Selling to a naive (agent, manager) pair *

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Abstract

A seller is selling a good to an (agent, manager) pair. The agent is budget constrained but the manager is not. Both value the good differently and want to jointly acquire it, but they take decisions in a lexicographic manner. In particular, for any pair of outcomes, the agent first compares using her valuation. If she cannot compare them (due to budget constraint), then the manager compares. We are interested in the optimal (expected revenue maximizing) mechanism under incentive and individual rationality constraints. We show that the optimal mechanism is either a posted price mechanism or a mechanism involving a pair of posted prices (a menu of three outcomes). In the latter case, the optimal mechanism involves randomization and *pools* types in the middle.

JEL Codes: D82, D40, D90

Keywords: behavioral mechanism design, optimal mechanism, posted-price mechanism, lexicographic choice, budget constraint.

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1 Introduction

An (agent, manager) pair needs to buy a good. The agent (she) is budget constrained, but the manager (he) is not budget constrained. A seller offers a menu of (quantity, price) bundles to them in a mechanism. If the agent's best bundle is within her budget, she buys it. Else, she contacts the manager. The manager is not budget constrained and can give any amount of funding as long as she respects his preference. Implicitly, the manager's payoff is linked to the agent's payoff in a monotone way and hence, the manager is willing to fund (without any side payments). This may be because both the manager and the agent need to acquire the good for the firm, and their payoff depends on the payoff of the firm. They have subjective valuation of the good for the firm. The valuations of the agent and the manager may be different because either there is inherent uncertainty about the valuation of the good and the agent and the manager may be differently informed about it or they use different attributes of the good to determine its valuation.

Our objective here is to capture a setting where an agent's behavior contradicts standard notions of rationality - ideally, the agent and the manager should get together and choose the best option according their joint estimate of the good's valuation. However, they are *naive*: (a) the agent only contacts the manager when she cannot choose the best bundle due to budget constraint; (b) whenever she contacts the manager, she respects his decision; and (c) the manager can impose his preference only when contacted by the agent. This makes the problem different from standard monopoly pricing problems. Sales to such an (agent, manager) pair who take decisions lexicographically, where the agent is budget constrained, is not uncommon: (child, parent) pair making decision to buy some product; (management, board) pair of a company making decisions to acquire another company; (department, dean) pair making decision to recruit a faculty candidate. A department (or, child or management) only contacts the dean (or, parent or board respectively) when it cannot take a decision about a new faculty candidate due to budget constraint. But once it contacts the dean, it has to respect the dean's preference. [1] We are interested in finding the optimal mechanism for

[1] The dean and the department cannot jointly evaluate a faculty candidate because the dean is time

selling to such an (agent, manager) pair.

The private information or *type* in our model is a pair of valuations: agent's own valuation and manager's valuation. There is no information transmission story here - even though the agent does not know the valuation of the manager, she can readily access the preference of the manager, but does so only when she cannot make a decision due to budget constraint. Hence, her decisions depend on her valuation *and* the manager's valuation. The incentive constraints in our model are quite different from a standard model of mechanism design. This is because the sequential nature of decision-making generates cyclic preference of the (agent, manager) pair. Hence, no utility representation is possible for such preferences, and the incentive constraints are *ordinal* in nature. In particular, if a mechanism assigns bundle (q, p) to a type, where q is quantity and p is price, then a manipulation to get another (quantity, price) pair (q', p') is possible if (a) the agent finds (q', p') more attractive than (q, p) and p' is less than the budget or (b) she cannot compare these two pairs (because the preferred pair is beyond budget) but the manager finds (q', p') more attractive than (q, p). An incentive compatible mechanism guards against all such manipulations.

Contributions. We fully characterize the optimal (expected revenue maximizing incentive compatible and individually rational) mechanism for the seller in our model. The optimal mechanism is either a posted-price mechanism (the no-haggling solution of Mussa and Rosen (1978); Riley and Zeckhauser (1983)) or a mechanism involving two posted-prices - we call it the POST-2 mechanism. The POST-2 mechanism has a pair of posted prices P_1 and P_2, both greater than the budget B. If the agent's valuation of the good is less than P_1, then the object is not sold (and no payments are made). If the agent's valuation of the good is more than P_1, then the object is sold with probability $\frac{B}{P_1}$ at per unit price P_1 (i.e., total payment is B). The remaining probability $(1 - \frac{B}{P_1})$ is sold at per unit price P_2 if the valuation of both the agent and the manager exceeds P_2. Hence, a POST-2 mechanism

constrained, and may be involved with a number of other such responsibilities. Similarly, the company board has delegated responsibility to the management with a budget constraint. Burkett (2015) shows that such arrangements can come out of an equilibrium contracting agreement between a *(principal, agent) pair* participating in a mechanism.

involves an extra layer of *pooling* of types in the middle and involves randomization. [2]

We provide a simple condition on the budget when a POST-2 mechanism is optimal. There are three special cases, where our problem reduces to a standard revenue maximization problem of a monopolist: (1) when the budget of the agent is sufficiently high (then the agent can make all the decisions); (2) when the budget of the agent is zero (then the manager makes all the decisions); and (3) when the preferences of the agent and the manager are identical. In all these cases, a posted-price mechanism is optimal (Mussa and Rosen, 1978; Riley and Zeckhauser, 1983) - call the optimal posted-price in such settings a *monopoly reserve price*. We show that if the budget of the agent is below the monopoly reserve price, a POST-2 mechanism is optimal.

Our optimal mechanism is simple since it can be described by a single parameter or a pair of parameters, and involves a menu of size two or three. Further, our result works for a rich class of priors (over values of the (agent, manager) pair), which allows for correlation. The nature of incentive constraints in our problem implies that there is no revenue equivalence theorem to work with. Compared to a standard multi-object monopolist, where one runs into difficulty even in the two-object case (Manelli and Vincent, 2007; Hart and Nisan, 2017), we still have tractability in our multidimensional model because of the nature of decision-making and the incentive constraints.

2 An illustration

We explain using a simple example why a posted price mechanism need not be optimal in our model. For simplicity, consider a setting where valuations of the agent and the manager,

[2] Randomization is often seen in practice: same product is sold with different quality levels; limited shares of a company are possible to acquire instead of complete acquisition; a faculty candidate considers different levels of teaching in the contract when being hired etc. However, our optimal mechanism design recommends a particular kind of randomization. We do not know if such particular randomization is seen in practical problems. Our results suggest that whenever a designer believes he is confronted with an (agent, manager) pair described in our model, it is optimal to offer such randomization in the menu.

$v \equiv (v_1, v_2)$, are distributed in $[0,1] \times [0,1]$. We assume that both the agent and the manager have quasilinear preferences. So, the agent evaluates options using v_1 and the manager evaluates options using v_2. Consider a budget $B > 0$. Suppose the seller uses a posted price mechanism with price $p > B$. We argue that such a posted price mechanism cannot be optimal. To see this, consider the menu in a posted price mechanism: $\{(1,p),(0,0)\}$, i.e., take the object with probability 1 at price p or get nothing at zero price. If $v \equiv (v_1, v_2)$ is such that $v_1 \leq p$ the agent will prefer $(0,0)$ to $(1,p)$ and she will take this decision without consulting the manager. If $v \equiv (v_1, v_2)$ is such that $v_2 \leq p$ and $v_1 \geq p$, then the agent prefers $(1,p)$ to $(0,0)$ but she cannot take this decision since $p > B$. Hence, she consults the manager who prefers $(0,0)$ to $(1,p)$. Hence, $(0,0)$ will be preferred over $(1,p)$ at such profiles. So, the only region where $(1,p)$ is preferred to $(0,0)$ is when $\min(v_1, v_2) \geq p$ - this is when both the agent and the manager prefers $(1,p)$ to $(0,0)$. This is shown in the left graph of Figure 1.

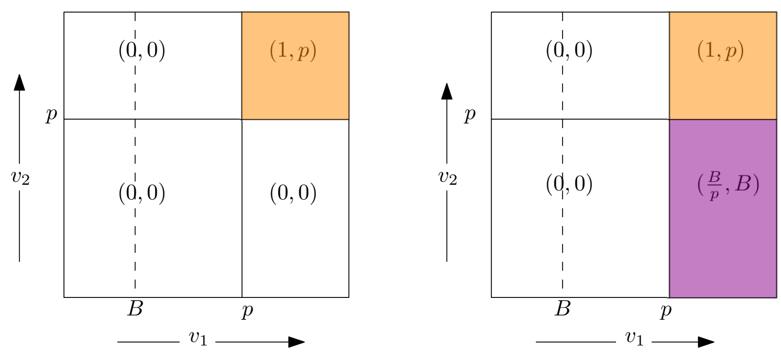

Figure 1: Non-optimality of posted prices

Now, consider another mechanism with a menu of three outcomes: $\{(1,p),(\frac{B}{p}, B),(0,0)\}$. So, the new menu contains an outcome that involves randomization and a payment of B. Consider the profile of values $v \equiv (v_1, v_2)$. Using the same argument as before, we see that if $\min(v_1, v_2) \geq p$, then the (agent,manager) pair prefers $(1,p)$ to the other two outcomes in the menu. Similarly, if $v_1 \leq p$, then the $(0,0)$ is preferred to the other two outcomes in the menu. However, if $v_1 \geq p$ but $v_2 \leq p$, then $v_1 - p \geq \frac{B}{p}(v_1 - p)$. But $p > B$ implies that the agent cannot compare $(1,p)$ and $(\frac{B}{p}, B)$ - i.e., the preferred outcome $(1,p)$ is beyond

beyond the budget. However, since $v_2 \leq p$, we see that $\frac{B}{p}(v_2 - p) \geq v_2 - p$. So, the manager prefers $(\frac{B}{p}, B)$ to $(1, p)$. The agent prefers $(\frac{B}{p}, B)$ to $(0, 0)$ because $\frac{B}{p}(v_1 - p) \geq 0$ and she can compare these outcomes (within budget). Hence, the $(\frac{B}{p}, B)$ is preferred to the other outcomes in the menu by the (agent, manager) pair when $v_1 \geq p$ but $v_2 \leq p$. This is shown the right graph of Figure 1. This graph has an extra positive measure region where revenue of B can be earned by the seller at every profile in this region. Hence, this mechanism generates strictly larger revenue than the posted price mechanism. As is apparent, the seller is able to exploit the lexicographic nature of decision-making of the (agent, manager) pair to extract more revenue than in a posted price mechanism. Our main result will show that it cannot exploit any more than this, i.e., such a mechanism will be optimal.

