Strategy-proof Partitioning *

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Abstract

We consider the problem of choosing a partition of a set of objects by a set of agents. The private information of each agent is a strict ordering over the set of partitions of the objects. A social choice function chooses a partition given the reported preferences of the agents. We impose a natural restriction on the allowable set of strict orderings over the set of partitions, which we call an *intermediate* domain. Our main result is a complete characterization of strategy-proof and tops-only social choice functions in the intermediate domain. We also show that a social choice function is strategy-proof and unanimous if and only if it is a *meet* social choice function.

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1 Introduction

The general mechanism design problem is concerned with choosing an alternative among a set of alternatives, when each agent has a preference ordering over the alternatives, which is his private information. The seminal work of Gibbard (1973) and Satterthwaite (1975) showed that if the preferences of agents over alternatives is unrestricted and the range of the social choice function has at least three alternatives, then the only strategy-proof social choice function is a dictatorship. A large body of literature has since focused on relaxing the underlying assumptions in the Gibbard-Satterthwaite theorem. One way to escape this impossibility result is to impose domain restrictions. Indeed, many real life problems have inherent domain restrictions.

We study one such model. In our model, a set of agents is faced with a set of objects. The agents have to collectively choose a partition of the set of objects. Each agent has a strict preference ordering over the set of partitions (of the set of objects), which is his private information. A social choice function asks for preference orderings of the agents, and based on the reported preference orderings, chooses a partition. Applications of this model are plenty: for example, creation of political districts by partitioning geographical districts when parties (agents) have preference over partitions; hedonic coalition formation problems with externalities.

We impose the standard notion of strategy-proofness on the social choice functions - it must be a dominant strategy for every agent to report his true preference ordering over partitions. If all possible preference orderings are allowed, then, under a mild range condition or unanimity, the Gibbard-Satterthwaite theorem will say that the only strategy-proof social choice function is a dictatorship.

We consider a restricted domain of preference orderings. Note that a partition must determine, for every pair of objects i and j, whether i and j should be together or separate. Call two partitions similar in i and j if they treat i and j similarly, i.e., either both of them put i and j together or both of them put i and j separately. Now, consider an agent who has A as the top partition in a preference ordering, and consider two other partitions B and C. Suppose whenever A and C are similar for any pair of objects, A and B are also similar for that pair of objects. This implies that overall, B is more similar to A than C is to A. Our domain restriction says that in such a case this agent must rank B over C, whenever A is his top ranked partition. We allow for all preference orderings which are consistent with such a restriction, and call such a domain an *intermediate domain*. Our objective is to investigate the consequence of strategy-proofness in this domain.

1.1 Our Contribution

We give a complete characterization of strategy-proof and tops-only social choice functions in the intermediate domain. Tops-onlyness property stipulates that at two preference profiles if the tops of the agents are the same, then the chosen partitions must also be the same. Because the number of partitions is quite large, tops-only property significantly reduces the communication requirement of each agent to the mechanism designer.

We prove our main result by proving another result, which is interesting in its own right. This result uses another mild property called *Pareto*$^+$, which requires that if all the agents want to put a pair of objects together in their top-ranked partition, then the social choice function must put them together. We show that if the number of objects is at least three, then a social choice function is strategy-proof, tops-only, and satisfies Pareto$^+$ if and only if it is a *meet** social choice function. A meet* social choice function identifies a subset of agents (may be empty) as oligarchs, and for every pair of objects, they are put together if and only if the top-ranked partition of each oligarch puts them together.

Our main result uses this characterization. It says that if a social choice function is strategy-proof and tops-only, then it can be *decomposed*. Decomposability roughly says that there exists a canonical partition such that a pair of objects belonging to different bundles (equivalence classes) of this partition are never put together. Further, the social choice function can be viewed as union of a set of strategy-proof social choice functions, each defined for a bundle of the canonical partition. We show that if a social choice function is strategy-proof and tops-only, we can decompose it into a set of social choice functions, each of which is strategy-proof, tops-only, and satisfies Pareto$^+$. As a result, we can invoke our earlier characterization to get a complete characterization of strategy-proof and tops-only social choice functions.

We show that if we impose unanimity, then we can get rid of tops-onlyness property. In particular, we show that if the number of objects is at least three, then a social choice function is strategy-proof and satisfies unanimity if and only if it is a meet social choice function (a meet social choice function is a meet* social choice function where the the oligarchs are non-empty). Hence, unanimity and strategy-proofness imply tops-onlyness in our domain.

On the other hand, if we impose Pareto efficiency, then also we can get rid of tops-onlyness property (Pareto efficiency implies unanimity), but it reduces the class of strategy-proof social choice functions significantly. In particular, we get a Gibbard-Satterthwaite-like impossibility - if the number of objects is at least three, then the only strategy-proof and Pareto efficient social choice function is a dictatorship.

2 Past Literature

Since the seminal work of Gibbard and Satterthwaite, many interesting restricted domains have been investigated - for a survey, see Barbera (2010) and Moulin (1983). Different restrictions bring out different possibilities, e.g., *median rules* and its generalizations are strategy-proof in various single-peaked domains (Moulin, 1980; Barbera et al., 1993).

As far as we know, there is no literature studying strategy-proof social choice functions in our model. A recent related paper is that of Duddy and Piggins (2010). They study strategy-proof social choice functions in a model where agents need to classify each object as 1 (good) or 0 (i.e., a partition into two bundles). Under some mild technical conditions and a range condition, Duddy and Piggins (2010) show that the only onto and strategy-proof social choice function in their domain is a dictatorship.

Our model has some resemblance to the *coalition formation* literature, where agents partition themseleves. However, the coalition formation literature usually does not consider externality between coalitions, i.e., focuses on coalition formation where an agent only cares about the coalition he is in (Bogomolnaia and Jackson, 2002).

There are two strands of literature which are closely related to our work. We describe them and their connection to our work in detail next.

2.1 Separable Preferences Literature

The literature on separable preferences is related to ours. Models with separable domains work in a multi-dimensional environment. Suppose there are k dimensions or components of an alternative. The set of alternatives in component j is A_j, and the set of alternatives is $A_1 \times A_2 \times \ldots \times A_k$ - a cartesian product of alternatives in each component. The separable domain roughly says that preferences over alternatives are separable over each component - a formal definition appears in Section 3.1. Barbera et al. (1991) and Barbera et al. (2005) studied the separable domains in the context of choosing a subset of objects from a set of objects. While these two papers consider the case where each object can have two alternatives (chosen or not chosen), Svensson and Tortstensson (2008) consider the case where each object can have more than two alternatives - see also Reffgen and Svensson (2010). Barbera et al. (1993) studied a separable domain in the context of multi-dimensional single-peaked preferences. The main insights of these papers is that in separable domains where an alternative is a product set of alternatives in each component, a strategy-proof social choice function (under some mild condition such as *non-imposition*) can be decomposed into strategy-proof social choice functions in each component. The most general version of this result is found in Le Breton and Sen (1999) - see also Le Breton and Weymark (1999) and Weymark (1999).

A crucial step of our result is very similar in spirit to this result. We show that in our domain, every strategy-proof and tops-only social choice function satisfies a property called *binary independence*. Binary independence is the same as decomposability in the separability literature - we use this terminology because it is a familiar terminology in the aggregation literature on partitions (Fishburn and Rubinstein, 1986). But our result cannot be deduced from these results - we give precise reasons for this later. Roughly, it is possible to imagine each pair of objects as a component in our model. The outcome for each pair of objects is either together or separate. But we *cannot* write the set of alternatives in our model (the set of partitions) as a product of possible outcomes for each pair of objects. This is because of the requirement that a partition is an *equivalence relation*, and must satisfy a transitivity property - if the pair of objects i and j is together, and the pair of objects j and k is together, then the outcome for the pair of objects i and k is fixed (i and k must be together). As a consequence, the set of alternatives is a *strictly smaller set* than the product of alternatives in each component.

There is another crucial difference. We show that our domain restriction, the intermediate domain, is a strictly smaller domain than the set of separable preferences. Also, the set of additively separable preferences, a smaller domain than the separable domain and where most of the results in the separability domain literature goes through, is neither a subset nor a superset of our intermediate domain. It is because of these two reasons, our results are not implied by the results in the separability literature. Moreover, our results complement the existing results in the separability literature by studying a specific problem where the set of alternatives is smaller and the domain restriction is more severe.

Barbera et al. (2005) study a model in separable domain with a set of alternatives smaller than the product of alternatives in each component. Their smaller set of alternatives is derived by putting constraints on the product of alternatives in each component. Since the constraints they consider are not specific, their results are very general. In our model, the constraints are specific - due to the transitivity requirement. Further, ours is a smaller domain than the separable domain. As a consequence, their results do not apply to our model.

Finally, our main result which characterizes the set of strategy-proof and tops-only social choice function uses our crucial result on binary independence to give a sharp characterization. This is a consequence of our domain restriction and the structure of our set of alternatives. We do not find any result of this flavor in the separability literature.

2.2 Aggregation Literature

There is a large body of literature studying Arrovian type aggregation in our model. This literature is inspired by Wilson (1978), who advocates Arrovian aggregation in abstract models such as those described in Rubinstein and Fishburn (1986). This literature considers *aggregators*, which takes as input a profile of partitions of agents, and gives as output a partition. This literature **does not consider** a preference ordering over partitions for every agent. Rather, each agent has a partition (an equivalence relation) of objects, and an aggregator chooses a collective partition. The main axioms used in that literature are binary independence and some form of unanimity (Mirkin, 1975; Leclerc, 1984; Barthélemy et al., 1986; Fishburn and Rubinstein, 1986; Barthélemy, 1988; Dimitrov et al., 2011). Broadly, this literature concludes that binary independence along with some form of unanimity gives us meet aggregators (when there are at least three objects) - see also Dimitrov et al. (2011) and Chambers and Miller (2011), who use a form of *separability* axiom instead of binary independence, and along with various forms of unanimity characterize the meet aggregators. As a model, there are two primary differences between this literature and our problem.

Strategic Model. While our main axiom is strategy-proofness, the aggregation literature uses other basic axioms, like some form of independence in the spirit of Arrow's independence of irrelevant alternatives. Our main contribution is to show that strategy-proofness in our model, along with an appropriate domain restriction and with some other mild axioms, imply the axioms used in aggregation literature. As a consequence, we can use the results in the aggregation literature to arrive at our first result, which is the building block of our main result. Much like Gibbard-Satterthwaite theorem is proved using Arrow's theorem (see Reny (2001) for a unified proof of Arrow's theorem and Gibbard-Satterthwaite theorem), we use one of the results in the aggregation literature of this model to prove one of our results. Thus, our results provide a strategic foundation to this literature on aggregating partitions.

