuation profiles where f^* is efficient, and f^* is efficient at the profiles mentioned in Properties P1 and P2.

Then, by Proposition 1, we have for all $\mathbf{v}$ with $v_1 \geq v_2 \geq \ldots \geq v_n$,

$$\sum_{i \in N} \tilde{p}_i(0, v_{-i}) = \sum_{i \in N} p_i^*(0, v_{-i}) = -\frac{1}{n}\Big[(n-2)v_2 + 2v_3\Big]. \tag{14}$$

Note that if $v_2 = v_3$, then Properties P1 and P2 imply that $\tilde{f}_1(\mathbf{v}) = f_1^*(\mathbf{v}) = 1$. Now suppose $v_2 > v_3$. If $v_1 = v_2$, then by revenue equivalence formula and using the fact that $\tilde{f}_1(x_1, v_{-1}) = \tilde{f}_2(x_2, v_{-2}) = 0$ for all $x_1, x_2 < v_1 (= v_2)$, we get

$$\begin{aligned}
p_1(\mathbf{v}) &= p_1(0, v_{-1}) + v_1 \tilde{f}_1(\mathbf{v}) \\
p_2(\mathbf{v}) &= p_2(0, v_{-2}) + v_2 \tilde{f}_2(\mathbf{v}) \\
p_j(\mathbf{v}) &= p_j(0, v_{-j}) \qquad \forall\, j \notin \{1, 2\}.
\end{aligned}$$

Adding and using budget-balance and symmetry, we have

$$\sum_{i \in N} p_i(0, v_{-i}) = -2v_1 \tilde{f}_1(\mathbf{v}) = -2v_2 \tilde{f}_1(\mathbf{v}).$$

Using Equation 14, we get

$$\tilde{f}_1(\mathbf{v}) = \tilde{f}_2(\mathbf{v}) = \frac{1}{2n}\Big[(n-2) + 2\frac{v_3}{v_2}\Big] = f_1^*(\mathbf{v}) = f_2^*(\mathbf{v}).$$

Hence, if $v_1 = v_2$ or $v_2 = v_3$, by top-only property $\tilde{f} = f^*$. Since Equation 13 holds strictly for some $\mathbf{v}$, such a valuation profile must satisfy $v_1 > v_2 > v_3$. By top-only property and Equation 13, we must have

$$\tilde{f}_1(\mathbf{v}) > f_1^*(\mathbf{v}). \tag{15}$$

But then,

$$\begin{aligned}
0 &= \sum_{i \in N} \tilde{p}_i(\mathbf{v}) \\
&= \sum_{i \in N} \tilde{p}_i(0, v_{-i}) + \sum_{i \in N} v_i \tilde{f}_i(\mathbf{v}) - \sum_{i \in N} \Big[\int_0^{v_i} \tilde{f}_i(x_i, v_{-i} dx_i\Big] \text{ (By revenue equivalence)} \\
&= \sum_{i \in N} \tilde{p}_i(0, v_{-i}) + v_1 \tilde{f}_1(\mathbf{v}) - \int_{v_2}^{v_1} \tilde{f}_1(x_1, v_{-1}) dx_1 \text{ (By top-only property of } \tilde{f}) \\
&\geq \sum_{i \in N} \tilde{p}_i(0, v_{-i}) + v_1 \tilde{f}_1(\mathbf{v}) - (v_1 - v_2)\tilde{f}_1(\mathbf{v}) \text{ (From monotonicity of } \tilde{f}_1) \\
&= \sum_{i \in N} \tilde{p}_i(0, v_{-i}) + v_2 \tilde{f}_1(\mathbf{v}) \\
&> -\frac{1}{n}\big[(n-2)v_2 + 2v_3\big] + v_2 f_1^*(\mathbf{v}) \text{ (From Equations 14 and Inequality 15)} \\
&= 0 \text{ (By definition of } f^*),
\end{aligned}$$

which is a contradiction.

This completes the proof of Theorem 1.

4 Welfare Comparision

In this section, we compare the welfare properties of our mechanism with some existing DSIC and *almost* efficient mechanisms.

4.1 Other Budget-balanced Mechanisms

The literature has exclusively dealt with DSIC and budget-balanced mechanisms that never burn probabilities but allocate the object with positive probability to non-highest-valued agents. One simple mechanism that achieves this is the GL mechanism.