The above discussion shows that a posted price mechanism which posts a price above the budget cannot be optimal. Our main result will formalize this intuition - for low enough budgets, we will show that the optimal mechanism will involve randomization but we can be precise about the nature of the randomization. The optimal mechanism will be a posted price mechanism for "high enough" budgets. But for budgets below a certain threshold, it will be a mechanism involving an extra layer of pooling in the middle.

3 The model

A seller is selling a single object to an agent who evaluates options along with her manager. She has a publicly observable budget $B \in (0, \beta)$, where $\beta > 0$. A consumption bundle is a pair (a, t), where $a \in [0, 1]$ is the allocation probability and $t \in \mathbb{R}$ is the transfer - amount *paid* by the agent. The set of all consumption bundles is denoted by $Z \equiv [0, 1] \times \mathbb{R}$. The agent and the manager evaluate the outcomes in Z using **quasilinearity**. Hence, their individual preference can be captured by valuations: a generic valuation of the agent is denoted as v_1 and a generic valuation of the manager is denoted by v_2. We assume that $v_1, v_2 \in V \equiv [0, \beta]$ - all our results extend even if we allow for the fact $v_i \in [0, \beta_i]$ for each $i \in \{1, 2\}$ and $\beta_1 \neq \beta_2$. Since the budget is publicly observable, the only private information in the model are the two valuations (v_1, v_2).

Preference (rationale) of the agent with valuation v_1 is denoted by $\succeq_{v_1}$. Formally, $\succeq_{v_1}$ is a binary relation (incomplete): $\forall\ (a,t),(a',t') \in Z$,

$$\big[(a,t) \succeq_{v_1} (a',t')\big] \quad \Leftrightarrow \quad \big[av_1 - t \geq a'v_1 - t' \text{ and } t \leq B\big].$$

Notice that t' need not be below B in the above definition. This is consistent with our story that the agent makes a decision whenever she can.

Preference of the manager with valuation v_2 is denoted by $\succeq_{v_2}$. Formally, $\forall\ (a,t),(a',t') \in Z$,

$$\big[(a,t) \succeq_{v_2} (a',t')\big] \quad \Leftrightarrow \quad \big[av_2 - t \geq a'v_2 - t'\big].$$

Hence, $\succeq_{v_2}$ is complete. Notice that both $\succeq_{v_1}$ and $\succeq_{v_2}$ are transitive.

We denote the **aggregate preference** of the (agent, manager) pair with type $v \equiv (v_1, v_2)$ as $\succeq_v$. The preference $\succeq_v$ is a complete binary relation derived from $\succeq_{v_1}$ and $\succeq_{v_2}$ as follows. For every $(a,t),(a',t') \in Z$,

$$\big[(a,t) \succeq_v (a',t')\big] \Leftrightarrow$$
$$\text{either } \big[(a,t) \succeq_{v_1} (a',t')\big] \text{ or } \big[(a,t) \not\succeq_{v_1} (a',t'), (a',t') \not\succeq_{v_1} (a,t), (a,t) \succeq_{v_2} (a',t')\big].$$

As is expected, $\succeq_v$ is intransitive for almost all $v \equiv (v_1, v_2)$ - for instance, it can be verified that for $v \equiv (v_1 = \frac{1}{2}, v_2 = 1)$, the aggregate preference over three outcomes $(\frac{1}{2}, B), (1, B + \frac{5}{16}), (\frac{11}{16}, B + \frac{1}{16})$ cycle. An important consequence of this observation is that there is *no utility representation* of the preference of our (agent, manager) pair. As discussed earlier, the aggregate preference captures the decision making process of the (agent, manager) pair. For every pair of outcomes, first the agent tries to compare. The manager compares only if the agent fails to compare due to budget constraint. We interpret this decision-making process further after defining the incentive constraints.

We assume that the random variable $v \equiv (v_1, v_2)$ over $V \times V$ follows a distribution G with G_1 being the marginal for agent's valuation and G_2 being the marginal for manager's valuation. Both G_1 and G_2 are assumed to be differentiable functions with positive densities g_1 and g_2 respectively. Notice that we allow for values of the agent and the manager to be correlated (but not perfectly correlated since densities g_1 and g_2 are positive). Our results will require some restrictions in G_1, which we will state later.

4 The optimal mechanism

4.1 Incentive compatibility

Since the preference of the (agent, manager) pair is completely captured by $v \equiv (v_1, v_2)$, we will refer to v as the **type** in our model. A (direct) **mechanism** is a pair of maps: an allocation rule $f : V^2 \to [0,1]$ and a payment rule $p : V^2 \to \mathbb{R}$. For every $v \in V^2$, $f(v)$ denotes the allocation probability and $p(v)$ denotes the payment of this type.

The restriction to such direct mechanisms is without loss of generality as a version of the revelation principle holds in our setting - see Section 5. [3] Hence, we can discuss about incentive compatibility of direct mechanisms.

Definition 1 *A mechanism* (f, p) *is* **incentive compatible (IC)** *if for all* $u, v \in V^2$,

$$(f(u), p(u)) \succeq_u (f(v), p(v)).$$

Fix a mechanism (f, p) and let the range of the mechanism be

$$R^{f,p} := \{(a,t) : (f(v), p(v)) = (a,t) \text{ for some } v \in V^2\}.$$

Consider a type $u \equiv (u_1, u_2)$. The designer has assigned the bundle $(f(u), p(u))$ to this type. For every $(a,t) \in R^{f,p}$, there are two possibilities of manipulation. First, the agent can manipulate - this is possible if $au_1 - t > f(u)u_1 - p(u)$ with $t \leq B$. Second, the manager can manipulate and this is possible if the agent could not take a decision, contacted the manager, and $au_2 - t > f(u)u_2 - p(u)$. Our notion of IC thus guards against two kinds of manipulations: one where the agent can take her own decision and manipulates, and the other where the agent cannot decide due to budget constraint and the manager manipulates.

In general, preferences over outcomes in $R^{f,p}$ may violate transitivity. However, our notion of IC requires that at every type u, the outcome $(f(u), p(u))$ is preferred to any other

[3] Though direct reporting of valuations of the agent and the manager may seem unrealistic in this setting, we can think of the direct mechanism as announcing a menu of outcomes and the agent choosing the best outcome from this menu (with the help of her manager).

outcome in $R^{f,p}$. This implies that if the designer wants type u to choose $(f(u), p(u))$ from the menu $R^{f,p}$, then it must be the case that for any other outcome (a, t) in $R^{f,p}$, the agent does not prefer (a, t) to $(f(u), p(u))$ or the agent cannot compare (a, t) and $(f(u), p(u))$, but the manager does not prefer (a, t) to $(f(u), p(u))$. Our notion of IC implies that the outcome chosen for every type is not involved in a cycle. This allows us to rule out Dutch book arguments (or money pump) using our notion of incentive compatibility.

Thus, our notion of IC can be broken down into two distinct cases. Fix $u, v \in V^2$. Then, there are two ways in which bundle $(f(u), p(u))$ can be (weakly) preferred over $(f(v), p(v))$ by a type u.

1. First, the agent prefers $(f(u), p(u))$ over $(f(v), p(v))$. This is possible if $p(u) \leq B$ and
$$u_1 f(u) - p(u) \geq u_1 f(v) - p(v).$$
2. Second, the agent cannot compare $(f(u), p(u))$ and $(f(v), p(v))$, but the manager prefers $(f(u), p(u))$ over $(f(v), p(v))$. This means $u_2 f(u) - p(u) \geq u_2 f(v) - p(v)$. Further, since the agent cannot compare these two outcomes, one of the following conditions must hold.
   (a) $u_1 f(u) - p(u) > u_1 f(v) - p(v)$ but $p(u) > B$.
   (b) $u_1 f(v) - p(v) > u_1 f(u) - p(u)$ but $p(v) > B$.
   (c) $u_1 f(v) - p(v) = u_1 f(u) - p(u)$ but $\min(p(u), p(v)) > B$.

Besides, IC, we will impose a natural participation constraint. For this, we will assume that outside option of the (agent, manager) pair is the outcome $(0, 0)$, where she receives nothing and pays nothing.

Definition **2** *A mechanism* (f, p) *is* **individually rational (IR)** *if for all* $v \in V^2$,
$$(f(v), p(v)) \ \succeq_v \ (0, 0).$$

It is useful to note that the above IR condition can be equivalently stated as follows. A mechanism (f, p) is IR if for all $v \in V^2$ (a) when $p(v) \leq B$, we have $v_1 f(v) - p(v) \geq 0$ and

(b) when $p(v) > B$, we have $v_1 f(v) - p(v) \geq 0$ and $v_2 f(v) - p(v) \geq 0$. This leads us to the following characterization of IR. Such characterizations are well known in standard settings and the result below shows that it extends to our model too.

Lemma **1** *Consider any IC mechanism (f, p). Then, (f, p) is IR if and only if $p(0, 0) \leq 0$.*

Proof: Suppose that $p(0, 0) \leq 0$. Consider any $u \in V^2$ with $p(u) \leq B$. IC implies that $(f(u), p(u)) \succeq_u (f(0, 0), p(0, 0))$. But $p(u) \leq B$ and $p(0, 0) \leq 0 < B$ implies that $u_1 f(u) - p(u) \geq u_1 f(0, 0) - p(0, 0)$. This combined with the fact that $u_1 f(0, 0) - p(0, 0) \geq 0$ (since $-p(0, 0), f(0, 0) \geq 0$), we conclude $(f(u), p(u)) \succeq_u (0, 0)$.