Preferences. The aggregation literature in partitioning does not consider preferences of agents over partitions. The aggregation problem is that each agent has a partition and they want to collectively choose a partition. Axioms such as binary independence and unanimity are imposed on this aggregation procedure to nail down the meet aggregators. On the other hand, we consider a *strategic environment*, and agents have *preferences* over partitions. We enrich our problem by a particular restriction on preferences, which enables us to derive the main axiom, binary independence, in the aggregation literature as a consequence of strategy-proofness.

3 The Domain of Preferences

Let $N = \{1, \ldots, n\}$ be the set of agents and $M = \{1, \ldots, m\}$ be the set of objects. A **partition** A of objects in M is an equivalence relation, and can be represented by an $m \times m$-$\{0, 1\}$ matrix satisfying (a) reflexivity: $A_{ii} = 1$ for all $i \in M$, (b) symmetry: $A_{ij} = A_{ji}$ for all $i, j \in M$, and (c) transitivity: $A_{ij} = A_{jk} = 1$ implies $A_{ik} = 1$ for all $i, j, k \in M$ [1]. The value of A_{ij} reflects whether objects i and j are together in partition A or not. In particular, $A_{ij} = 1$ indicates that the objects i and j are together in partition A, whereas $A_{ij} = 0$ indicates that the objects i and j are separate in partition A. A **bundle** of a partition A is a set of objects $S \subseteq M$ such that for all $i, j \in S$, $A_{ij} = 1$ and for all $i \in S$ and $j \notin S$, $A_{ij} = 0$ [2]. Hence, a partition can be written as a collection of bundles. Let $\mathbb{M}$ be the set of all partitions of M. Two kinds of partitions will be of particular interest: (1) the *empty* partition which puts every object in a unique bundle and (2) the *complete* partition which puts all the objects in the same bundle.

A preference ordering is a complete, transitive, and anti-symmetric binary relation over $\mathbb{M}$. Let $\mathcal{P}$ be the set of all strict orderings over $\mathbb{M}$. An agent $h \in N$ has a preference ordering $\succ_h$ over $\mathbb{M}$, where $\succ_h (k)$ denotes the k-th ranked partition according to $\succ_h$. We impose a

[1] Here, A_{ij} specifies the value of the entry in the ith row and jth column.

[2] The conventional mathematical terminology for partition is "equivalence relation" and for bundle is "equivalence class". We use partition and bundle for convenience.

natural restriction on the allowable set of preference ordering. Let $\mathcal{C} := \{\{i,j\} : i \in M, j \in M, i \neq j\}$.

DEFINITION **1** *A domain of preferences $\mathcal{D} \subseteq \mathcal{P}$ is* **intermediate** *if for every $\succ_h \in \mathcal{D}$ with $A = (\succ_h (1))$ and every $B, C \in \mathbb{M}$ such that*

$$\{\{i,j\} \in \mathcal{C} : C_{ij} = A_{ij}\} \subsetneq \{\{i,j\} \in \mathcal{C} : B_{ij} = A_{ij}\},$$

we have $B \succ_h C$.

In some sense, if for every $i, j \in M$, $C_{ij} = A_{ij}$ implies $B_{ij} = A_{ij}$, then B is more similar to A than C is to A. A strict ordering in $\succ_h$ belonging to $\mathcal{D}$ must satisfy the property that if a partition B is "more similar" to the top partition of $\succ_h$ than a partition C, then $B \succ_h C$. We assume that the preference ordering $\succ_h$ of every agent $h \in N$ must belong to the domain of intermediate preferences $\mathcal{D}$.

Our intermediate domain uses a familiar notion of *betweenness* for any relation. Its use can be traced back to Grandmont (1978). A partition is an equivalence relation. Using the terminology of Grandmont (1978), our domain restriction says that if a partition B is between partitions A and C, then if A is at the top, then B must be ranked above C.

We give an example to clarify some of the nuances of the intermediate domain.

EXAMPLE **1** *Suppose $M = \{a, b, c, d\}$. Consider a preference ordering $\succ_h \in \mathcal{D}$ with $\succ_h (1) = A$, where A refers to the partition with the following bundles: $\{a, b, c\}$ and $\{d\}$. Consider four more partitions and their corresponding bundles:*

- *B is the partition with bundles $\{a, b\}$, $\{c\}$, and $\{d\}$.*
- *C is the partition with bundles $\{a, b, d\}$ and $\{c\}$.*
- *D is the partition with bundles $\{a\}$, $\{b\}$, and $\{c, d\}$.*
- *E is the partition with bundles $\{a\}$ and $\{b, c, d\}$.*

Comparing B and C with A, we see that whenever C and A agree on a pair of objects, B and A also agree on the same pair. Hence, $B \succ_h C$.

However, for B and E, we cannot make such a comparison - a and b are together in A and B but separate in E, but b and c are together in A and E but separate in B. So, it is possible that $B \succ_h E$ or $E \succ_h B$. Similarly, D and E can be ranked either way when A is the top.

On the other hand, wherever A and D agree on a pair of objects, A and B also agree on that pair. Hence, $B \succ_h D$.

3.1 Separability and Intermediate Domain

We highlight the relationship of our model and results to the separable and the additive separable domains. Let $\mathcal{B} = \{0,1\}^{|\mathcal{C}|}$ be the extreme points of the multidimensional cube. We can view each pair $\{i,j\} \in \mathcal{C}$ as a *component* with two possible alternatives $\{0,1\}$, where alternative 0 indicates objects i and j belong to different bundles and alternative 1 indicates objects i and j are in the same bundle. An alternative is a product of the alternatives chosen for each component, and hence, lies in $\mathcal{B}$. However, we require an alternative to be a partition, which must satisfy the transitivity requirement: for any alternative if the component $\{i,j\}$ is 1 and the component $\{j,k\}$ is 1, then the component $\{i,k\}$ has to be 1 also. This implies that the set of alternatives in our model is a strictly smaller subset of $\mathcal{B}$. Denote the set of alternatives in our model as $\mathcal{T}$ [3]. Now, an alternative A, a partition, will be an element of $\mathcal{T}$, and $A_{ij} \in \{0,1\}$ will denote whether the component $\{i,j\}$ of A is 0 or 1, i.e., whether objects i and j are together or separate. For any partition A, let A^+ denote all the components $\{i,j\} \in \mathcal{C}$ such that $A_{ij} = 1$.

Our intermediate domain restriction is like the separability requirement in Barbera et al. (1991) [4]. To be precise, we first remind what separability means in this context - of course, we will have to adapt the definition of Barbera et al. (1991) since their definition was for the case when the set of alternatives is $\mathcal{B}$. A preference ordering $\succ_h$ on the set of all partitions induces an ordering on the alternatives in $\mathcal{T}$ - this follows from the fact that every partition can be mapped to a unique alternative in $\mathcal{T}$.

Definition 2 *A preference ordering $\succ_h$ is* **separable** *if for every pair of partitions $B, C \in \mathcal{T}$ with $B^+ = C^+ \cup \{\{i,j\}\}$ we have $B \succ_h C$ if and only if $A_{ij} = 1$, where $\succ_h (1) \equiv A$.*

This is a straightforward adaptation of the definition of separability in Barbera et al. (1991) to our model. The intermediate domain is a separable domain.

Claim 1 *If a preference ordering belongs to the intermediate domain, then it is separable.*

Proof: Consider a preference ordering $\succ_h$ in the intermediate domain with $\succ_h (1) = A$. Let B and C be a pair of partitions such that $B^+ = C^+ \cup \{\{i,j\}\}$. Hence, B and C only differ in how they group the pair of objects i and j. In that case

$$\{\{k,l\} \in \mathcal{C} : C_{kl} = A_{kl}\} \subsetneq \{\{k,l\} \in \mathcal{C} : B_{kl} = A_{kl}\},$$

if and only if $A_{ij} = B_{ij} = 1 \neq C_{ij} = 0$. Hence, $\succ_h$ is separable. ■

However, the intermediate domain is a strict subset of the separable domain. Consider an example with three objects $M := \{i,j,k\}$. Now, consider a partition $\succ_h$ with $\succ_h (1) = A$ and

[3] Earlier, we used the notation $\mathbb{M}$ to denote the set of partitions. Here, we use $\mathcal{T}$ to denote the representation of the partitions in the multidimensional cube $\mathcal{B}$.

[4] We thank an anonymous referee for pointing this out.

$A_{ij} = A_{ik} = A_{jk} = 0$. Consider two other partitions B and C with $B_{ij} = 1, B_{jk} = B_{ik} = 0$ and $C_{ij} = C_{jk} = C_{ik} = 1$. Note that separability imposes no restriction on how B and C should be ranked in $\succ_h$. On the other hand, intermediate domain requires that $B \succ_h C$. Note the role of transitivity in this example. In other words, if the set of alternatives was $\mathcal{B}$, separability would have coincided with the intermediate domain restriction, but in our problem where the set of alternatives is $\mathcal{T} \subsetneq \mathcal{B}$, the intermediate domain turns out to be strictly smaller than the separable domain.

We now show how our domain restriction is related to *additive separability*. Formally, the definition of additive separability can be adapted from Barbera et al. (1991) as follows.

Definition 3 *A preference ordering $\succ_h$ is additively separable if there exists a utility function $u_h : \mathcal{C} \times \{0, 1\} \to \mathbb{R}$ such that $u_h(\{i, j\}, 0) = 0$ for all $\{i, j\} \in \mathcal{C}$ and for any pair of partitions B and C, we have $B \succ_h C$ if and only if*

$$\sum_{\{i,j\}\in\mathcal{C}} u_h(\{i, j\}, B_{ij}) > \sum_{\{i,j\}\in\mathcal{C}} u_h(\{i, j\}, C_{ij}).$$

The following example shows that a preference ordering in the intermediate domain need not be additively separable. Now, consider an example with four objects $M := \{i, j, k, l\}$. Let $\succ_h$ be a preference ordering with $\succ_h (1) = A$, where $A_{pq} = 1$ for all $\{p, q\} \in \mathcal{C}$. Now, consider the following six partitions.