We discuss efficiency of the GL mechanism and our mechanism. For this, we fix a generic valuation profile $v_1 > v_2 > v_3 > \ldots > v_n$. The welfare from the GL mechanism is $v_1(1 - \frac{1}{n}) + v_2 \frac{1}{n}$, and the welfare from our mechanism is $v_1(1 - \frac{2}{n}) + \frac{2}{n}\frac{v_1 v_3}{v_2}$. Hence, our mechanism generates more welfare than the GL mechanism if and only if $\frac{2}{n}\frac{v_1 v_3}{v_2} \geq \frac{1}{n}(v_1 + v_2)$. This is equivalent to requiring

$$2\frac{v_3}{v_2} \geq 1 + \frac{v_2}{v_1}.$$

Notice that if valuations are drawn from some compact interval $[0, \beta]$, where $\beta > 0$, the set of profiles where this condition is satisfied has positive Lebesgue measure. In particular, from an ex-ante perspective, it is not clear which of these two simple mechanisms can give higher expected welfare - it will depend on the prior distribution being considered. We refrain from doing such a prior-based analysis and leave it for future research.

A similar analysis reveals that our mechanism also welfare dominates the *optimal* mechanism described in Mishra and Sharma (2016) at a positive measure of valuation profiles. Hence, within the class of DSIC and budget-balanced mechanisms, it seems unclear whether burning probabilities or sacrificing the top-only property is more useful.

4.2 A Class of No-deficit Mechanisms

Now, we return to the issue of burning money instead of burning probabilities to escape the Green-Laffont impossibility. We describe a class of *no-deficit* mechanisms that welfare dominates our mechanism. This essentially hints that burning money may be a better than burning probabilities to increase welfare - we are being careful here because we have not explored the entire class of top-only mechanisms. However, we stress here that asymptotically these mechanisms have similar welfare properties. Moreover, these mechanisms are impractical in settings where budget-balance is a hard constraint.

Before describing our new class of mechanisms, we first give a formal definition of no-deficit mechanisms.

DEFINITION **6** *A mechanism* $(f, \mathbf{p})$ *satisfies* ***no-deficit*** *if for every* $\mathbf{v} \in V^n$, *we have*

$$\sum_{i \in N} p_i(\mathbf{v}) \geq 0.$$

We use the idea of our mechanism to construct a class of no-deficit mechanism. The extremes of this class is our mechanism and the mechanism by Cavallo (2006). As we go from our mechanism to the Cavallo mechanism inside this class, the utility of every agent increases, achieving the maximum at the Cavallo mechanism. At the same time, as we go from our mechanism to the Cavallo mechanism inside this class, (a) the amount money burning increases and (b) the amount of probability burning decreases.

The class of mechanisms we define are parametrized by $\lambda \in [0, 1]$. We call such a mechanism λ-**Vickrey-Redistribution** mechanism.

1. Agents are asked to report their values, and suppose the reported values are $v_1 > v_2 > \ldots > v_n$ - we consider reported values to be strictly ordered for simplicity.

2. Probability
$$\pi^\lambda(v_2, v_3) = \lambda \Big[(1 - \frac{2}{n}) + \frac{2}{n} \frac{v_3}{v_2} \Big] + (1 - \lambda)$$
is auctioned using a second-price auction. In particular,

   (a) Agent 1 wins the probability $\pi^\lambda(v_2, v_3)$.

   (b) Agent 1 pays $v_2 \pi^\lambda(v_2, v_3) \equiv \lambda \big[(1 - \frac{2}{n}) v_2 + \frac{2}{n} v_3 \big] + (1 - \lambda) v_2$.

3. Part of the generated revenue from the second-price auction, $v_2 \pi^\lambda(v_2, v_3)$, is redistributed among agents as follows:

   (a) Agents 1 and 2 receive an amount $\frac{v_3}{n}$ each.

   (b) Each agent j, where $j > 2$, receives an amount $\frac{v_2}{n}$.

The 1-Vickrey-redistribution mechanism is our mechanism and 0-Vickrey-redistribution mechanism is the Cavallo mechanism. In the Cavallo mechanism, a Vickrey auction of the entire unit of resource is conducted. The revenue raised from the auction is redistributed exactly like our auction, but this leaves some surplus, which is burnt.

We can formally break ties in our class of no-deficit mechanisms by maintaining symmetry - this can be analogously done to the formal definition of our mechanism $\mathcal{M}^\lambda$.