Similarly, consider any $v = (v_1, v_2) \in V^2$ with $p(v) > B$. IC and the fact that $p(0, 0) \leq 0 < B$, $p(v) > B$ imply that the agent cannot compare $(f(v), p(v))$ and $(f(0, 0), p(0, 0))$ but the manager prefers $(f(v), p(v))$ to $(f(0, 0), p(0, 0))$. This implies that $v_1 f(v) - p(v) \geq v_1 f(0, 0) - p(0, 0)$ and $v_2 f(v) - p(v) \geq v_2 f(0, 0) - p(0, 0)$. These inequalities imply that $v_1 f(v) - p(v) \geq 0$ and $v_2 f(v) - p(v) \geq 0$ as $-p(0, 0), f(0, 0) \geq 0$. From this we conclude $(f(v), p(v)) \succeq_v (0, 0)$.

For the other direction, consider the type $(0, 0) \in V$. IR implies that $(f(0, 0), p(0, 0)) \succeq_{(0,0)} (0, 0)$. This implies that $-p(0, 0) \geq 0$. ■

4.2 New mechanisms

Incentive compatibility has different implications in our model because of the sequential nature of decision-making. There are some simple mechanisms that are IC and resemble similar mechanisms in standard settings where decisions are taken using a single preference relation.

Definition **3** *A mechanism (f, p) is a* post-1 *mechanism if there exists a $K_1 \in [0, B]$ such that*

$$(f(v), p(v)) = \begin{cases} (0, 0) & \text{if } v_1 \leq K_1 \\ (1, K_1) & \text{otherwise.} \end{cases}$$

A POST-1 mechanism is a mechanism where the object is allocated by only considering the value of the agent. So, it can be thought of as a posted price mechanism *for* the agent. This is because it posts a price K_1 which is less than the budget B, and hence, the agent can make a decision using her preference. So, if her value is less than K_1, then the object is not allocated. Else, the object is allocated with probability 1. It is easy to see that such a mechanism is IC and IR.

We now introduce a new class of mechanisms that we call the POST-2 mechanisms. Unlike the POST-1 mechanism, the POST-2 mechanism considers the values of both the agent and the manager.

DEFINITION 4 *A mechanism (f,p) is a* POST-2 *mechanism if there exists a $K_1, K_2 \in [B, \beta]$ with $K_1 \leq K_2$, such that*

$$(f(v), p(v)) = \begin{cases} (0,0) & \text{if } v_1 \leq K_1 \\ (1, B + K_2(1 - \frac{B}{K_1})) & \text{if } \min(v_1, v_2) > K_2 \\ (\frac{B}{K_1}, B) & \text{otherwise} \end{cases}$$

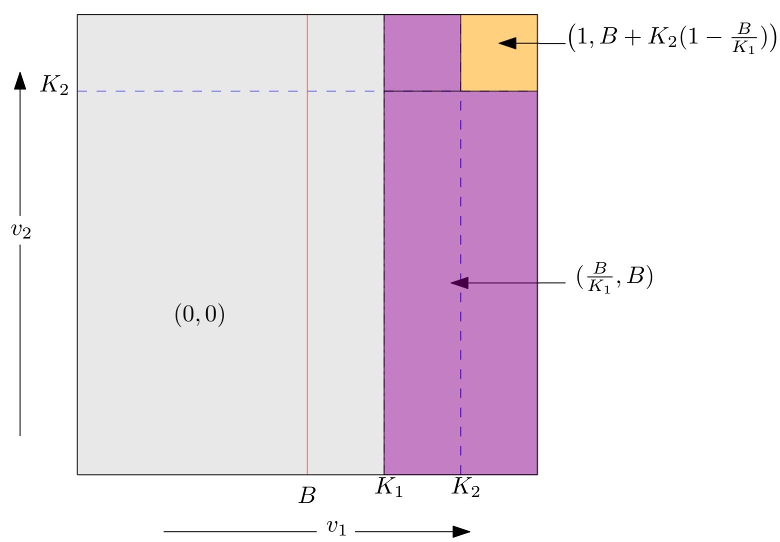

Figure 2: POST-2 mechanism

The POST-2 mechanism has a pair of posted prices. The first posted price K_1 is for the agent. If the value of the agent is below K_1, then the object is not sold. Else, the the object is sold with probability $\frac{B}{K_1}$ at per unit price of K_1, i.e., the total price paid equals K_1 times the probability of winning, which is $K_1 \times \frac{B}{K_1} = B$. The remaining probability $(1 - \frac{B}{K_1})$ is sold at per unit price K_2 if the values of both the agent and the manager exceed K_2. Figure 2 gives a graphical illustration of a POST-2 mechanism. We show below that a POST-2 mechanism is IC and IR.

PROPOSITION **1** *Every* POST-2 *mechanism is IC and IR.*

Though, we provide a formal proof of this result (and all subsequent omitted proofs) in the Appendix, we explain how the notion of incentive compatibility and the lexicographic decision-making make the result possible. There are three outcomes in the "menu" (range) of a POST-2 mechanism. The outcomes $(0,0)$ and $(\frac{B}{K_1}, B)$ are outcomes which can be compared using preference of the agent. On the other hand, outcome $(1, B + K_2(1 - \frac{B}{K_1}))$ has payment more than B. So, if a type $v \equiv (v_1, v_2)$ is assigned this outcome, IC requires that $(1, B + K_2(1 - \frac{B}{K_1}))$ is preferred to $(0,0)$ and $(\frac{B}{K_1}, B)$ by *both* the agent and the manager. It is easy to verify that this is possible if $v_1, v_2 \geq K_2$ and $K_2 \geq K_1$. Similarly, the other incentive constraints can be shown to hold.

A POST-2 mechanism uses the naivety of the (agent, manager) pair by posting a pair of prices. There are other kinds of mechanisms that can be IC. Our main result below shows that the optimal mechanism can be either a POST-1 or a POST-2 mechanism.

4.3 Main results

The expected (ex-ante) revenue of a mechanism (f, p) is given by

$$\text{REV}(f,p) = \int_{V^2} p(v) dG(v)$$

We say that a mechanism (f, p) is **optimal** if (a) (f, p) is IC and IR, and (b) $\text{REV}(f, p) \geq \text{REV}(f', p')$ for any other IC and IR mechanism (f', p').

For the optimality of our mechanisms, we will need a condition on the marginal distribution of the agent. Define the function H_1 as follows:

$$H_1(x) = xG_1(x) \ \forall \ x \in [0, \beta],$$

where G_1 is the marginal distributin function of value of the agent.

THEOREM **1** *Suppose H_1 is a strictly convex function. Then, either a* POST-1 *or a* POST-2 *mechanism is an optimal mechanism.*

Our results are slightly stronger than what Theorem 1 suggests. We prove that among all mechanisms which has a positive measure of types where the payment is more than the budget, a POST-2 mechanism is optimal. In the remaining class of mechanisms, a POST-1 mechanism is optimal. The strict convexity assumption of H_1 is satisfied by a variety of distributions, including the uniform distribution. [4]

We can be more precise about the optimization programs that need to be solved to get the optimal mechanism in Theorem 1. In particular, we either need to solve a one-variable or a two-variable optimization program.

PROPOSITION **2** *Suppose H_1 is strictly convex. Then, the expected revenue from the optimal mechanism is* $\max(R_1, R_2)$*, where*

$$R_1 = \max_{K_1 \in [0,B]} K_1(1 - G_1(K_1))$$

$$R_2 = \max_{K_2 \in [B,\beta], K_1 \in [B,K_2]} B\Big[1 - G_1(K_1)\Big] + K_2\big(1 - \frac{B}{K_1}\big)\Big[1 - G_1(K_2) - G_2(K_2) + G(K_2, K_2)\Big].$$

The maximization expressions for R_1 and R_2 reflect the expected revenue from a POST-1 and POST-2 mechanism respectively.

If the budget B is high enough, then the POST-1 mechanism becomes optimal - intuitively, the agent makes more decisions and screening along her valuation becomes optimal. It is more interesting to see how much restriction on budget we need to get POST-2 mechanism to be optimal. Below, we derive such a sufficient condition on the budget.

Define the optimal monopoly reserve price as $\bar{K}$

$$\bar{K} := \arg \max_{r \in [0,\beta]} r(1 - G_1(r)).$$

[4] Such a distributional assumption has appeared in the context of mechanism design before (Che and Gale, 2000). The strict convexity of H_1 requires that the function $G_1(x) + xg_1(x)$ is strictly increasing. This is equivalent to requiring $g_1(x)\big(x - \frac{1-G_1(x)}{g_1(x)}\big)$ being strictly increasing. The standard regularity condition in mechanism design requires increasingness of the bracketed term only.

If H_1 is a strictly convex function, $\bar{K}$ is uniquely defined since $x - xG_1(x)$ is a strictly concave function. The interpretation of $\bar{K}$ is that if the agent was *not* budget-constrained, then the optimal mechanism would have involved a posted-price of $\bar{K}$. Our other main result shows that if the budget constraint is less than $\bar{K}$, then the optimal mechanism is a POST-2 mechanism.

PROPOSITION **3** *Suppose H_1 is strictly convex and $B \leq \bar{K}$. Then, the optimal mechanism is a* POST-2 *mechanism. In particular, it is a solution to the following program.*

$$\max_{K_2 \in [B,\beta],\ K_1 \in [B,K_2]} B\Big[1 - G_1(K_1)\Big] + K_2\big(1 - \frac{B}{K_1}\big)\Big[1 - G_1(K_2) - G_2(K_2) + G(K_2, K_2)\Big].$$

Proof: Since H_1 is strictly convex, $r(1 - G_1(r))$ is strictly increasing for all $r \leq \bar{K}$. Using $B \leq \bar{K}$, we get that $B(1 - G_1(B)) \geq r(1 - G_1(r))$ for all $r \leq B$. Hence, R_1 defined as the maximum possible revenue in a posted-price mechanism in our problem (Proposition 2) is

$$R_1 = \max_{K_1 \in [0,B]} K_1(1 - G_1(K_1)) = B(1 - G_1(B)).$$

But the POST-2 mechanism with $K_1 = K_2 = B$ generates a revenue of $B(1 - G_1(B))$. This proves the theorem. ■

The optimality of POST-2 mechanism is possible even for $B > \bar{K}$. Proposition 3 only gives a sufficient condition on the budget for optimality of a POST-2 mechanism. The exact optimal mechanism is difficult to describe in general. Section 4.5 works out the exact optimal mechanism for the uniform distribution prior.