1. $B : B_{ik} = 1$ but $B_{pq} = 0$ for all $\{p, q\} \neq \{i, k\}$,
2. $C : C_{il} = 1$ but $C_{pq} = 0$ for all $\{p, q\} \neq \{i, l\}$,
3. $D : D_{jk} = 1$ but $D_{pq} = 0$ for all $\{p, q\} \neq \{j, k\}$,
4. $E : E_{jl} = 1$ but $E_{pq} = 0$ for all $\{p, q\} \neq \{j, l\}$,
5. $F : F_{ij} = F_{jl} = F_{il} = 1$ but $F_{pq} = 0$ for all $\{p, q\} \notin \{\{i, j\}, \{j, l\}, \{i, l\}\}$, and
6. $G : G_{ij} = G_{jk} = G_{ik} = 1$ but $G_{pq} = 0$ for all $\{p, q\} \notin \{\{i, j\}, \{j, k\}, \{i, k\}\}$.

Note that we can have $B \succ_h C$, $D \succ_h E$, and $F \succ_h G$ if $\succ_h$ is in the intermediate domain since the intermediate domain does not put any restriction on these pairs of partitions if A is the top ranked partition. However, $\succ_h$ is not additively separable. To see this, suppose $\succ_h$ is additively separable. Then, there is a utility function $u_h : \mathcal{C} \times \{0, 1\} \to \mathbb{R}$ such that $u_h(\{p, q\}, 0) = 0$ for all $\{p, q\}$. Further $F \succ_h G$ implies that $u_h(\{i, l\}, 1) + u_h(\{j, l\}, 1) > u_h(\{i, k\}, 1) + u_h(\{j, k\}, 1)$. But $B \succ_h C$ implies that $u_h(\{i, k\}, 1) > u_j(\{i, l\}, 1)$ and $D \succ_h E$ implies that $u_h(\{j, k\}, 1) > u_h(\{j, l\}, 1)$, and adding these two inequalities contradicts the earlier inequality.

We also show that an additively separable preference ordering need not lie in the intermediate domain. To see this, consider an example with $M = \{i, j, k\}$. Let u_h be a utility function satisfying $u_h(\{i,j\},1) = -1, u_h(\{j,k\},1) = 2, u_h(\{i,k\},1) = 4$. Let $\succ_h$ be the additively separable preference ordering induced by this utility function. Let A be the partition where $A_{ij} = A_{jk} = A_{ik} = 1$, B be the partition where $B_{ij} = 1, B_{ik} = B_{jk} = 0$, and C be the partition where $C_{ij} = C_{jk} = C_{ik} = 0$. Note that $\succ_h (1) = A$ and $C \succ_h B$. But in the intermediate domain, any preference ordering will put B above C if A is the top-ranked partition.

4 Strategy-proof, Tops-only, Pareto Properties

A **social choice function (SCF)** is a mapping $F : \mathcal{D}^n \to \mathbb{M}$, i.e., given the intermediate preference orderings of agents, it selects a partition. For a profile $(\succ_1, \ldots, \succ_n) \in \mathcal{D}^n$, the output of F is denoted by $F(\succ_1, \ldots, \succ_n)$, and $F(\succ_1, \ldots, \succ_n)_{ij} \in \{0, 1\}$ denotes whether $i, j \in M$ belong to the same bundle or not in $F(\succ_1, \ldots, \succ_n)$. Often, we write the profile $(\succ_1, \ldots, \succ_n)$ as $\succ$ and the profile $(\succ'_1, \ldots, \succ'_n)$ as $\succ'$, and so on.

We impose the usual strategy-proofness requirement on an SCF - every agent must have a dominant strategy to submit his true preference ordering. An agent h **manipulates** an SCF F at $(\succ_h, \succ_{-h}) \in \mathcal{D}^n$ via $\succ'_h \in \mathcal{D}$ if $F(\succ'_h, \succ_{-h}) \succ_h F(\succ_h, \succ_{-h})$.

Definition 4 *An SCF F is* **strategy-proof** *if no agent $h \in N$ can manipulate at any preference profile $(\succ_h, \succ_{-h}) \in \mathcal{D}^n$ via any preference ordering $\succ'_h \in \mathcal{D}$.*

The following is a well-known requirement on SCFs.

Definition 5 *An SCF F is* **tops-only** *if for every pair of profiles $\succ, \succ' \in \mathcal{D}^n$ such that $\succ_h (1) = \succ'_h (1)$ for all $h \in N$, then $F(\succ) = F(\succ')$.*

Tops-onlyness is a well-studied axiom in social choice theory. It comes as a consequence of unanimity in various domains (Chatterji and Sen, 2011). We discuss this issue further for our model later. For tops-only SCFs, we can focus on *aggregators* instead of SCFs.

An **aggregator** v is a mapping $v : \mathbb{M}^n \to \mathbb{M}$. So, for a profile of partitions $(A^1, \ldots, A^n)$, an aggregator gives a partition $v(A^1, \ldots, A^n)$.

An agent h **manipulates** an aggregator v at (A^h, A^{-h}) via $B^h \in \mathbb{M}$ if for some preference ordering $\succ_h \in \mathcal{D}$ with $\succ_h (1) = A^h$, we have $v(B^h, A^{-h}) \succ_h v(A^h, A^{-h})$.

Definition 6 *An aggregator v is* **strategy-proof** *if no agent $h \in N$ can manipulate at any $(A^h, A^{-h}) \in \mathbb{M}^n$ via any B^h.*

A tops-only SCF induces an aggregator. Suppose F is a tops-only SCF. Then, define $v^F(A^1, \ldots, A^n) = F(\succ_1, \ldots, \succ_n)$, where $\succ \in \mathcal{D}^n$ is such that $\succ_h (1) = A^h$ for every $h \in N$ - note that for every $A^h \in \mathbb{M}$, there exists $\succ_h \in \mathcal{D}$ such that $\succ_h (1) = A^h$ (such domains are called **minimally rich**). Clearly, if F is strategy-proof and tops-only, then the induced aggregator v^F is strategy-proof as well.

Similarly, an aggregator v induces an SCF F^v as follows. For every profile of preference orderings $\succ \in \mathcal{D}^n$, define $F^v(\succ) = v(\succ_1(1), \ldots, \succ_n(1))$. Clearly, F^v is tops-only, and if v is strategy-proof, then F^v is strategy-proof too.

4.1 An Implicit Characterization

We will now establish an implicit characterization of strategy-proof *aggregators*. This characterization will identify a simple property of an aggregator which is equivalent to strategy-proofness. This of course implies a characterization of strategy-proof SCFs which are tops-only.

Definition **7** *An aggregator v is responsive, if for every $h \in N$, for every A^{-h}, for every $A^h \in \mathbb{M}$, and every $i, j \in M$ we have that $A^h_{ij} \neq v(A^h, A^{-h})_{ij}$ implies that*

$$v(A^h, A^{-h})_{ij} = v(B^h, A^{-h})_{ij} \qquad \forall\ B^h \in \mathbb{M}.$$

Responsiveness requires that if an agent's preference for a pair of objects is not fulfilled for some partition (keeping profile of other agents fixed), then the outcome for that pair of objects do not change by changing the partition.

Proposition **1** *An aggregator is strategy-proof if and only if it is responsive.*

Proof: Suppose v is strategy-proof. Now, fix an agent $h \in N$, and fix the profile of other agents at A^{-h}. Let $A^h \in \mathbb{M}$ be such that $A^h_{ij} \neq v(A^h, A^{-h})_{ij}$ for some $i, j \in M$. Consider another partition $B^h \in \mathbb{M}$. Let $v(A^h, A^{-h}) = B$, and $v(B^h, A^{-h}) = C$. Assume for contradiction $B_{ij} \neq C_{ij}$. By definition, $A^h_{ij} \neq B_{ij}$ and $A^h_{ij} = C_{ij}$. We claim that there is a preference ordering $\succ''_h$ such that $\succ''_h(1) = A^h$, and $C \succ''_h B$. If this was not true, then $B \succ_h C$ for all $\succ_h \in \mathcal{D}$ with $\succ_h(1) = A^h$. By our intermediate domain assumption, if $C_{ij} = A^h_{ij}$, then $B_{ij} = A^h_{ij}$. This is a contradiction. But $C \succ''_h B$ implies that agent h will manipulate at (A^h, A^{-h}) via B^h. This is a contradiction to the fact that v is strategy-proof.

Suppose v is responsive. Fix an agent $h \in N$, and a profile A^{-h} of other agents. Consider partitions $A^h, B^h \in \mathbb{M}$. Let $v(A^h, A^{-h}) = A$ and $v(B^h, B^{-h}) = B$. If $A = B$, agent h cannot manipulate at (A^h, A^{-h}) via B^h. Else, $A \neq B$. If $A = A^h$, then agent h cannot manipulate at (A^h, A^{-h}) via B^h. So, assume $A \neq A^h$. Consider any $i, j \in M$, such that $A^h_{ij} \neq A_{ij}$. By responsiveness, $A_{ij} = B_{ij}$. This implies that whenever $B_{ij} = A^h_{ij}$ for some $i, j \in M$, $A_{ij} = A^h_{ij}$. Now, for any preference ordering $\succ_h \in \mathcal{D}$ with $\succ_h(1) = A^h$, we must have $A \succ_h B$. Hence, agent h cannot manipulate at (A^h, A^{-h}) via B^h. This implies that v is strategy-proof. ■

We now define an independence axiom for an aggregator. For every $i, j \in M$ and any profile of partitions $(A^1, \ldots, A^n)$, denote the n-dimensional vector $(A^1_{ij}, \ldots, A^n_{ij})$ as $\mathbf{A}_{ij}$.

Definition 8 *An aggregator v satisfies* **binary independence** *if for every $i, j \in M$ and for every pair of profiles $(A^1, \ldots, A^n)$ and $(B^1, \ldots, B^n)$ such that $\mathbf{A}_{ij} = \mathbf{B}_{ij}$, we have*

$$v(A^1, \ldots, A^n)_{ij} = v(B^1, \ldots, B^n)_{ij}.$$

Binary independence points at some kind of separability of aggregation. In particular, it says that whether a pair of objects remains separate or together must depend *only* on agents' preferences about that pair of objects. It is a widely studied axiom in Arrovian aggregation literature of this model (Mirkin, 1975; Fishburn and Rubinstein, 1986; Dimitrov et al., 2011). Below, we show that every strategy-proof aggregator satisfies binary independence.