Definition 7 *The mechanism $\mathcal{M}^\lambda \equiv (f^\lambda, \mathbf{p}^\lambda)$ for any $\lambda \in [0,1]$ is defined as follows. The allocation rule f^λ is defined as: for every $\mathbf{v}$ with $v_1 \geq v_2 \geq v_3 \geq \ldots \geq v_n$, we have*

$$f_i^\lambda(\mathbf{v}) := \begin{cases} \frac{1}{|\mathbf{v}[1]|}\Big[\lambda\big((1-\frac{2}{n}) + (\frac{2}{n})\frac{v_3}{v_2}\big) + (1-\lambda)\Big] & \textit{if } i \in \mathbf{v}[1] \\ 0 & \textit{otherwise} \end{cases}$$

where $\frac{0}{0}$ is assumed to be 1. The payment of each agent $i \in N$ is given by

$$p_i^\lambda(\mathbf{v}) := p_i^*(0, v_{-i}) + v_i f_i(\mathbf{v}) - \int_0^{v_i} f_i^\lambda(x_i, v_{-i})dx_i,$$

where $p_i^*(0, v_{-i})$ is defined as

$$p_i^*(0, v_{-i}) = \begin{cases} -\frac{v_3}{n} & \textit{if } i \in \{1,2\} \\ -\frac{v_2}{n} & \textit{otherwise} \end{cases}$$

Notice that the redistribution amounts p_i^* remains the same irrespective of the value of λ. The proof that any such mechanism is DSIC follows arguments similar to Theorem 1, and is skipped - it can also be shown using the fact that each mechanism in $\mathcal{M}^\lambda$ is a convex combination of $\mathcal{M}^*$ and the Cavallo mechanism. It clearly satisfies ex-post individual rationality. The surplus generated by such a λ-Vickrey redistribution mechanism is the following at valuation profile $\mathbf{v}$.

$$\lambda\Big[(1-\frac{2}{n})v_2 + \frac{2}{n}v_3\Big] + (1-\lambda)v_2 - \frac{2}{n}v_3 - (1-\frac{2}{n})v_2 = \frac{2}{n}(1-\lambda)(v_2 - v_3) \geq 0.$$

Hence, each λ-Vickrey-redistribution mechanism satisfies no-deficit, and for $\lambda = 1$, we have budget-balance. We summarize these conclusions in the following proposition.

Proposition 2 *Every λ-Vickrey redistribution mechanism is DSIC and satisfies ex-post individual rationality and no-deficit.*

Fix any λ-Vickrey-redistribution mechanism. At any valuation profile $\mathbf{v}$ (consider a strict valuation profile $v_1 > v_2 > \ldots > v_n$), the utility of agent j, where $j \notin \{1,2\}$ is $\frac{v_2}{n}$. The utility of agent 2 is $\frac{v_3}{n}$. The utility of agent 1 is

$$\begin{aligned}(v_1 - v_2)\pi^\lambda(v_2, v_3) + \frac{v_3}{n} &= \lambda(v_1 - v_2)\Big[(1-\frac{2}{n}) + \frac{2}{n}\frac{v_3}{v_2}\Big] + (1-\lambda)(v_1 - v_2) + \frac{v_3}{n} \\ &= (v_1 - v_2)\Big[(1-\frac{2}{n}) + \frac{2}{n}\frac{v_3}{v_2}\Big] + \frac{v_3}{n} + (1-\lambda)(v_1 - v_2)\frac{2}{n}(1-\frac{v_3}{v_2})\end{aligned}$$

Hence, the utility of agent 1 is strictly increasing with decreasing λ. On the other hand, the utilities of other agents are unchanged. Hence, by reducing λ, we increase the surplus that needs to be burnt but make the highest valued agent better off. This illustrates that the ability to burn some surplus allows one greater flexibility to increase welfare. The budget-balance condition constraints our mechanism, though asymptotically both the mechanisms have similar welfare.

5 Conclusion

Besides providing with a new asymptotically efficient, DSIC, budget-balanced, and ex-post individually rational mechanism, we provide insights into some technical issues on designing DSIC and budget-balanced mechanism.

First, our mechanism $\mathcal{M}^*$ cannot be expressed as a convex combination of deterministic, DSIC, and budget-balanced mechanisms [2] - note that the GL mechanism can be expressed in that form.

Second, our mechanism is a *non-ranking* DSIC and budget-balanced mechanism - Mishra and Sharma (2016) define a ranking mechanism as one which allocates a fixed probability π_k to the k-th highest valued agent at every valuation profile. Our mechanism is a non-ranking mechanism because it allocates different probabilities to the highest-valued agent. Ours is the first paper to carefully analyze a non-ranking DSIC and budget-balanced mechanism and establish its optimality and asymptotic properties.

Finally, ours is the first paper to explore the power of a top-only mechanism and illustrate that probability burning may help in partially overcoming the Green-Laffont impossibility result.

We provide one foundation for using our mechanism in the main result of this paper. The main result shows that our mechanism is not welfare-dominated by another top-only mechanism. It remains to be shown whether some other stronger optimality property of our mechanism can be established.

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[2] This follows from the top-only property of our mechanism.

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