Our results are for the case when the budget B is observed by the seller. We can *partially* extend our results to the case when B is also a private information of the (agent, manager) pair. Under a reasonable assumption on the set of mechanisms, we can completely describe the optimal mechanism with private budgets. The projection of such an optimal mechanism for *low* budget is POST-1 mechanism and for *high* budgets, it is a POST-2 mechanism - this shows that our results are robust to our public budget assumption. We are not able provide a formal statement and proof of these results due to space constraints.

4.4 Limiting cases

It is interesting to see what our result says in three extreme cases. First, as $B \to \beta$, then the expected revenue from any POST-2 mechanism tends to 0 (since $K_1, K_2 \geq B$). As a result, a POST-1 mechanism becomes optimal.

Second, as $B \to 0$, the expected revenue from a POST-1 mechanism is zero (since posted price is not more than B in a POST-1 mechanism), but using the expression of revenue for optimal POST-2 mechanism given by Proposition 2 and the fact that $B \to 0$, we see that it is independent of K_1:

$$\max_{K_2 \in [0,\beta]} K_2 \Big(1 - G_1(K_2) - G_2(K_2) + G(K_2, K_2)\Big)$$

Hence, the optimal POST-2 mechanism can have $K_1 = K_2$ and chooses K_2 that maximizes the product of K_2 and the probability measure of the square on the north-east corner of Figure 2 (where $v_1 \geq K_2$ and $v_2 \geq K_2$). Note that since $\frac{B}{K_1} \to 0$, there are only two outcomes in the menu such a mechanism: $(0, 0)$ and $(1, K_2)$. Thus the optimal mechanism converges to the optimal posted-price mechanism for the *manager* - just as we described in Section 2, only types in the north-east square will choose outcome $(1, K_2)$ in a posted-price mechanism with a posted-price K_2. Note that such a posted price mechanism is *not* a POST-1 mechanism because a POST-1 mechanism has a posted price less than or equal to the budget. It is just a POST-2 mechanism with one posted price and two outcomes in the range.

Finally, though our results require that we *do not* have perfect correlation (since densities are assumed to be positive), it is interesting to see what happens as we approach the perfect correlation case. As we approach perfect correlation, we have for all x, $G(x, x) \to G_i(x)$ for each $i \in \{1, 2\}$. Hence, using Proposition 2, we conclude that the optimal POST-2 mechanism revenue is given by

$$\max_{K_2 \in [B,\beta], K_1 \in [B,K_2]} B\Big[1 - G_1(K_1)\Big] + K_2\big(1 - \frac{B}{K_1}\big)\Big[1 - G_1(K_2) - G_2(K_2) + G(K_2, K_2)\Big]$$
$$= \max_{K_2 \in [B,\beta], K_1 \in [B,K_2]} B\Big[1 - G_1(K_1)\Big] + K_2\big(1 - \frac{B}{K_1}\big)\Big[1 - G_1(K_2)\Big].$$

The above expression is just maximizing the expected revenue of the following class of mechanisms. Pick any $K_2 \in [B, \beta]$ and $K_1 \in [B, K_2]$ and define a mechanism (f, p) as follows:

$$(f(v), p(v)) = \begin{cases} (0,0) & \text{if } v_1 \leq K_1 \\ (1, B + K_2(1 - \frac{B}{K_1})) & \text{if } v_1 > K_2 \\ (\frac{B}{K_1}, B) & \text{otherwise} \end{cases}$$

A straightforward calculation reveals that the revenue from this mechanism is exactly the expression in the maximization term above. Of course, this mechanism is an IC mechanism in a standard model where there is just the agent with type v_1. But, we know that the optimal mechanism in such a model is a posted-price mechanism with some posted-price p^* and revenue $p^*(1-G_1(p^*))$. Hence, the revenue R_2 from the optimal POST-2 mechanism must satisfy $R_2 \leq p^*(1-G_1(p^*))$. The expression on the right can be achieved by the revenue of a POST-1 mechanism if $p^* \leq B$. In that case, a POST-1 mechanism is an optimal mechanism. If $p^* > B$, then the expression on the right is the revenue from a POST-2 mechanism with $K_1 = K_2 = p^*$. Note that since $G(x,x) \to G_i(x)$ for each i and for each x, the probability measure of the rectangle $\{v : v_1 > K_2, v_2 < K_2\}$ tends to zero. Hence, such a POST-2 mechanism approaches a standard posted-price mechanism with two outcomes.

Sketch of proof. We give an overview of the proof of Theorem 1 now - the detailed proofs are in Appendix. Fix a mechanism (f,p), and define the following partitioning of the type space: $V^+(f,p) := \{v : p(v) > B\}$ and $V^-(f,p) = \{u : p(u) \leq B\}$. The proof considers two classes of mechanisms, those (f,p) where $V^+(f,p)$ has non-zero Lebesgue measure and those where $V^+(f,p)$ has zero Lebesgue measure. Define the following partitioning of the class of IC and IR mechanisms:

$$M^+ := \{(f,p) \text{ is IC and IR} : V^+(f,p) \text{ has positive Lebesgue measure}\}$$
$$M^- := \{(f,p) \text{ is IC and IR} : V^+(f,p) \text{ has zero Lebesgue measure}\}.$$

The proof of Theorem 1 is completed by proving the following proposition.

PROPOSITION **4** *Suppose H_1 is strictly convex. Then, the following are true.*

1. *There exists a* POST-1 *mechanism $(f,p) \in M^-$ such that*

$$\text{Rev}(f,p) \geq \text{Rev}(f',p') \qquad \forall\ (f',p') \in M^-.$$

2. There exists a POST-2 *mechanism* $(f,p) \in M^+$ *such that*

$$\textsc{Rev}(f,p) \geq \textsc{Rev}(f',p') \qquad \forall\ (f',p') \in M^+.$$

The proof of (1) in Proposition 4 uses somewhat familiar ironing arguments. However, proof of (2) in Proposition 4 is quite different, and requires a lot of work to get to a simpler class of mechanisms where ironing can be applied. The proof proceeds by deriving some necessary conditions for IC and reducing the space of mechanisms. In this smaller class of mechanisms, we show that ironing arguments lead to a POST-2 mechanism. Though the proof does not introduce new tools to deal with multidimensional mechanism design problems, it illustrates that multidimensional mechanism design problems may be tractable under certain behavioral assumptions.

4.5 Uniform distribution

In this section, we work out the exact optimal mechanism for the uniform distribution case. We assume that $\beta = 1$ and G is the uniform distribution over $[0,1] \times [0,1]$. We first show that the optimal POST-2 mechanism has only one posted price (i.e., $K_1 = K_2$).

LEMMA **2** *Suppose* $\beta = 1$ *and* G *is the uniform distribution over* $[0,1] \times [0,1]$*. Then, the optimal* POST-2 *mechanism must satisfy:*

1. *if* $B \geq \frac{1}{2}(3-\sqrt{5})$*, then* $K_1 = K_2 = B$*,*
2. *if* $B < \frac{1}{2}(3-\sqrt{5})$*, then* $K_1 = K_2 = \frac{1}{3}\big(B + 2 - \sqrt{(B^2+B+1)}\big)$*.*

Using this lemma, we can provide a complete description of the optimal mechanism for the uniform distribution case.

PROPOSITION **5** *Suppose* $\beta = 1$ *and* G *is the uniform distribution over* $[0,1] \times [0,1]$*. Then, the optimal mechanism is the following.*

1. *If* $B > \frac{1}{2}$*, then a* POST-1 *mechanism with* $K_1 = \frac{1}{2}$ *is optimal.*

2. *If* $B \in [\frac{1}{2}(3-\sqrt{5}), \frac{1}{2}]$, *then a* POST-1 *mechanism with* $K_1 = B$ *is optimal. In this case, a* POST-2 *mechanism with* $K_1 = K_2 = B$ *is also optimal.*

3. *If* $B \in (0, \frac{1}{2}(3-\sqrt{5}))$, *then a* POST-2 *mechanism with* $K_1 = K_2 = \frac{1}{3}\big(B + 2 - \sqrt{(B^2+B+1)}\big)$ *is optimal.*

Proof: To do the proof, we first compute the optimal POST-1 mechanism, which is the solution to the following optimization program:

$$\max_{K_1 \in [0,B]} K_1(1-K_1).$$

It is clear the optimal POST-1 mechanism is $K_1 = \frac{1}{2}$ if $B > \frac{1}{2}$ and $K_1 = B$ if $B \leq \frac{1}{2}$. Now, we consider the three cases separately.

CASE 1 - $B > \frac{1}{2}$. Optimal POST-1 mechanism generates a revenue of $\frac{1}{4}$. By Lemma 2, optimal POST-2 mechanism generates a revenue of $B(1-B)$, which is less than $\frac{1}{4}$. Hence, the optimal mechanism is a POST-1 mechanism with $K_1 = \frac{1}{2}$.

CASE 2 - $B \in [\frac{1}{2}(3-\sqrt{5}), \frac{1}{2}]$. In this case, both the optimal POST-1 mechanism and the optimal POST-2 mechanism (due to Lemma 2) generates a revenue of $B(1-B)$. Hence, the optimal POST-1 mechanism with $K_1 = B$ is optimal.