Proposition 2 *If an aggregator is strategy-proof, then it satisfies binary independence.*

Proof: Let v be an aggregator which is strategy-proof. By our characterization in Proposition 1, v is responsive. Consider any $i, j \in M$. Let $(A^1, \ldots, A^n)$ and $(B^1, \ldots, B^n)$ be two profiles such that $\mathbf{A}_{ij} = \mathbf{B}_{ij}$. Consider the profile $(B^1, A^2, \ldots, A^n)$. Assume for contradiction that $v(A^1, \ldots, A^n)_{ij} \neq v(B^1, A^2, \ldots, A^n)_{ij}$. Since $A^1_{ij} = B^1_{ij}$, either $v(A^1, \ldots, A^n)_{ij} \neq A^1_{ij}$ or $v(B^1, A^2, \ldots, A^n)_{ij} \neq B^1_{ij}$. By responsiveness, if $v(A^1, \ldots, A^n)_{ij} \neq A^1_{ij}$, then $v(A^1, \ldots, A^n)_{ij} = v(B^1, A^2, \ldots, A^n)_{ij}$, and if $v(B^1, A^2, \ldots, A^n)_{ij} \neq B^1_{ij}$, then $v(A^1, \ldots, A^n)_{ij} = v(B^1, A^2, \ldots, A^n)_{ij}$. This is a contradiction.

We can repeat this argument by changing the preference of one agent at a time to reach the profile $(B^1, \ldots, B^n)$, and conclude $v(A^1, \ldots, A^n)_{ij} = v(B^1, \ldots, B^n)_{ij}$. ■

We make some remarks on the results of this section.

Binary Independence is not sufficient. Although binary independence is implied by strategy-proofness, it is not sufficient for strategy-proofness if $|M| \geq 3$. Consider the following aggregator v^*. Suppose $|M| \geq 3$. Fix a pair of objects $i, j \in M$. For any pair of objects $\{k, l\} \neq \{i, j\}$ and every profile $(A^1, \ldots, A^n)$,

$$v^*(A^1, \ldots, A^n)_{kl} = 0.$$

For every profile $(A^1, \ldots, A^n)$,

$$v^*(A^1, \ldots, A^n)_{ij} = 1 \text{ if and only if } A^1_{ij} = 1 \text{ and } A^h_{ij} = 0 \ \forall \ h \in N \setminus \{1\}.$$

Clearly, v^* satisfies binary independence. But it is not strategy-proof. To see this, consider agent 2, and fix the profile of other agents at (A^{-2}) such that $A^1_{ij} = 1$ and $A^h_{ij} = 0$ for all $h \in N \setminus \{1, 2\}$. Consider A^2 such that $A^2_{ij} = 0$. By definition $v^*(A^2, A^{-2})_{ij} = 1 \neq A^2_{ij}$. Now, consider B^2 such that $B^2_{ij} = 1$. By definition $v^*(B^2, A^{-2})_{ij} = 0 \neq v^*(A^2, A^{-2})_{ij}$. Hence, v^* is not responsive, and by Proposition 1, it is not strategy-proof.

More on Proposition 2. It is worth noting that one direction of Proposition 1 (responsiveness implies strategy-proofness) holds even when the set of alternatives is *any* subset of $\mathcal{B}$, i.e., it does not require the set of alternatives to be $\mathcal{T}$ (satisfying transitivity requirement). As a consequence, Proposition 2 also holds when the set of alternatives is *any* subset of $\mathcal{B}$.

Binary Independence and Decomposability in Barbera et al. (1991). Our binary independence property is the same as the *decomposability* property in Barbera et al. (1991), who show that a tops-only SCF in separable domain is strategy-proof **if and only if** it is decomposable. Since our intermediate domain is strictly smaller than the separable domain and the set of alternatives is strictly smaller than the points in $\mathcal{B}$, we cannot invoke their result or other standard results in the separability literature (Le Breton and Sen, 1999; Weymark, 1999; Svensson and Tortstensson, 2008; Reffgen and Svensson, 2010). Also, note that in the intermediate domain, strategy-proofness implies binary independence but binary independence does not imply strategy-proofness.

Responsiveness in Unrestricted Domain. Though responsiveness is sufficient for strategy-proofness in the intermediate domain, it is not sufficient in the unrestricted domain. For instance, consider an aggregator v which puts any pair of objects together if and only if the top partition of each agent puts these two objects together. This aggregator belongs to a family of aggregators which are responsive as we will show in Proposition 3. But, it is not strategy-proof in the unrestricted domain - this is immediate because this aggregator satisfies unanimity, and in the unrestricted domain the only strategy-proof and unanimous SCF is a dictatorship.

4.2 A Rich Family of Aggregators

In this section, we define a rich class of aggregators, and the corresponding SCFs.

Definition 9 *An aggregator v is a* **meet*** *aggregator if there exists a set of agents (called an oligarchy) $S \subseteq N$ such that for all $(A^1, \ldots, A^n)$, we have for all $i, j \in M$, $v(A^1, \ldots, A^n)_{ij} = 1$ if and only if $A^h_{ij} = 1$ for all $h \in S$. If the oligarchy is the empty set, then we call the aggregator a* **trivial aggregator**. *An aggregator is a* **meet** *aggregator if it is a meet* aggregator but not a trivial aggregator. An aggregator is a* **dictatorship** *if it is a meet aggregator with a unique oligarch.*

An SCF F is a **meet*** *SCF if there exists a meet* aggregator v such that $F = F^v$. An SCF F is a* **trivial** *SCF if there exists a trivial aggregator v such that $F = F^v$. An SCF F is a* **meet** *SCF if it is a meet* SCF but not a trivial SCF. An SCF F is a* **dictatorship** *if there exists an aggregator v which is dictatorship, and $F = F^v$.*

Note that a trivial aggregator always gives the partition where all the objects in M are put in one bundle.

Not every meet* aggregator is strategy-proof when the domain of preferences is unrestricted. However, in intermediate domains, every meet* aggregator is strategy-proof.

Proposition **3** *If v is a meet* aggregator, then it is strategy-proof.*

Proof: Let v be a meet* aggregator with $S \subseteq N$ being the oligarchy. We will show that v is responsive, and hence by Proposition 1, it is strategy-proof.

For this, pick any agent $h \in N$ and fix the profile of partitions of other agents at A^{-h}. Choose some $\{i, j\} \in \mathcal{C}$. Suppose $v(A^h, A^{-h})_{ij} \neq A^h_{ij}$. If $h \notin S$, then $v(B^h, A^{-h})_{ij} = v(A^h, A^{-h})_{ij} \neq A^h_{ij}$. If $h \in S$, then this means that $A^h_{ij} = 1$ and $v(A^h, A^{-h})_{ij} = 0$. This further implies that there is some agent $h' \in S$ such that $A^{h'}_{ij} = 0$. But, this means that for any other partition B^h, we must have $v(B^h, A^{-h})_{ij} = 0$. ■

As we will show later, the entire set of strategy-proof aggregators is much larger than the meet* family of aggregators. Our first result is a characterization of strategy-proof aggregators in the presence of the following weak requirement.

Definition **10** *An aggregator v satisfies* **Pareto$^+$** *if for every $i, j \in M$ and for every profile $(A^1, \ldots, A^n)$ with $A^h_{ij} = 1$ for all $h \in N$, we have $v(A^1, \ldots, A^n)_{ij} = 1$.*

A social choice function F satisfies **Pareto$^+$** *if for every $i, j \in M$ and for every preference profile $\succ \in \mathcal{D}^n$ with $(\succ_h (1))_{ij} = 1$ for all $h \in N$, we have $F(\succ)_{ij} = 1$.*

Pareto$^+$ says that if each agent puts objects i and j together, then the aggregator must put them together.

Theorem **1** *Suppose $|M| \geq 3$. A social choice function is strategy-proof, tops-only, and satisfies Pareto$^+$ if and only if it is a meet* social choice function.*

Proof: A meet* social choice function is tops-only. By Proposition 3, every meet* aggregator is strategy-proof, and hence, every meet* social choice function is also strategy-proof. Clearly, a meet* social choice function also satisfies Pareto$^+$.

For the converse, let F be a strategy-proof and tops-only social choice function satisfying Pareto$^+$. Since F is tops-only, v^F is well-defined. Further v^F is strategy-proof and satisfies Pareto$^+$. By Proposition 2, v^F must satisfy binary independence. Finally, Dimitrov et al. (2011) show that if an aggregator satisfies binary independence and Pareto$^+$, then it must be a meet* aggregator - we give a proof of this fact in the Appendix for completeness. Hence, v^F is a meet* aggregator, and F is a meet* social choice function. ■

Another way to state Theorem 1 is that an aggregator is strategy-proof and satisfies Pareto$^+$ if and only if it is a meet* aggregator.

Couple of remarks are in order.

Tops-onlyness is not implied. We remark that strategy-proofness and Pareto^{+} property of a social choice function does not imply tops-onlyness. The following example illustrates that.

Example **2** *Let $\hat{A}$ be the partition where each bundle is a singleton (i.e., all the objects are put separately). Consider the social choice function which chooses agent 1's top ranked partition from the set $\mathbb{M} \setminus \{\hat{A}\}$ at every preference profile. This social choice function is clearly strategy-proof and satisfies Pareto^{+}, but it is not tops-only.*

Pareto^{-}. A property analogous to Pareto^{+} is Pareto^{-}. It can be found, for example, in Fishburn and Rubinstein (1986).

Definition **11** *An aggregator v satisfies* **Pareto^{-}** *if for every $i, j \in M$ and for every profile $(A^1, \ldots, A^n)$ with $A^h_{ij} = 0$ for all $h \in N$, we have $v(A^1, \ldots, A^n)_{ij} = 0$.*

A social choice function F satisfies **Pareto^{-}** *if for every $i, j \in M$ and for every preference profile $\succ \in \mathcal{D}^n$ with $(\succ_h (1))_{ij} = 0$ for all $h \in N$, we have $F(\succ)_{ij} = 0$.*

One wonders if there is an analogue of Theorem 1 if we use Pareto^{-} instead of Pareto^{+}. Define an *empty SCF* as an SCF which generates the empty partition at every profile. One way to read Theorem 1 is that every strategy-proof and tops-only SCF satisfying Pareto^{+} is either a meet SCF or a trivial SCF. Is it then true that every strategy-proof and tops-only SCF satisfying Pareto^{-} is either a meet SCF or an empty SCF? The answer is no. In Theorem 2, we characterize the entire class of strategy-proof and tops-only SCF. A corollary of that result is that the set of strategy-proof, tops-only, and Pareto^{-} SCFs is larger than the set of meet SCFs combined with the empty SCF.

This apparent asymmetry between Pareto^{+} and Pareto^{-} may be due to the fact that the definition of a partition requires transitivity in objects being "together" and not in objects being "separate". Since Pareto^{+} is a requirement for a pair of objects to be together, it has stronger implications with transitivity then Pareto^{-}. We touch on this issue further after Theorem 2.

Connection to the aggregation literature. As pointed out in Section 2, our problem is the strategic counterpart of the aggregation literature on partitions (Fishburn and Rubinstein, 1986; Mirkin, 1975; Barthélemy et al., 1986; Chambers and Miller, 2011; Dimitrov et al., 2011). The proof of Theorem 1 helps to establish the exact relationship. Using a model where agents have preference over partitions in the intermediate domain, and using strategy-proofness and tops-onlyness, we are able to reduce the problem to a problem of aggregating partitions satisfying axioms like binary independence and Pareto^{+}. We can then use the results in the aggregation literature to get Theorem 1, which will serve as the building block for our main result in Theorem 2.

4.3 The Two Objects Case

If $|M| = 2$, then we have many more aggregators which are strategy-proof and satisfy Pareto$^+$. Suppose $M = \{i, j\}$. Consider the following family of aggregators. Let $\mathbb{S} = \{S_1, \ldots, S_k\}$, where S_p with $p \in \{1, \ldots, k\}$ is a subset of agents (may be empty), called an oligarchy. A set of oligarchies $\mathbb{S}$ is **non-nested** if for every $S_p, S_q \in \mathbb{S}$, S_p is not a subset of S_q. Note that if a non-nested set of oligarchies $\mathbb{S}$ contains $\emptyset$, then it is the only element of $\mathbb{S}$.

Definition **12** *Suppose $M = \{i, j\}$. An aggregator v is a* **meet*-join aggregator** *if there exists a set of non-nested oligarchies $\mathbb{S}$ such that for every $(A^1, \ldots, A^n) \in \mathbb{M}^n$, we have*

$$v(A^1, \ldots, A^n)_{ij} = 1 \text{ if and only if } A^h_{ij} = 1 \ \forall \ h \in S_p \text{ for some } S_p \in \mathbb{S}.$$

A social choice function F is a **meet*-join social choice function** *if there is a meet*-join aggregator v such that $F = F^v$.*

Note that a meet*-join aggregator is a meet aggregator if $\mathbb{S}$ is a singleton.

Proposition **4** *Suppose $|M| = 2$. An aggregator is strategy-proof and satisfies Pareto$^+$ if and only if it is a meet*-join aggregator. Further, a social choice function is strategy-proof and satisfies Pareto$^+$ if and only if it is a meet*-join social choice function.*

Note that when $|M| = 2$, every social choice function is tops-only. Also, when $|M| = 2$ in our model, we are in the standard Gibbard-Satterthwaite model with two alternatives. The characterization of strategy-proof social choice functions is well-known in that case - see, for example, Moulin (1983) and Barbera et al. (1991). Applying Pareto$^+$, we get Proposition 4 immediately. We provide a proof below for completeness.

Proof: With $|M| = 2$, every social choice function is tops-only. So, without loss of generality, we focus on aggregators instead of social choice functions. Consider a meet*-join aggregator v. Consider agent $h \in N$, and fix the profile of other agents at A^{-h}. Let A^h be a partition such that $A^h_{ij} \neq v(A^h, A^{-h})_{ij}$. Consider another partition B^h. If agent h is not an oligarch (i.e., $h \notin S_p$ for some $S_p \in \mathbb{S}$), then $v(A^h, A^{-h})_{ij} = v(B^h, A^{-h})_{ij}$. If h is an oligarch, then there are two cases to consider.

- $A^h_{ij} = 0$ and $v(A^h, A^{-h})_{ij} = 1$ implies that there is some oligarchy $S_p \in \mathbb{S}$ such that $h \notin S_p$, and $A^{h'}_{ij} = 1$ for all $h' \in S_p$. In that case, $v(A^h, A^{-h})_{ij} = v(B^h, A^{-h})_{ij} = 1$.
- $A^h_{ij} = 1$ and $v(A^h, A^{-h})_{ij} = 0$ implies that in every oligarch $S_p \in \mathbb{S}$ there is some agent $h' \in S_p \setminus \{h\}$ such that $A^{h'}_{ij} = 0$. In that case, $v(A^h, A^{-h})_{ij} = v(B^h, A^{-h})_{ij} = 0$.

This shows that v is responsive. Hence, it is strategy-proof by Proposition 1.

Suppose v is strategy-proof and satisfies Pareto$^+$. Then, call a set of agents $S \subseteq N$ decisive if for every $(A^1, \ldots, A^n)$, $v(A^1, \ldots, A^n)_{ij} = 1$ if and only if $A^h_{ij} = 1$ for all $h \in S$.

A decisive set exists because v satisfies Pareto$^+$. Also, for any $S \subsetneq T \subseteq N$, if S is decisive, then T is also decisive. This follows from responsiveness of v (since v is strategy-proof). Let $\mathbb{S} = \{S_1, \ldots, S_k\}$ be such that each $S_p \in \mathbb{S}$ is *minimally* decisive. Since each $S_p \in \mathbb{S}$ is minimally decisive, for any $S_q, S_r \in \mathbb{S}$, S_q cannot be a subset of S_r. Hence, by definition $\mathbb{S}$ is a set of oligarchies, and v is a meet*-join aggregator.

Hence, a social choice function is a meet*-join aggregator if and only if it is strategy-proof and satisfies Pareto$^+$. ■

The proof illustrates how we get a set of oligarchies when there are exactly two objects. When there are more than two objects, the proof of Theorem 1 shows how the set of oligarchies (decisive sets) collapse into a single oligarchy.

5 Strategy-proof and Tops-Only SCFs

The Pareto$^+$ property used in Theorem 1 may not be completely appealing in our model. Consider the example of building a network, where each node/city in the network represents an object. Even if all the agents agreed to connect a pair of cities, it may be infeasible for the government to connect them because of their distance or budget constraints.

In this section, we drop the Pareto$^+$ requirement of a social choice function. We provide a complete characterization of strategy-proof aggregators. In other words, we provide a complete characterization of strategy-proof and tops-only social choice functions. Our result comes as a consequence of a decomposability result we are able to prove in our model. To define decomposability, we require some notation. For every subset of objects $X \subseteq M$, let $\mathbb{X}$ be the set of all partitions of objects in X. For any partition A^h of agent h, we can look at the restriction of A^h to some subset of objects X, and that restriction is denoted as $A^{h,X}$.

Definition 13 *An aggregator $v : \mathbb{M}^n \to \mathbb{M}$ is* **decomposable** *if there exists a partition $\bar{A}$ of M with bundles $\bar{A}_1, \ldots, \bar{A}_k$ and k aggregators $v_1, \ldots, v_k$, where $v_p : \bar{\mathbb{A}}_p^n \to \bar{\mathbb{A}}_p$ for all $p \in \{1, \ldots, k\}$, with the following two conditions holding for every profile $(A^1, \ldots, A^n)$:*

- $v(A^1, \ldots, A^n)_{ij} = v_p(A^{1,\bar{A}_p}, \ldots, A^{n,\bar{A}_p})_{ij}$ *if $i, j \in \bar{A}_p$ for some $p \in \{1, \ldots, k\}$.*
- $v(A^1, \ldots, A^n)_{ij} = 0$ *if $i \in \bar{A}_p$ and $j \in \bar{A}_q$ for some $p \neq q$ and $p, q \in \{1, \ldots, k\}$.*

In such a case, we say v can be decomposed into $v_1, \ldots, v_k$ via partition $\bar{A}$ with bundles $(\bar{A}_1, \ldots, \bar{A}_k)$.

Every strategy-proof aggregator is decomposable - it can be decomposed into itself via the complete partition with the unique bundle M. However, we can say something non-trivial about decomposing any strategy-proof aggregator.

Proposition 5 *If v is a strategy-proof aggregator, then it can be decomposed into strategy-proof aggregators $v_1, \ldots, v_k$ via some partition $\bar{A}$ with bundles $(\bar{A}_1, \ldots, \bar{A}_k)$ such that for all $p \in \{1, \ldots, k\}$, each v_p satisfies Pareto$^+$.*

Proof: Since v is strategy-proof, it is responsive due to Proposition 1, and satisfies binary independence by Proposition 2. Let $(D^1, \ldots, D^n)$ be a profile of partitions where D^h is a complete partition for every $h \in N$. Suppose $v(D^1, \ldots, D^n) = \bar{A}$, where $\bar{A}$ is a partition with bundles $(\bar{A}_1, \ldots, \bar{A}_k)$.

Consider the restriction of v to $\bar{A}_p$ for all $p \in \{1, \ldots, k\}$, and denote it by v_p. In particular, define for every $(X^1, \ldots, X^n)$, where $X^h \in \bar{\mathbb{A}}_p$ for all $h \in N$, and for every $i, j \in \bar{A}_p$

$$v_p(X^1, \ldots, X^n)_{ij} = v(A^1, \ldots, A^n)_{ij}.$$

Since v satisfies binary independence, each v_p is well-defined. Further, since v is responsive and satisfies binary independence, each v_p is also responsive. By Proposition 1, each v_p is strategy-proof.

Next, we will show that for every $i \in \bar{A}_p$ and $j \in \bar{A}_q$ where $p \neq q$ and $p, q \in \{1, \ldots, k\}$, $v(A^1, \ldots, A^n)_{ij} = 0$ for all profiles $(A^1, \ldots, A^n)$. Fix an $i \in \bar{A}_p$ and $j \in \bar{A}_q$ where $p \neq q$ and $p, q \in \{1, \ldots, k\}$, and a profile $(A^1, \ldots, A^n)$. By definition, $\bar{A}_{ij} = 0$ [5]. Assume for contradiction $v(A^1, \ldots, A^n)_{ij} = 1$.

Construct another profile $(B^1, \ldots, B^n)$ such that $B^h_{ij} = A^h_{ij}$ and $B^h_{st} = 0$ for all $\{s, t\} \neq \{i, j\}$. By binary independence, $v(B^1, \ldots, B^n)_{ij} = 1$. Let $S = \{h \in N : B^h_{ij} = 0\}$. If $S \neq \emptyset$, then choose $h \in S$, and consider C^h such that $C^h_{ij} = 1$ and $C^h_{st} = B^h_{st} = 0$ for all $\{s, t\} \neq \{i, j\}$. By responsiveness, $v(C^h, B^{-h})_{ij} = v(B^h, B^{-h})_{ij} = 1$. Continuing in this manner by choosing a new agent from S in every iteration, we will get to a profile $(C^1, \ldots, C^n)$, where $C^h_{ij} = 1$ and $C^h_{st} = 0$ for all $\{s, t\} \neq \{i, j\}$, and $v(C^1, \ldots, C^n)_{ij} = 1$. But, by binary independence $v(D^1, \ldots, D^n)_{ij} = 1$, whereas, by definition, $v(D^1, \ldots, D^n) = \bar{A}$ and $\bar{A}_{ij} = 0$. This is a contradiction.