CASE 3 - $B \in (0, \frac{1}{2}(3-\sqrt{5}))$. In this case, the optimal POST-1 mechanism generates a revenue of $B(1-B)$, which is also the revenue generates by a POST-2 mechanism with $K_1 = K_2 = B$. But the optimal POST-2 is unique and has $K_1 = K_2 = \frac{1}{3}\big(B + 2 - \sqrt{(B^2+B+1)}\big)$ due to Lemma 2. Hence, the result follows. ■

Notice that as $B \to 0$, the optimal mechanism is a posted price mechanism with price $\frac{1}{3}$. So, in the limiting case when the agent has zero budget to make decisions, the optimal mechanism is *not* a posted price mechanism with posted price $\frac{1}{2}$ - which is the optimal posted price in the standard model. To see why, consider the limiting case $B = 0$. Suppose the seller uses a posted price mechanism with price p. Who are the types who will accept this price? This is shown in the left graph in Figure 1. All the types (v_1, v_2) such that $v_1 < p$ will choose outcome $(0, 0)$. All types (v_1, v_2) with $v_1 > p$ but $v_2 < p$ will also choose outcome $(0, 0)$ - this is because even though the agent prefers $(1, p)$ over $(0, 0)$, it cannot make a decision because of budget constraint. Thus, the only types (v_1, v_2) which will prefer $(1, p)$ to $(0, 0)$ are those with $v_1 > p, v_2 > p$. Hence, the expected revenue from a posted price mechanism is $p(1-p)^2$, which is maximized at $\frac{1}{3}$. This argument establishes the optimal posted price mechanism. Proposition 5 shows that it is the optimal mechanism.

On the other extreme, when $B \to \beta$, the optimal mechanism is a posted price mechanism with price $\frac{1}{2}$. This is because the agent makes all the decisions now and for any price p, the types that accept this price are just the types with $v_1 > p$. An optimal solution thus gives a posted price of $\frac{1}{2}$ as in a standard model.

5 Notion of incentive compatibility

In this section, we discuss some issues related to the revelation principle and our notion of incentive compatibility. We show here a version of the revelation principle holds in our setting. To define an arbitrary mechanism, let M be a message space and $\mu : M \to Z$ be a mechanism. A strategy of the (agent, manager) pair is a map $s : V \to M$. We say that mechanism μ **implements** the direct revelation mechanism (f, p) if there exists a strategy $s : V \to M$ such that (a) $\mu(s(v)) \succeq_v \mu(m)\ \forall\ v \in V,\ \forall\ m \in M$ and (b) $\mu(s(v)) = (f(v), p(v))\ \forall\ v \in V$. Suppose μ implements (f, p). Then, fix some $v, v' \in V$ and note that $(f(v), p(v)) = \mu(s(v)) \succeq_v \mu(s(v')) = (f(v'), p(v'))$, which proves IC of (f, p). Hence, the revelation principle holds in this setting. It is well known that with behavioral agents, the revelation principle may not hold in general (de Clippel, 2014). There are at least two assumptions in our model which allows the revelation principle to work. The first is the completeness of our relation $\succeq_v$ (even though it may be intransitive). The second, and more important one, is the notion of IC we use.

The primitives of our model involves how the (agent, manager) pair chooses from pairs of outcomes. We are silent about how it chooses from a subset of alternatives. This is consistent with Tversky (1969) and most of the literature which works on binary choice models (Rubinstein, 1988; Tadenuma, 2002; Houy and Tadenuma, 2009). Our incentive constraints are appropriate for this binary choice model.

There are two main reasons why we use our existing notions of incentive compatibility instead of a choice-theoretic version of (i.e., a model where we specify how the (agent, manager) pair chooses from subsets of outcomes) incentive compatibility. First, to be able to use choice-incentive compatibility, we have to *assume* that the (agent, manager) pair chooses from subsets of outcomes using some choice procedure. The current primitives of our model are much simpler - it just makes assumptions on how we choose between pairs of outcomes. Importantly, our notion of incentive compatibility allows us tractability using minimal assumptions about deviations from rationality. Second, if the primitives of the model are choice correspondences, then a revelation principle need not hold - see de Clippel (2014). This implies that the space of mechanisms are more complex than the set of direct revelation mechanisms. In summary, it is not clear how an optimal mechanism will look like if we considered a model assuming certain choice behavior of agents over subsets of outcomes and choice-incentive compatibility as the notion of our incentive compatibility. We leave this issue for future research.

6 Related literature

Our paper is related to a couple of strands of literature in mechanism design. We go over them in some detail. Before doing so, we relate our work to two papers which seem directly related to our work. The first is the work of Burkett (2016), who studies a principal-agent model where the agent is participating in an auction mechanism. In his model, there is a third-party which has proposed a mechanism for selling a single good. After the third-party announces a mechanism, the principal in his model announces another mechanism, which he terms as a *contract*, to the agent. The sole purpose of the contract is to determine the amount the agent will bid in the third-party mechanism. In his model, the value of the good to the agent is the *only* private information - the value of the good to the principal can be determined from the value of the agent. The main result in this paper is that the optimal contract for the principal is a "budget-constraint" contract, which specifies a cap on the report of each type of the agent to the third-party mechanism and involves no side-payments between the principal and the agent. [5]

Though related, our model is quite different. In our model, the values of the agent and the manager can be completely different (at a technical level, Burkett (2016) has a one-dimensional mechanism design problem, whereas ours is a two-dimensional mechanism design problem). Further, we do not model decision-making by our (agent, manager) pair via a contract. In other words, the naive decision-making in our model makes it quite different from Burkett (2015, 2016).

Another closely related paper is Malenko and Tsoy (Forthcoming), who study a model where a single good is sold to a set of buyers. Each buyer is advised by a unique advisor. Each buyer does not know her value but the advisor knows. However, the advisor has some bias, which is commonly known. Before the start of the auction, there is communication from the advisor to the buyer, which influences how much the buyer bids in the auction. The aim of Malenko and Tsoy (Forthcoming) is to compare standard auction formats in the presence of such uncertain buyers being advised by biased consultants. They find that standard sealed-bid auctions are revenue equivalent, but ascending-price auction generates more expected revenue than sealed-bid auctions. While their focus is on the effect of communication on equilibrium of standard form auctions, ours is a mechanism design problem where the (agent, manager) pair do not engage in any communication. Our novelty is to solve for the optimal contract of a seller in the presence of a naive (agent, manager) pair.

Behavioral mechanism design. We discuss some literature in mechanism design which looks at specific models of behavioral agents and designing optimal contracts for selling to such agents. A very detailed survey with excellent examples can be found in Koszegi (2014).

[5] In a related paper, Burkett (2015) considers first-price and second-price auctions and compares their revenue and efficiency properties when a seller is faced with such principal-agent pairs.

Our literature survey is limited in nature as we focus on models which are closer to ours.

A stream of papers investigate the optimal contract for a firm to a consumer in a two-period model, where the consumer has time inconsistent preferences. These papers differ in the way it treats inconsistent preferences and non-common priors between firm and the consumer.

Eliaz and Spiegler (2006) consider a model where the type of the agent is his "cognitive" ability. In their model, there are two periods and the agent enjoys a valuation for an action in each period. In period 2, the agent's valuation may change to another value. Agents differ in their subjective assessment of the probability of that transition. So, in their model, the type is the subjective probability of the agent. They show how the optimal contract treats sophisticated and naive agents. While this paper allows agents to be time-inconsistent, in another paper, Eliaz and Spiegler (2008) study a similar model but do not allow time inconsistency. There, they allow the monopolist to have a separate belief about the change of state. They characterize the optimal contract and show the implications of non-common priors on the menu of optimal contract and ex-post efficiency. Grubb (2009) considers a two period model where a firm is selling a divisible good to consumers. The private type of the consumer is his demand in period 2. In period 1, the firm offers them a tariff which is accepted or rejected. If accepted, the consumers buy the quantity in period 2 once they realize their demand. The key innovation in his paper is again the lack of common prior between consumers and the firm - in particular, he shows that if the prior of the consumers is such that it *underestimates* the variance of the actual prior (for instance, if the consumer prior has the same mean as the firm, then consumer prior is a mean-preserving spread of the firm prior), then the optimal tariff of the firm must have three parts (with non-zero quantities offered at zero marginal cost).

de Clippel (2014) studies complete information implementation with behavioral agents - his main results extend Maskin's characterization (Maskin, 1999) to environments with behavioral agents. Esteban et al. (2007) consider a model where agents have temptation and self control preferences as in Gul and Pesendorfer (2001), and characterize the optimal contract - also see related work on self control preferences in DellaVigna and Malmendier (2004). There are several other papers who consider time inconsistent preferences and analyze the optimal contracting problem. Carbajal and Ely (2016) consider a model of optimal price discrimination when buyers have loss averse preferences with state dependent reference points. They characterize the optimal contract in their model.

Multidimensional mechanism design. The type space of our agent is two-dimensional. It is well known that the problem of finding an optimal mechanism for selling multiple goods (even to a single buyer) is notorious. A long list of papers have shown the difficulties involved in extending the one-dimensional results in Mussa and Rosen (1978); Myerson (1981); Riley and Zeckhauser (1983) to multidimensional framework - see Armstrong (2000); Manelli and Vincent (2007) as examples. Even when the seller has just *two* objects and there is just one buyer with additive valuations (i.e., value for both the objects is sum of values of both the objects), the optimal mechanism is difficult to describe (Manelli and Vincent, 2007; Daskalakis et al., 2017; Hart and Nisan, 2017). This has inspired researchers to consider *approximately* optimal mechanisms (Chawla et al., 2007, 2010; Hart and Nisan, 2017) or additional robustness criteria for design (Carroll, 2017). Compared to these problems, our two-dimensional mechanism design problem becomes tractable because of the nature of incentive constraints, which in turn is a consequence of the preference of the agent.

Mechanism design with budget constraints. In our model, the agent is budget constrained but the manager is not. We compare this with the literature in the standard model when there is a single object and the buyer(s) is budget constrained. The space of mechanisms is restricted to be such that payment is no more than the budget. This feasibility requirement on the mechanisms essentially translates to a violation of quasilinearity assumption of the buyer's preference for prices above the budget (utility assumed to be $-\infty$) but below the budget the utility is assumed to be quasilinear. This introduces additional complications for finding the optimal mechanism. Laffont and Robert (1996) show that an all-pay-auction with a suitable reserve price is an optimal mechanism for selling an object to multiple buyers who have publicly known budget constraints. When the budget is private information, the problem becomes even more complicated - see Che and Gale (2000) for a description of the optimal mechanism for the single buyer case and Pai and Vohra (2014) for a description of the optimal mechanism for the multiple buyers case. All these mechanisms involve randomization but the nature of randomization is quite different from ours. This is because the source of randomization in all these papers is either due to budget being private information (hence, part of the type, as in Che and Gale (2000); Pai and Vohra (2014)) or because of multiple agents with budget being common knowledge (as in Laffont and Robert (1996); Pai and Vohra (2014)). Indeed, with a single agent and public budget, the optimal mechanism in a standard single object allocation model is a posted price mechanism. This can be contrasted with our result where we get randomized optimal mechanism even with one (agent, manager) pair and budget being common knowledge. This shows that the lexicographic decision making using two rationales plays an important role in making a POST-2 mechanism optimal. Also, the set of menus in the optimal mechanism in the standard single object auction with budget constraint may have more than three outcomes. Further, the outcomes in the menu of these optimal mechanisms are not as simple as our POST-2 mechanism. Finally, like us, these papers assume that budget is exogenously determined by the agent. If the buyer can choose his budget constraint, then Baisa and Rabinovich (2016) show that the optimal mechanism in a multiple buyers setting allocates the object efficiently whenever it is allocated - this is in contrast to the exogenous budget case (Laffont and Robert, 1996; Pai and Vohra, 2014).