Finally, we show that each v_p for $p \in \{1, \ldots, k\}$ satisfies Pareto$^+$. Assume for contradiction some v_p does not satisfy Pareto$^+$. Then, by the definition of v_p, for some preference profile $(A^1, \ldots, A^n)$ such that $A^h_{ij} = 1$ for all $h \in N$ for some $i, j \in \bar{A}_p$, we have $v(A^1, \ldots, A^n)_{ij} = 0$. By binary independence, $v(D^1, \ldots, D^n)_{ij} = 0$ (since v is strategy-proof, it satisfies binary independence by Proposition 2). This is a contradiction by the definition of $\bar{A}$. ■

Proposition 5 says that every strategy-proof aggregator can be decomposed into aggregators that satisfy Pareto$^+$. This is non-trivial since we did not impose Pareto$^+$ for the main aggregator. As a consequence of this, we can say precisely how a strategy-proof aggregator must look like.

We are now ready to define a general family of aggregators, which includes the meet* family.

[5] We let $\bar{A}_p$ (single superscript) to denote a bundle in $\bar{A}$, but $\bar{A}_{ij}$ (double superscript) to denote the value of the i-th row and j-th column entry corresponding to the $0-1$ matrix of partition $\bar{A}$. We apologize for this notational clumsiness.

Definition **14** *An aggregator v is a* **decomposed meet* aggregator** *if there exists a partition $\bar{A}$ with bundles $\bar{A}_1, \ldots, \bar{A}_k$, and for every $\bar{A}_p$ with $p \in \{1, \ldots, k\}$, we have*

- *an oligarchy $S_p \subseteq N$ if $|\bar{A}_p| > 2$,*
- *and a set of non-nested oligarchies $\mathbb{S}_p$ if $|\bar{A}_p| = 2$,*

such that

- *if $i, j \in \bar{A}_p$ for some $p \in \{1, \ldots, k\}$ with $|\bar{A}_p| > 2$, then $v(A^1, \ldots, A^n)_{ij} = 1$ if and only if $A^h_{ij} = 1$ for all $h \in S_p$ (meet*),*
- *if $i, j \in \bar{A}_p$ for some $p \in \{1, \ldots, k\}$ with $|\bar{A}_p| = 2$, then $v(A^1, \ldots, A^n)_{ij} = 1$ if and only if $A^h_{ij} = 1$ for all $h \in S$ for some $S \in \mathbb{S}_p$,*
- *and, $v(A^1, \ldots, A^n)_{ij} = 0$, if $i \in \bar{A}_p$ and $j \in \bar{A}_q$, where $p \neq q$.*

An SCF F is a decomposed meet* SCF if there exists a decomposed meet* aggregator v such that $F = F^v$.

Different choices of $\bar{A}$ result in interesting aggregators. If we choose $\bar{A}$ to be the complete partition, then the resulting decomposed meet* aggregator is a meet* aggregator if $|M| > 3$ and meet*-join aggregator if $|M| = 2$. Choosing any $\bar{A}$, and choosing $\emptyset$ as oligarchs in each bundle of $\bar{A}$ gives $\bar{A}$ as the output in every profile. Intuitively, $\bar{A}$ reflects the bias of the mechanism designer towards a particular partition. Such bias may be inherent in some applications.

We now state the main result of the paper.

Theorem **2** *A social choice function is strategy-proof and tops-only if and only if it is a decomposed meet* social choice function.*

Proof: Let F be a strategy-proof and tops-only social choice function. Define the aggregator v^F as follows. For every $(A^1, \ldots, A^n) \in \mathbb{M}^n$, let $v^F(A^1, \ldots, A^n) = F(\succ_1, \ldots, \succ_n)$ such that $(\succ_1, \ldots, \succ_n) \in \mathcal{D}^n$ and $\succ_h (1) = A^h$ for all $h \in N$. Since F is tops-only, v^F is well-defined. Further, v^F is strategy-proof. By Proposition 5, v^F can be decomposed into $v_1, \ldots, v_k$ via partition $\bar{A}$ with bundles $\bar{A}_1, \ldots, \bar{A}_k$. For each $p \in \{1, \ldots, k\}$, v_p is strategy-proof and satisfies Pareto$^+$. Using Theorem 1 and Proposition 4, we conclude that v^F is a decomposed meet* aggregator. Hence, F is a decomposed meet* social choice function.

Suppose F is a decomposed meet* social choice function. By definition, F only uses information in the top-ranked partition of each agent. So, it is tops-only. Let v^F be the decomposed meet* aggregator induced by F. Consider the partition $\bar{A}$ corresponding to this decomposed meet* aggregator, let $(\bar{A}_1, \ldots, \bar{A}_k)$ be the bundles in this partition. Denote the restriction of v^F to $\bar{A}_p$ for every $p \in \{1, \ldots, k\}$ as v_p. By definition, each v_p is well-defined. Further, each v_p is either a meet* aggregator or a meet*-join aggregator. By Theorem 1 and Proposition 4, each v_p is strategy-proof. Strategy-proofness of each v_p implies strategy-proofness of v^F (by definition of v^F). ■

Tops-onlyness is needed. The tops-onlyness property in Theorem 2 is essential for the characterization. For instance, consider the aggregator in Example 2. This is an aggregator which is not tops-only, but strategy-proof. Hence, it is not a decomposed meet* aggregator.

More on Pareto$^-$. Earlier, we discussed how Pareto$^+$ and Pareto$^-$ have different implications, and the absence of a "dual" result to Theorem 1 when we substitute Pareto$^+$ with Pareto$^-$. Using Theorem 2, we see that if an aggregator is strategy-proof and satisfies Pareto$^-$, then it must be a decomposed meet* aggregator where the set of oligarchies does not contain the empty set. This is not very "symmetric" to our result using Pareto$^+$ in Theorem 1.

We discussed earlier some plausible reasons on why this could be happening. Here, we will like to point out that our main result (Theorem 2) has a symmetric interpretation (in terms of Pareto$^+$ and Pareto$^-$). Consider a decomposed meet* SCF with partition $\bar{A}$. Let $\bar{A}_1, \ldots, \bar{A}_q$ be the bundles for which the corresponding aggregators $v_1, \ldots, v_q$ are all non-trivial - this means for bundles $\bar{A}_{q+1}, \ldots, \bar{A}_p$, the corresponding aggregators are all trivial. Now, note that objects inside bundle $\bar{A}_k$ for $k \in \{q+1, \ldots, p\}$ are always together, and the aggregators $v_1, \ldots, v_q$ satisfy **both** Pareto$^+$ and Pareto$^-$. The only place where Pareto$^-$ may be violated is for the aggregators $v_{q+1}, \ldots, v_p$, and these aggregators always put all the objects in their respective bundles.

So, an implication of Theorem 2 is that if an aggregator v is strategy-proof, then there exists a partition $\bar{A}$ with bundles $\bar{A}_1, \ldots, \bar{A}_p$ such that

- for some bundles $\bar{A}_1, \ldots, \bar{A}_q$ there exists corresponding aggregators $v_1, \ldots, v_q$ (as in Definition 13) which are strategy-proof and satisfy Pareto$^+$ and Pareto$^-$, and the outcome of these aggregators coincide with v,
- for the remaining bundles $\bar{A}_{q+1}, \ldots, \bar{A}_p$, if $i, j \in \bar{A}_k$ where $k \in \{q+1, \ldots, p\}$, then i and j are always put together,
- if $i \in \bar{A}_k$ and $j \in \bar{A}_r$, where $k \neq r$, then i and j are never put together.

Informally, a strategy-proof aggregator must always put some objects together and some objects separate, but where such impositions are absent, the corresponding decomposed aggregators satisfy Pareto$^+$ and Pareto$^-$.

6 Unanimity and Tops-Only Property

The tops-only property says that the only relevant information in the preference orderings of agents are their tops. It is a useful tool, and often the biggest obstacle in establishing characterization results in social choice theory (Chatterji and Sen, 2011; Weymark, 2008). Tops-only property is critical in our characterizations of Theorems 1 and 2. At the same time, it may not be entirely appealing. However, we have already seen that tops-only property is not implied by Pareto$^+$. In this section, we introduce the well known property unanimity for social choice functions, and show its connection to tops-onlyness. It says that whenever agents have the same top-ranked partition, the social choice function must choose that partition. It is an appealing property, and used extensively in social choice theory.

Definition **15** *An aggregator v satisfies* **unanimity** *if for every $i, j \in M$ and for every profile $(A^1, \ldots, A^n)$ with $A^1 = \ldots = A^n = A$, we have $v(A^1, \ldots, A^n) = A$.*

A social choice function F satisfies **unanimity** *if for every profile $(\succ_1, \ldots, \succ_n) \in \mathcal{D}^n$ such that $\succ_1 (1) = \ldots = \succ_n (1) = A$, we have $F(\succ_1, \ldots, \succ_n) = A$.*

The main result of this section is the following.

Theorem **3** *Suppose $|M| \geq 3$. Then, the following statements are equivalent.*

1. *A social choice function is a meet social choice function.*
2. *A social choice function is strategy-proof and satisfies unanimity.*
3. *A social choice function is strategy-proof and satisfies Pareto$^+$ and Pareto$^-$.*

As discussed earlier, when $|M| \geq 3$, Fishburn and Rubinstein (1986) characterized meet aggregators using binary independence, Pareto$^+$, and Pareto$^-$. Similarly, Mirkin (1975) characterized meet aggregators using binary independence and unanimity. Theorem 3 is the strategic counterpart of these results.

The main hurdle in proving Theorem 3 is to show that unanimity implies the tops-only property for a strategy-proof social choice function. To establish this property for our domain, we use a general result in Chatterji and Sen (2011). Chatterji and Sen (2011) identify a sufficient condition on domains such that every strategy-proof function which satisfies unanimity in that domain is tops-only. We show that this sufficient condition is satisfied in our domain.

To be able to use their result, we need to first explore some structure of our domain. First, we define the notion of *betweenness*. For any pair of distinct partitions A and B, let

$$\beta(A, B) = \{C : \forall \succ_h \in \mathcal{D} \text{ with } \succ_h (1) = A, C \notin \{A, B\}, C \succ_h B\}.$$

So, $\beta(A, B)$ contains all partitions which will lie between A and B, whenever A is the top ranked partition. An alternate way to define $\beta(A, B)$ is the following. To remind, $\mathcal{C} := \{\{i, j\} : i, j \in M, i \neq j\}$, i.e., all pairs of objects such that the objects are distinct. For any pair of partitions, let $L(A, B) = \{\{i, j\} \in \mathcal{C} : A_{ij} = B_{ij}\}$.