A Appendix: Omitted Proofs of Section 4

A.1 Proof of Proposition 1

Proof: Consider a POST-2 mechanism (f, p) defined by parameters K_1 and K_2 with $B \leq K_1 \leq K_2$. Since $p(0,0) = 0$, Lemma 1 implies that (f, p) is IR if it is IC. We show IC of

(f,p). We will denote by $\bar{u} \to \tilde{u}$ the incentive constraint associated with type $\bar{u}$ when it cannot misreport $\tilde{u}$.

Consider types u, v, s taken from three different regions in Figure 2 with three different outcomes. In particular, u, v, s satisfy: $u_1 \leq K_1$, $\min(v_1, v_2) \leq K_2$ but $v_1 > K_1$, and $\min(s_1, s_2) > K_2$. Note that

$$(f(u), p(u)) = (0, 0),\ (f(v), p(v)) = (\frac{B}{K_1}, B),\ \text{and}\ (f(s), p(s)) = (1, B + K_2(1 - \frac{B}{K_1})).$$

We consider IC of each of these types.

(1). $u \to v, u \to s$. Note that since $u_1 \leq K_1$, we have $u_1 \frac{B}{K_1} - B \leq 0$. Hence, type u weakly prefers $(0,0)$ to $(\frac{B}{K_1}, B)$. Similarly,

$$u_1 - B - K_2\big(1 - \frac{B}{K_1}\big) \leq K_1 - B - K_2 + \frac{K_2}{K_1}B = (K_2 - K_1)\Big(\frac{B}{K_1} - 1\Big) \leq 0,$$

where first inequality is due to $u_1 \leq K_1$ and the second is due to $K_2 \geq K_1$ and $B \leq K_1$. Hence, u prefers $(0,0)$ to $(f(s), p(s))$.

(2). $v \to u, v \to s$. For $v \to u$, we note that $v_1 \frac{B}{K_1} - B \geq 0$. This follows from the fact that $v_1 > K_1$. Hence, $v \to u$ holds as $p(v) = B$. For $v \to s$, we note that

$$\begin{aligned}\min(v_1, v_2) - B - K_2\Big(1 - \frac{B}{K_1}\Big) &\leq \min(v_1, v_2) - B - \min(v_1, v_2)\Big(1 - \frac{B}{K_1}\Big) \\ &= \frac{B}{K_1}\min(v_1, v_2) - B.\end{aligned}$$

If $\min(v_1, v_2) = v_1$, then we see that $(f(v), p(v))$ is preferred to $(f(s), p(s))$. Else, $\min(v_1, v_2) = v_2$. In that case since $p(s) > B$, even if the agent prefers $(f(s), p(s))$ to $(f(v), p(v))$, she cannot compare. But the manager prefers $(f(v), p(v))$ to $(f(s), p(s))$. Hence, $v \to s$ holds.

(3). $s \to u, s \to v$. Note that for $x \in \{s_1, s_2\}$, we have

$$0 \leq \frac{K_2}{K_1}B - B \leq \frac{B}{K_1}x - B = x - B - x\Big(1 - \frac{B}{K_1}\Big) \leq x - B - K_2\Big(1 - \frac{B}{K_1}\Big),$$

where the inequalities follow from the fact that $x \geq \min(s_1, s_2) > K_2 \geq K_1 \geq B$. This shows that the aggregate preference at s prefers $(f(s), p(s))$ to $(f(v), p(v))$ and $(f(u), p(u))$. Because $p(s) > B$, $s \to v$ and $s \to u$ hold. ■

A.2 Proofs of Theorem 1 and Propositions 2 and 4

In this section, we provide the proof of the main results - Theorem 1 and Propositions 2 and 4. It is clear that Proposition 4 immediately implies Theorem 1. So, we first provide a proof of Proposition 4, followed by a proof of Proposition 2.

A.2.1 Preliminary Lemmas

We start off by proving a series of necessary conditions for IC. These are use in proving our main result. The first lemma is a monotonicity condition of allocation rule: for every IC mechanism, type with higher payment implies higher allocation probability. Hence, the outcomes in the range of an IC mechanism are ordered in a natural sense.

Lemma **3** *For any IC mechanism (f, p), if $p(u) < p(v)$ for any u, v, then $f(u) < f(v)$.*

Proof: Take any u, v such that $p(u) < p(v)$. IC implies that $(f(v), p(v)) \succeq_v (f(u), p(u))$. If $p(v) \leq B$, then we must have $v_1 f(v) - p(v) \geq v_1 f(u) - p(u) > v_1 f(u) - p(v)$, where the last inequality uses $p(v) > p(u)$. This implies $f(u) < f(v)$.

If $p(v) > B$, then we have $v_2 f(v) - p(v) \geq v_2 f(u) - p(u) > v_2 f(u) - p(v)$, where the last inequality uses $p(v) > p(u)$. This implies $f(u) < f(v)$. ■

Lemma **4** *For any IC mechanism (f, p), for all u, v*

1. *if $p(u), p(v) \leq B$ and $u_1 > v_1$, then $f(u) \geq f(v)$,*
2. *if $p(u), p(v) > B$ and $u_2 > v_2$, then $f(u) \geq f(v)$.*

Proof: Take any u, v. If $p(u), p(v) \leq B$, then adding the incentive constraints using $\succeq_{v_1}$ and $\succeq_{u_1}$gives us the desired result and if $p(u), p(v) > B$, then adding the incentive constraints using $\succeq_{v_2}$ and $\succeq_{u_2}$ gives us the desired result. ■

LEMMA **5** *For any IC mechanism (f, p), for all u, v the following holds:*

$$\Big[p(u) \leq B < p(v)\Big] \Rightarrow \Big[\min(v_1, v_2) \geq \min(u_1, u_2)\Big].$$

Proof: Since $p(u) \leq B < p(v)$, by Lemma 3, $f(v) > f(u)$. We consider $v \to u$ first. This gives us

$$v_2 f(v) - p(v) \geq v_2 f(u) - p(u). \tag{1}$$

$$v_1 f(v) - p(v) > v_1 f(u) - p(u). \tag{2}$$

Using $f(v) > f(u)$, and aggregating Inequalities 1 and 2 gives us

$$\min(v_1, v_2)\big(f(v) - f(u)\big) \geq p(v) - p(u). \tag{3}$$

IC from u to v implies one of the following two conditions to holds:

CASE 1. $\succeq_{u_1}$ prefers $(f(u), p(u))$ to $(f(v), p(v))$: this gives

$$u_1 f(u) - p(u) \geq u_1 f(v) - p(v) \text{ or } p(v) - p(u) \geq u_1(f(v) - f(u)).$$

Adding with Inequality 3, we get, $(\min(v_1, v_2) - u_1)(f(v) - f(u)) \geq 0$. Then, $f(v) > f(u)$ implies that $\min(v_1, v_2) \geq u_1$.

CASE 2. $\succeq_{u_1}$ does not prefer $(f(u), p(u))$ to $(f(v), p(v))$ but budget has a bite - so, $\succeq_{u_2}$ prefers $(f(u), p(u))$ to $(f(v), p(v))$: this gives

$$u_2 f(u) - p(u) \geq u_2 f(v) - p(v). \tag{4}$$

Adding Inequalities (4) and (3), we get $(\min(v_1, v_2) - u_2)(f(v) - f(u)) \geq 0$. Since $f(v) > f(u)$, we get $\min(v_1, v_2) \geq u_2$.

Combining both the cases, $\min(v_1, v_2) \geq \min(u_1, u_2)$. ■

Now, fix a mechanism (f, p), and define as before: $V^+(f, p) := \{v : p(v) > B\}$ and $V^-(f, p) := \{u : p(u) \leq B\}$. We now prove some properties of IC mechanisms using these sets.

LEMMA **6** *Fix an IC mechanism (f, p). If $V^+(f, p)$ and $V^-(f, p)$ are non-empty, then the following holds:*

$$\inf_{v \in V^+(f,p)} \min(v_1, v_2) = \sup_{u \in V^-(f,p)} \min(u_1, u_2).$$

Proof: Since $V^+(f, p)$ is non-empty and $\min(v_1, v_2) \geq 0$, we have that $\inf_{v \in V^+(f,p)} \min(v_1, v_2)$ is a non-negative real number - we denote it as $\underline{v}$. By Lemma 5, $\sup_{u \in V^-(f,p)} \min(u_1, u_2)$ is also a non-negative real number as it is bounded above - we denote this as $\bar{v}$.

First, we show that $\underline{v} \geq \bar{v}$. If not, then $\underline{v} < \bar{v}$. Then, there is some v such that $\underline{v} < \min(v_1, v_2) < \bar{v}$. By definition of $\underline{v}$, there is a v' such that $\min(v'_1, v'_2)$ is arbitrarily close to $\underline{v}$ and $p(v') > B$. Since $\min(v'_1, v'_2) < \min(v_1, v_2)$, Lemma 5 gives us $p(v) > B$. Similarly, by definition of $\bar{v}$, there is a u' such that $\min(u'_1, u'_2)$ is arbitrarily close to $\bar{v}$ and $p(u') \leq B$. Since $\min(u'_1, u'_2) > \min(v_1, v_2)$, Lemma 5 gives us $p(v) \leq B$, giving us the desired contradiction.

Next, we show that $\underline{v} = \bar{v}$. If not, $\underline{v} > \bar{v}$. But this is not possible since for any v with $\underline{v} > \min(v_1, v_2) > \bar{v}$, we will have both $p(v) \leq B$ and $p(v) > B$, giving us a contradiction. ■

For any mechanism (f, p), we will denote by $K_{(f,p)}$ the following:

$$K_{(f,p)} := \inf_{v \in V^+(f,p)} \min(v_1, v_2) = \sup_{u \in V^-(f,p)} \min(u_1, u_2). \tag{5}$$

By Lemma 6, this is well-defined if $V^+(f, p)$ and $V^-(f, p)$ is non-empty.

Lemma **7** *If (f,p) is an IC and IR mechanism, then $V^-(f,p)$ is non-empty.*

Proof: Lemma 1 ensures that $(0,0) \in V^-(f,p)$ if (f,p) is IC and IR. ■

We remind the following partitioning of the class of IC and IR mechanisms:

$$M^+ := \{(f,p) \text{ is IC and IR} : V^+(f,p) \text{ has positive Lebesgue measure}\}$$
$$M^- := \{(f,p) \text{ is IC and IR} : V^+(f,p) \text{ has zero Lebesgue measure}\}.$$

We now prove a series of Lemmas for M^+ class of mechanisms.

A.2.2 Lemmas for M^+

The following lemma shows that $K_{(f,p)}$ is well defined if $(f,p) \in M^+$.

Lemma **8** *Suppose (f,p) is an IC and IR mechanism.*

1. *If $V^+(f,p)$ is non-empty, then $K_{(f,p)}$ defined in Equation (5) exists and satisfies: for all $v \in V$,*
$$\Big[\min(v_1,v_2) > K_{(f,p)}\Big] \Rightarrow \Big[p(v) > B\Big],$$
$$\Big[\min(v_1,v_2) < K_{(f,p)}\Big] \Rightarrow \Big[p(v) \le B\Big].$$

2. *If $(f,p) \in M^+$, then $\beta > K_{(f,p)} > B$.*

Proof: The first part follows from Lemma 6, Lemma 7, and the definition of $K_{(f,p)}$.

For the second part, we first argue that $K_{(f,p)} \ge B$. Suppose $K_{(f,p)} < B$. Then, for some v with $K_{(f,p)} < \min(v_1,v_2) \le B$, we have $p(v) > B$. But this violates IR.

Now, assume for contradiction $K_{(f,p)} = B < \beta$. In that case, fix some $\epsilon \in (0,1)$ and positive integer k, and consider the type $v^{k,\epsilon} \equiv (B+\epsilon^k, B+\epsilon^k)$. By (1), we know that $p(v^{k,\epsilon}) > B$. By IR, $(B+\epsilon^k)f(v^{k,\epsilon}) \ge p(v^{k,\epsilon}) > B$. This gives us $f(v^{k,\epsilon}) > \frac{B}{B+\epsilon^k}$. Since $B+\epsilon > B+\epsilon^k$ for all $k > 1$, by (1) of Lemma 4, we have $f(v^{1,\epsilon}) \ge f(v^{k,\epsilon}) > \frac{B}{B+\epsilon^k}$. As $\frac{B}{B+\epsilon^k}$ can be made arbitrarily close to 1, we conclude that $f(v^{1,\epsilon}) = 1$ - notice that $v^{1,\epsilon} \equiv (B+\epsilon, B+\epsilon)$ and the claim holds for *all* $\epsilon \in (0,1)$. By Lemma 3, for all $\epsilon, \epsilon' \in (0,1)$, since $f(v^{1,\epsilon}) = f(v^{1,\epsilon'}) = 1$, we get that $p(v^{1,\epsilon}) = p(v^{1,\epsilon'})$. Denote $p(v^{1,\epsilon}) = B + \delta$, where $\epsilon \in (0,1)$. By definition, $\delta > 0$. Now, IR requires that for *every* $\epsilon \in (0,1)$, $(B+\epsilon)f(v^{1,\epsilon}) - p(v^{1,\epsilon}) = (B+\epsilon) - (B+\delta) \geq 0$. But this will mean $\epsilon > \delta$ for all $\epsilon \in (0,1)$. Since $\delta > 0$ is fixed, this is a contradiction.

Finally, we know that $(f,p) \in M^+$ implies $V^+(f,p)$ has positive Lebesgue measure. If $\beta = K_{(f,p)}$, then by (1), we know that $V^+(f,p)$ has zero Lebesgue measure, which is a contradiction. ■

Next, we show a useful inequality involving $K_{(f,p)}$ for any $(f,p) \in M^+$.

Lemma **9** *Suppose (f,p) is an IC and IR mechanism. If $(f,p) \in M^+$, then for all types $u \in V$ with $B < p(u)$, we must have*

$$K_{(f,p)} f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) \geq K_{(f,p)} f(u) - p(u).$$

Proof: First, consider two types $v \equiv (K_{(f,p)}, 0)$ and $v' \equiv (K_{(f,p)}, K_{(f,p)} - \epsilon)$, where $\epsilon > 0$ such that $K_{(f,p)} - \epsilon > 0$. Notice that $\min(v_1, v_2) < K_{(f,p)}$ and $\min(v'_1, v'_2) < K_{(f,p)}$. Hence, by Lemma 8, $p(v) \leq B$ and $p(v') \leq B$. As a result, $v \to v'$ and $v' \to v$ imply that

$$K_{(f,p)} f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) = K_{(f,p)} f(K_{(f,p)}, K_{(f,p)} - \epsilon) - p(K_{(f,p)}, K_{(f,p)} - \epsilon). \quad (6)$$

Now, assume for contradiction that for some u with $p(u) > B$ we have

$$K_{(f,p)} f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) < K_{(f,p)} f(u) - p(u).$$

Using Equation 6 and chooseing $\epsilon > 0$ sufficiently small, we get,

$$K_{(f,p)} f(K_{(f,p)}, K_{(f,p)} - \epsilon) - p(K_{(f,p)}, K_{(f,p)} - \epsilon) < \big(K_{(f,p)} - \epsilon\big) f(u) - p(u).$$

But then $(K_{(f,p)} - \epsilon) f(K_{(f,p)}, K_{(f,p)} - \epsilon) - p(K_{(f,p)}, K_{(f,p)} - \epsilon) < \big(K_{(f,p)} - \epsilon\big) f(u) - p(u) < K_{(f,p)} f(u) - p(u)$. Hence, $(K_{(f,p)}, K_{(f,p)} - \epsilon) \to u$ does not hold - a contradiction. ■

Lemma **10** *Suppose $(f,p) \in M^+$ is an IC and IR mechanism. Then the following limits exist:*

$$\lim_{\delta \to 0^+} f(K_{(f,p)} + \delta, \beta) \quad \textit{and} \quad \lim_{\delta \to 0^+} p(K_{(f,p)} + \delta, \beta).$$

Further, if $A_{(f,p)}$ and $P_{(f,p)}$ are the respective limits above, then the following equations hold:

$$K_{(f,p)}A_{(f,p)} - P_{(f,p)} = K_{(f,p)}f(K_{(f,p)},0) - p(K_{(f,p)},0) \tag{7}$$

$$\beta A_{(f,p)} - P_{(f,p)} = \beta f(\beta,\beta) - p(\beta,\beta). \tag{8}$$

Proof: Fix any $\delta > 0$ such that $K_{(f,p)} + \delta \leq \beta$ - by Lemma 8, such $\delta > 0$ exists. Consider two types $v \equiv (K_{(f,p)} + \delta, \beta)$ and $v' \equiv (\beta,\beta)$. By Lemma 8, $p(v), p(v') > B$. The pair of incentive constraints between v and v' give us

$$\beta f(v) - p(v) \geq \beta f(v') - p(v') \quad \text{and} \quad \beta f(v') - p(v') \geq \beta f(v) - p(v).$$

Combining these and using the definition of v' gives us

$$\beta f(K_{(f,p)} + \delta, \beta) - p(K_{(f,p)} + \delta, \beta) = \beta f(\beta,\beta) - p(\beta,\beta). \tag{9}$$

Now, consider $v'' \equiv (K_{(f,p)}, 0)$. By Lemma 8, $p(v'') \leq B$. But $p(v) > B$ implies that $v \to v''$ must give us

$$(K_{(f,p)} + \delta)f(v) - p(v) \geq (K_{(f,p)} + \delta)f(v'') - p(v'') \geq K_{(f,p)}f(v) - p(v) + \delta f(v''),$$

where the second inequality comes from Lemma 9 and the fact that $p(v) > B$. Using Equation 9, we replace $p(v)$ in the previous equation to get,

$$(K_{(f,p)} + \delta)f(v) \geq (K_{(f,p)} + \delta)f(v'') - p(v'') + \beta f(v) - \beta f(\beta,\beta) + p(\beta,\beta) \geq K_{(f,p)}f(v) + \delta f(v'').$$

Rearranging terms, we get

$$\begin{aligned}\Big[\beta - K_{(f,p)}\Big]f(v) &\leq \Big[\beta f(\beta,\beta) - p(\beta,\beta)\Big] - \Big[K_{(f,p)}f(v'') - p(v'')\Big] \\ &\leq \Big[\beta - K_{(f,p)}\Big]f(v) - \delta\big(f(v) - f(v'')\big)\end{aligned}$$

Since v'' is independent of δ and $v \equiv (K_{(f,p)} + \delta, \beta)$, we get that

$$\Big[\beta - K_{(f,p)}\Big] \lim_{\delta \to 0^+} f(K_{(f,p)} + \delta, \beta) = \Big[\beta f(\beta,\beta) - p(\beta,\beta)\Big] - \Big[K_{(f,p)}f(K_{(f,p)},0) - p(K_{(f,p)},0)\Big].$$

This gives us the desired expression for $A_{(f,p)}$. Using Equation 9, we also get the desired expression for $P_{(f,p)}$. Then, it is routine to check that Equations (7) and (8) hold. ■

The final preparatory lemma is the following.