Lemma **1** *Consider a pair of partitions A, B. A partition $C \in \beta(A, B)$ if and only if for every $\{i, j\} \in \mathcal{C}$, $A_{ij} = B_{ij}$ implies $A_{ij} = C_{ij}$ (i.e., $L(A, B) \subseteq L(A, C)$).*

Proof: This follows from the definition of $\beta(A, B)$. ■

The following lemma says that the partitions that lie between A and B when A is the top are also the partitions that lie between A and B when B is the top.

LEMMA **2** *For any pair of partitions A and B, $\beta(A, B) = \beta(B, A)$.*

Proof: Suppose $C \in \beta(A, B)$. This means for every $\{i, j\} \in \mathcal{C}$, if $B_{ij} = A_{ij}$ then $C_{ij} = A_{ij}$. But this also implies that if $B_{ij} = A_{ij}$ then $C_{ij} = B_{ij}$. Hence, $C \in \beta(B, A)$. A symmetric argument establishes that $C \in \beta(B, A)$ implies $C \in \beta(A, B)$. ■

Given a preference ordering $\succ_h$ and a partition B, let

$$\alpha(B, \succ_h) = \{A \in \mathbb{M} : A \succ_h B\}.$$

So, $\alpha(B, \succ_h)$ are all the partitions which are above B in preference ordering $\succ_h$. The following lemma says that if we take a pair of partitions A and B, we can find a preference ordering $\succ_h$ with A being the top-ranked partition such that the partitions between A and B in $\succ_h$ are exactly the partitions in $\beta(A, B)$.

LEMMA **3 (Squeezing)** *For every pair of partitions A and B, there exists a preference ordering $\succ_h \in \mathcal{D}$ such that $\succ_h (1) = A$ and $\alpha(B, \succ_h) = \{A\} \cup \beta(A, B)$.*

Proof: Fix a pair of partitions A and B. Define $T(A) = \{\succ_h \in \mathcal{D} : \succ_h (1) = A\}$. For every $\succ_h \in T(A)$, define $S(\succ_h) = \alpha(B, \succ_h) \setminus (\{A\} \cup \beta(A, B))$. Choose $\succ'_h \in T(A)$ such that $|S(\succ'_h)| \leq |S(\succ_h)|$ for all $\succ_h \in T(A)$. If $|S(\succ'_h)| = 0$, we are done. Assume for contradiction $|S(\succ'_h)| = r > 0$.

Let C be a partition above B in $\succ'_h$ such that $C \notin \beta(A, B)$, i.e., $C \in \alpha(B, \succ'_h) \setminus (\{A\} \cup \beta(A, B))$, and for all $D \neq C$ and $D \in \alpha(B, \succ'_h) \setminus (\{A\} \cup \beta(A, B))$ we have $D \succ'_h C$. In other words, C is the lowest ranked partition above B which does not lie in $\beta(A, B)$. Such a C exists since $|S(\succ'_h)| = r > 0$.

We construct another profile $\succ''_h$ by moving C just below B and keeping all the other partitions at the same position. We claim that $\succ''_h \in \mathcal{D}$. Assume for contradiction that $\succ''_h \notin \mathcal{D}$. Then, there must exist two partitions $X, Y \in \mathbb{M} \setminus \{A\}$ such that $X \succ''_h Y$ but $L(A, X) \subsetneq L(A, Y)$. Since $\succ'_h \in \mathcal{D}$, we must have $Y = C$ and $X \in \alpha(B, \succ'_h)$ but $X \notin \alpha(C, \succ'_h)$, i.e., X must be a partition between B and C in $\succ'_h$, and Y must be C. Then, by definition of C, $X \in \beta(A, B)$. Since $C \notin \beta(A, B)$, there is a preference ordering $\hat{\succ}_h \in \mathcal{D}$ with $\hat{\succ}_h(1) = A$ and $X \hat{\succ}_h C$. This is a contradiction to the fact $L(A, X) \subsetneq L(A, C)$.

Hence, there exists a preference ordering $\succ''_h \in T(A)$ such that $|S(\succ''_h)| = r - 1$. This is a contradiction. ■

We are now ready to state the tops-only result. Before stating the result, we state one notation, which we use in the proof. For every $\succ_h \in \mathcal{D}$ and for every $A \in \mathbb{M}$, define $\omega(A, \succ_h) = \{B \in \mathbb{M} : A \succ_h B\}$, i.e., all the partitions in $\succ_h$ that are worse than A.

PROPOSITION **6** *If a social choice function is strategy-proof and satisfies unanimity, then it is tops-only.*

Proof: We are going to use a result due to Chatterji and Sen (2011). They show that the following two conditions are sufficient for tops-onlyness if a social choice function is strategy-proof and satisfies unanimity.

- A domain $\mathcal{Z} \subseteq \mathcal{P}$ is **minimally rich** if for every partition $A \in \mathbb{M}$, there exists $\succ_h \in \mathcal{Z}$ such that $\succ_h (1) = A$.

- Let $A \in \mathbb{M}$ and $\succ_h \in \mathcal{P}$ such that $A \neq \succ_h (1) = B$. We say A is **satisfactory** in domain $\mathcal{Z} \subseteq \mathcal{P}$ for $\succ_h \in \mathcal{Z}$ if for all $C \in \{B\} \cup \beta(A, B)$ there exists a $\succ'_h \in \mathcal{Z}$ such that $\succ'_h (1) = A$ and $C \succ'_h D$ for all $D \in \omega(A, \succ_h)$. In other words, A is satisfactory for $\succ_h$ if for every C in $\{B\} \cup \beta(A, B)$, there is a preference ordering where A is the top and C is better than all the alternatives worse than A in $\succ_h$.

  A domain $\mathcal{Z} \subseteq \mathcal{P}$ satisfies **Property** T^* if for all $\succ_h \in \mathcal{Z}$ and $A \in \mathbb{M} \setminus \{\succ_h (1)\}$, A is satisfactory for $\succ_h$.

The intermediate domain $\mathcal{D}$ is clearly minimally rich. We show that our intermediate domain $\mathcal{D}$ satisfies Property T^*. Pick an arbitrary $\succ_h \in \mathcal{D}$, and let $\succ_h (1) = B$. Pick $A \in \mathbb{M} \setminus \{B\}$. We need to show that A is satisfactory for $\succ_h$. Pick a $C \in \{B\} \cup \beta(A, B)$. By the squeezing lemma (Lemma 3), and using the fact $\beta(A, B) = \beta(B, A)$ (Lemma 2), there is a preference ordering $\succ'_h$ such that $\succ'_h (1) = A$ and $\alpha(B, \succ'_h) = \{A\} \cup \beta(B, A)$. Clearly, for all $D \in \omega(A, \succ_h)$, we have $B \succ'_h D$. Since $C \in \{B\} \cup \beta(B, A)$, we get that $C \succ'_h D$. Hence, the intermediate domain $\mathcal{D}$ satisfies Property T^*.

Finally, Chatterji and Sen (2011) show that if F is strategy-proof and $\mathcal{D}$ is minimally rich and satisfies Property T^*, then F is tops-only [6]. Hence, F is tops-only. ■

We can now state the proof of Theorem 3.

PROOF OF THEOREM 3

Proof: (1) $\Rightarrow$ (2), (3): Clearly, any meet social choice function is strategy-proof (Theorem 1), and satisfies unanimity, Pareto$^+$, and Pareto$^-$.

(2) $\Rightarrow$ (3): Let F be a strategy-proof social choice function satisfying unanimity. By Proposition 6, F is tops-only. Hence, the aggregator v^F is well-defined. Further, v^F is strategy-proof, and satisfies unanimity. By Proposition 2, v^F satisfies binary independence. Unanimity and binary independence implies that v^F must also satisfy Pareto$^+$ and Pareto$^-$.

[6] Chatterji and Sen (2011) consider a general model, a la Gibbard (1973) and Satterthwaite (1975), and provide sufficient conditions on domains for a strategy-proof social choice functions to be tops-only.

(3) $\Rightarrow$ (1): Let F be a strategy-proof social choice function satisfying Pareto$^+$ and Pareto$^-$. Then, it must satisfy unanimity. By Proposition 6, F is tops-only. Hence, the aggregator v^F is well-defined. Further, v^F is strategy-proof, and satisfies Pareto$^+$ and Pareto$^-$. By Theorem 1, v^F must be a meet aggregator, and F must be a meet social choice function. ■

There are other ways to prove Theorem 3 once we establish that tops-only property holds. For instance, Mirkin (1975) has shown that for $|M| \geq 3$, an aggregator satisfying unanimity and binary independence must be a meet aggregator. This establishes the equivalence between (1) and (2). Similarly, Fishburn and Rubinstein (1986) have shown that for $|M| \geq 3$, an aggregator satisfying Pareto$^+$, Pareto$^-$, and binary independence must be a meet aggregator. Since Pareto$^+$ and Pareto$^-$ imply unanimity, this establishes the equivalence between (1) and (3).

The analogue of Theorem 3 for $|M| = 2$ can be derived too using Theorem 2. Note that when $|M| = 2$, every social choice function is tops-only. Using unanimity and Theorem 2, we conclude that every strategy-proof social choice function satisfying unanimity must be a *meet-join* social choice function, where a meet-join SCF is a meet*-join SCF where the empty set is not part of the set of oligarchies.

7 Pareto Efficiency and Dictatorship

It is well known that in the presence of strategy-proofness, unanimity is equivalent to a social choice function being onto, which in turn is equivalent to the following definition Pareto efficiency in many domains, including the unrestricted domain in Gibbard (1973) and Satterthwaite (1975) and the single-peaked domain in Moulin (1980).

Definition **16** *A social choice function F is* **Pareto efficient** *if for every profile $\succ \in \mathcal{D}^n$, there exists no partition $A \in \mathbb{M}$ such that $A \succ_h F(\succ)$ for all $h \in N$.*

A meet SCF need not be Pareto efficient as the following example illustrates.

Example **3** *Let $M = \{a, b, c\}$ and $N = \{1, 2\}$. Consider the meet SCF where the oligarchy is $\{1, 2\}$. Consider a profile $(\succ_1, \succ_2)$ such that*

- *$\succ_1 (1)$ is the partition with bundles $\{a, b\}$ and $\{c\}$,*
- *and $\succ_2 (1)$ is the partition with bundles $\{a, c\}$ and $\{b\}$.*

By definition, $F(\succ)$ is the empty partition. Let A be the complete partition. Note that A and $\succ_1 (1)$ agree that a and b should be together but A and $F(\succ)$ do not agree on that. Similarly, A and $\succ_2 (1)$ agree that a and c should be together but A and $F(\succ)$ do not agree on that. Hence, we can assume, without loss of generality, that $\succ_1$ is such that $A \succ_1 F(\succ)$ and $\succ_2$ is such that $A \succ_2 F(\succ)$. So, F is not Pareto efficient.

The intuition of Example 3 carries over more generally.

Theorem 4 *Suppose $|M| \geq 3$. A social choice function is strategy-proof and Pareto efficient if and only if it is a dictatorship.*

Proof: Clearly, a dictatorship is Pareto efficient and strategy-proof. Suppose F is a strategy-proof and Pareto efficient SCF. Since Pareto efficiency implies unanimity, by Theorem 3, F is a meet social choice function, and v^F is a meet aggregator. Let $S \subseteq N$ be the oligarchy of v^F. We complete the proof by showing $|S| = 1$. Suppose $|S| > 1$. Fix three objects $i, j, k \in M$. Consider the following partitions for each agent. For some $h \in S$, let the top-ranked partition of agent h be A^h, and $A^h_{ik} = 1$ and $A^h_{st} = 0$ for all $\{s, t\} \neq \{i, k\}$. For every $h' \in S \setminus \{h\}$, let the top-ranked partition of agent h' be $A^{h'}$, and $A^{h'}_{ij} = 1$ and $A^{h'}_{st} = 0$ for all $\{s, t\} \neq \{i, j\}$. If $N \setminus S$ is non-empty, then for every $h'' \in N \setminus S$, let the top-ranked partition of agent h'' be $A^{h''} = A$, where A is the complete partition.

Now, by definition $v^F(A^1, \ldots, A^n) = B$, where B is the empty partition. Now, note that $A^h_{ik} = A_{ik} = 1$ but $A^h_{ik} \neq B_{ik}$. Also, $A^{h'}_{ij} = A_{ij}$ but $A^{h'}_{ij} \neq B_{ij}$ for all $h' \in S \setminus \{h\}$. Hence, there exists a preference ordering $\succ_h \in \mathcal{D}$ such that $\succ_h (1) = A^h$ and $A \succ_h B$. Also, for every $h' \in S \setminus \{h\}$, there exists a preference ordering $\succ'_{h'} \in \mathcal{D}$ such that $\succ'_{h'} (1) = A^{h'}$ and $A \succ_{h'} B$. By definition, for every $h'' \in N \setminus S$, $A \succ_{h''} B$ for any preference ordering $\succ_{h''} \in \mathcal{D}$. This implies that F is not Pareto efficient, which is a contradiction. Hence, $|S| = 1$, and F is a dictatorship. ■

Theorem 4 shows that we are back to Gibbard-Satterthwaite type impossibility if we impose Pareto efficiency in addition to strategy-proofness. This is in sharp contrast to various possibilities we have seen in the presence of unanimity or tops-onlyness. We also note here that similar impossibility using Pareto efficiency and strategy-proofness has been derived in the separable domain studied in Barbera et al. (1991). As we have shown, ours is a smaller domain than the separable domain, and hence, our result cannot be deduced from Barbera et al. (1991).

8 Conclusion

This paper adds a new restricted domain to the literature on strategic social choice theory initiated by the papers of Gibbard (1973) and Satterthwaite (1975). We provide characterizations of strategy-proof social choice functions (a) under tops-onlyness property, (b) under unanimity, and (c) under Pareto efficiency. Our model and the restricted domain we consider are natural, and has some plausible applications (as discussed in Section 1).

It will be interesting to consider even more restriction of preferences in our model. For instance, one natural notion of comparing two partitions with respect to a reference partition is using "distance" between them. If A is the top-ranked partition in a preference ordering, then a partition B may be preferred over partition C if and only if the distance between A and B is less than the distance between A and C. One can verify that this is a smaller domain than our intermediate domain. It will be interesting to find a characterization of strategy-proof social choice functions in this domain.

Rubinstein and Fishburn (1986) consider an abstract model of algebraic aggregation. The current paper, specially Theorem 3, gives a strategic foundation to their result on partitions (Fishburn and Rubinstein, 1986). It will be interesting to extend this result to the abstract model of algebraic aggregation.

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Appendix

We give a proof of the fact that every aggregator which satisfies binary independence and Pareto* must be a meet * aggregator. The proof is based on Fishburn and Rubinstein (1986) - Dimitrov et al. (2011) show this using an indirect proof. To remind, $\mathbb{M}$ is the set of all partitions and an aggregator is a mapping $v : \mathbb{M}^n \to \mathbb{M}$. We denote a profile of partitions as $\mathbf{A} \equiv (A^1, \ldots, A^n)$ and the n dimensional $0-1$ vector $(A^1_{ij}, \ldots, A^n_{ij})$ as $\mathbf{A}_{ij}$.

Proposition **7** *Suppose $|M| \geq 3$. An aggregator satisfies binary independence and Pareto$^+$ if and only if it is a meet* aggregator.*

Proof: It is easy to see that a meet* aggregator satisfies Pareto$^+$ and binary independence. Suppose v is an aggregator which satisfies Pareto$^+$ and binary independence. We start the proof with two claims.

Claim **2** *Consider $i, j, k, l \in M$ such that $i \neq j$ and $k \neq l$ and two profiles of partitions $\boldsymbol{A}$ and $\boldsymbol{B}$ such that $\boldsymbol{A}_{ij} = \boldsymbol{B}_{kl}$. If v satisfies Pareto$^+$ and binary independence, then $v(\boldsymbol{A})_{ij} = v(\boldsymbol{B})_{kl}$.*

Proof: If $\{i, j\} = \{k, l\}$, then the claim follows from binary independence. First, consider distinct $i, j, k \in M$ and two partitions $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A}_{ij} = \mathbf{B}_{jk}$. Consider another profile of partition $\mathbf{C}$ such that $\mathbf{C}_{ij} = \mathbf{A}_{ij} = \mathbf{B}_{jk}$, $\mathbf{C}_{jk} = \mathbf{B}_{jk} = \mathbf{A}_{ij}$, and $\mathbf{C}_{ik} = (1, \ldots, 1)$. By Pareto$^+$, $v(\mathbf{C})_{ik} = 1$ Transitivity implies that $v(\mathbf{C})_{ij} = v(\mathbf{C})_{jk}$. But binary independence implies that $v(\mathbf{C})_{ij} = v(\mathbf{A})_{ij}$ and $v(\mathbf{C})_{jk} = v(\mathbf{B})_{jk}$. This means that $v(\mathbf{A})_{ij} = v(\mathbf{B})_{jk}$.

Now consider distinct i, j, k, l and $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A}_{ij} = \mathbf{B}_{kl}$. Consider a profile $\mathbf{D}$ such that $\mathbf{D}_{ij} = \mathbf{D}_{jk} = \mathbf{D}_{kl} = \mathbf{A}_{ij} = \mathbf{B}_{kl}$. Repeating the above argument, we get that $v(\mathbf{D})_{ij} = v(\mathbf{D})_{kl}$. Using binary independence, $v(\mathbf{A})_{ij} = v(\mathbf{B})_{kl}$. ■

Denote by $\mathbf{A}_{ij} = \mathbf{1}_K$ if $A^h_{ij} = 1$ if and only if $h \in K$.

Claim **3** *Consider three profiles of partitions $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, and $i, j \in M$ such that $\boldsymbol{A}_{ij} = \boldsymbol{1}_K$ for some $K \subseteq N$ and $\boldsymbol{B}_{ij} = \boldsymbol{1}_L$ for some $L \subseteq N$ and $\boldsymbol{C}_{ij} = \boldsymbol{1}_{K \cap L}$. Suppose $v(\boldsymbol{A})_{ij} = v(\boldsymbol{B})_{ij} = 1$. If v satisfies Pareto$^+$ and binary independence, then $v(\boldsymbol{C})_{ij} = 1$.*

Proof: If $K = \emptyset$ or $L = \emptyset$, then the claim follows from binary independence. Assume $K \neq \emptyset$ and $L \neq \emptyset$. Consider a profile $\mathbf{D}$ and three distinct $i, j, k \in M$ such that $\mathbf{D}_{ij} = \mathbf{1}_K$, $\mathbf{D}_{jk} = \mathbf{1}_L$, and $\mathbf{D}_{ik} = \mathbf{1}_{K \cap L}$. By binary independence, we get $v(\mathbf{D})_{ij} = v(\mathbf{D})_{jk} = 1$. By transitivity, $v(\mathbf{D})_{ik} = 1$. By Claim 2, $v(\mathbf{C})_{ij} = 1$. ■

Define the minimal decisive set of agents $S(i,j) \subseteq N$ for the pair $\{i,j\}$ as follows: there exists a profile $\mathbf{A}$ such that $\mathbf{A}_{ij} = \mathbf{1}_{S(i,j)}$ and $v(\mathbf{A})_{ij} = 1$ and there exists no profile $\mathbf{B}$ such that $\mathbf{B}_{ij} = \mathbf{1}_T$ fpr some $T \subsetneq S(i,j)$ and $v(\mathbf{B})_{ij} = 1$. Since v satisfies Pareto$^+$, $S(i,j)$ exists for every pair $\{i,j\}$. Now, consider the following claim.

Claim **4** *Suppose v satisfies Pareto$^+$ and binary independence. For every profile $\boldsymbol{A}$ such that $\boldsymbol{A}_{ij} = \boldsymbol{1}_T$ for some $T \supseteq S(i,j)$, we have $v(\boldsymbol{A})_{ij} = 1$.*

Proof: Like in the proof of Claim 3, construct a profile $\mathbf{D}$ using three distinct i,j,k such that $\mathbf{D}_{ij} = \mathbf{1}_{S(i,j)}$, $\mathbf{D}_{jk} = \mathbf{1}_T$, and $\mathbf{D}_{ik} = \mathbf{1}_{S(i,j)}$. By binary independence, $v(\mathbf{D})_{ij} = 1$. By Claim 2, $v(\mathbf{D})_{ik} = 1$. By transitivity, $v(\mathbf{D})_{jk} = 1$. By Claim 2, $v(\mathbf{A})_{ij} = 1$. ■

By Claim 3, for every pair $\{i,j\}$, the minimal decisive set $S(i,j)$ is unique. Further, by Claim 2, for every $\{i,j\} \neq \{k,l\}$, $S(i,j) = S(k,l)$. Hence, we call it the minimal decisive set, and denote it as S. This along with Claim 4 shows that v is a meet* aggregator. ■