Lemma **11** *Suppose $(f,p) \in M^+$ is an IC and IR mechanism. Then the following are true.*

1. $P_{(f,p)} \geq p(u) > B$ *for all* u *with* $u_2 < \beta$ *and* $\min(u_1, u_2) > K_{(f,p)}$.

2. $A_{(f,p)} > f(K_{(f,p)}, 0) + \frac{1}{K_{(f,p)}}\Big[B - p(K_{(f,p)}, 0)\Big]$.

Proof: Proof of (1). Consider a type $(K_{(f,p)}+\delta, \beta)$ for some $\delta > 0$ but close to zero, and u such that $u_2 < \beta$ and $\min(u_1, u_2) > K_{(f,p)}$. By Lemma 8, we know that $p(K_{(f,p)}+\delta, \beta), p(u) > B$. By Lemma 4(2), we get $f(K_{(f,p)} + \delta, \beta) \geq f(u)$. Now, $u \to (K_{(f,p)} + \delta, \beta)$ implies

$$u_2 f(u) - p(u) \geq u_2 f(K_{(f,p)} + \delta, \beta) - p(K_{(f,p)} + \delta, \beta)$$
$$\Rightarrow p(K_{(f,p)} + \delta, \beta) - p(u) \geq u_2\Big[f(K_{(f,p)} + \delta, \beta) - f(u)\Big] \geq 0.$$

Since this holds for all $\delta > 0$ arbitrarily close to zero, $P_{(f,p)} = \lim_{\delta \to 0^+} p(K_{(f,p)} + \delta, \beta) \geq p(u)$.

Proof of (2). Assume for contradiction that

$$A_{(f,p)} \leq f(K_{(f,p)}, 0) + \frac{1}{K_{(f,p)}}\Big[B - p(K_{(f,p)}, 0)\Big].$$
$$\Leftrightarrow K_{(f,p)}A_{(f,p)} - B \leq K_{(f,p)}f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0).$$

But Equation (7) in Lemma 10 says that the RHS above is equal to $K_{(f,p)}A_{(f,p)} - P_{(f,p)}$. This gives us $P_{(f,p)} \leq B$, which contradicts (1). ■

Next, we will look at a subclass of mechanisms which fixes some regions of the type space. Further, we will show that such a restriction is also without loss of generality for optimal mechanisms. To show this property, we consider an arbitrary IC and IR mechanism $(f, p) \in M^+$. We then construct a new IC and IR mechanism which generates more expected revenue and has the property we require. The new mechanism, which we denote as (f', p') is defined as follows.

$$(f'(v), p'(v)) = \begin{cases} (A_{(f,p)}, P_{(f,p)}) & \text{if } \min(v_1, v_2) > K_{(f,p)} \\ (f(v), p(v)) & \text{if } v_1 < K_{(f,p)} \\ \Big(f(K_{(f,p)}, 0) + \frac{1}{K_{(f,p)}}\big(B - p(K_{(f,p)}, 0)\big), B\Big) & \text{otherwise} \end{cases}$$

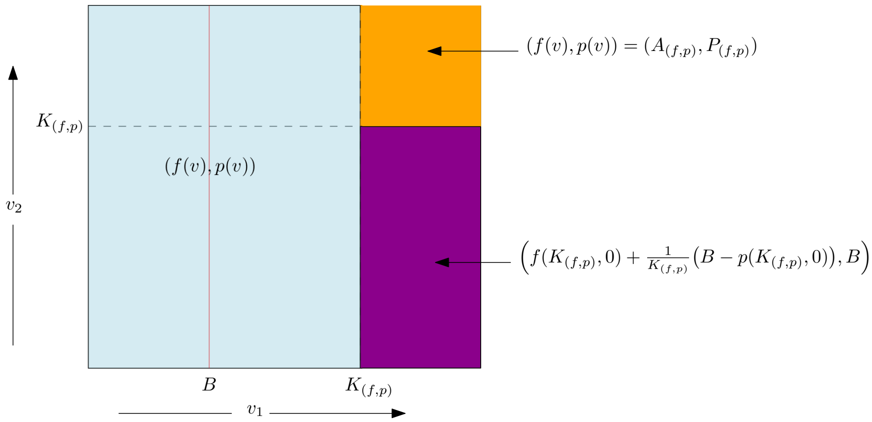

Figure 3: New mechanism

The new mechanism is shown in Figure 3. The rectangle at the top-right corner of the type space (excluding the lower boundaries) have the outcome $(A_{(f,p)}, P_{(f,p)})$. The outcomes in the big white rectangle to the left (but excluding the right boundary) is left unchanged. Note that $v_1 < K_{(f,p)}$ implies $p'(v) = p(v) \leq B$ by Lemma 8 in this region. The outcomes along the vertical line corresponding to $K_{(f,p)}$ value of the agent and the outcomes for all types such that $v_1 > K_{(f,p)}$ and $v_2 \leq K_{(f,p)}$ is assigned value

$$\left(f(K_{(f,p)}, 0) + \frac{1}{K_{(f,p)}}\big(B - p(K_{(f,p)}, 0)\big), B\right)$$

We prove the following.

Lemma **12** *If $(f, p) \in M^+$ is an IC and IR mechanism, then the mechanism (f', p') is IC, IR, and $p'(v) \geq p(v)$ for almost all v.*

Proof: First, we establish that $p'(v) \geq p(v)$ for **almost** all $v \in V$. To see this, note that if $v \in V$ is such that $v_2 < \beta$ and $\min(v_1, v_2) > K_{(f,p)}$, then Lemma 11 implies that $p'(v) = P_{(f,p)} \geq p(v)$. Next, if $v \in V$ such that $\min(v_1, v_2) < K_{(f,p)}$ and $v_1 \geq K_{(f,p)}$, then $p'(v) = B \geq p(v)$. For $v \in V$ with $v_1 < K_{(f,p)}$, we have $p'(v) = p(v)$. Hence, the only profiles where we cannot compare $p(v)$ and $p'(v)$ have Lebesgue measure zero. So, for almost all v, we have $p'(v) \geq p(v)$.

For IC, we consider three possible types belonging to the three regions: (a) s such that

$s_1 < K_{(f,p)}$; (b) t such that $t_1 > K_{(f,p)}, t_2 \leq K_{(f,p)}$ or $t_1 = K_{(f,p)}$; and (c) u such that $\min(u_1, u_2) > K_{(f,p)}$.

For any two types belonging to the same region we need not verify the incentive constraints in the new mechanism as it is either unchanged compared to original mechanism or they have the same outcome. We note couple of properties that we will use often in establishing these incentive constraints: (i) $(f(s), p(s)) = (f'(s), p'(s))$; (ii) $p(K_{(f,p)}, 0) \leq B$ by Lemma 8.

(1) $s \to t$. Note that $p(K_{(f,p)}, 0) \leq B$ and since $p(s) \leq B$, incentive constraint $s \to (K_{(f,p)}, 0)$ in (f,p) implies that

$$\begin{aligned} s_1 f(s) - p(s) &\geq s_1 f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) \\ &\geq s_1 f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) - \Big[B - p(K_{(f,p)}, 0)\Big]\Big(1 - \frac{s_1}{K_{(f,p)}}\Big), \end{aligned}$$

where the second inequality follows because $p(K_{(f,p)}, 0) \leq B$ and $s_1 < K_{(f,p)}$. Using $f(s) = f'(s)$, $p(s) = p'(s)$, and a slight rearrangement of RHS of the above inequality gives us

$$s_1 f'(s) - p'(s) \geq s_1 \Big[f(K_{(f,p)}, 0) + \frac{1}{K_{(f,p)}}\big(B - p(K_{(f,p)}, 0)\big)\Big] - B = s_1 f'(t) - p'(t). \tag{10}$$

Hence, $s \to t$ holds for (f', p').

(2) $t \to s$. Since $p(s) \leq B$, $(K_{(f,p)}, 0) \to s$ in (f,p) implies that

$$\begin{aligned} & K_{(f,p)} f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) \geq K_{(f,p)} f(s) - p(s) \\ \Rightarrow\ & K_{(f,p)} \Big[f(K_{(f,p)}, 0) + \frac{1}{K_{(f,p)}}\big(B - p(K_{(f,p)}, 0)\big)\Big] - B \geq K_{(f,p)} f(s) - p(s) \\ \Rightarrow\ & K_{(f,p)} f'(t) - p'(t) \geq K_{(f,p)} f'(s) - p'(s) \\ \Rightarrow\ & K_{(f,p)} \Big[f'(t) - f'(s)\Big] \geq p'(t) - p'(s). \end{aligned} \tag{11}$$

But $p'(t) = B \geq p'(s) = p(s)$ implies that $f'(t) \geq f'(s)$. Using the fact that $t_1 \geq K_{(f,p)}$, we get $t_1 \Big[f'(t) - f'(s)\Big] \geq p'(t) - p'(s)$. Since $p'(t) = B$ and $p'(s) \leq B$, this is the desired $t \to s$ in (f', p').

(3) $t \to u$, $u \to t$. By Lemma 10, we know that

$$\begin{aligned} K_{(f,p)} f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) &= K_{(f,p)} A_{(f,p)} - P_{(f,p)} \\ \Leftrightarrow K_{(f,p)} \Big[f(K_{(f,p)}, 0) - \frac{1}{K_{(f,p)}} \big(B - p(K_{(f,p)}, 0) \big) \Big] - B &= K_{(f,p)} A_{(f,p)} - P_{(f,p)}. \end{aligned}$$

Hence, we get

$$K_{(f,p)} \Big[f'(u) - f'(t) \Big] = p'(u) - p'(t). \tag{12}$$

Using Lemma 11, we know that $f'(u) > f'(t)$ and $p'(u) > p'(t)$. Using $\min(u_1, u_2) > K_{(f,p)}$, we get

$$u_1 f'(u) - p'(u) \geq u_1 f'(t) - p'(t) \quad \text{and} \quad u_2 f'(u) - p'(u) \geq u_2 f'(t) - p'(t).$$

Hence, $u \to t$ holds in (f', p').

Similarly, we now use the fact that $\min(t_1, t_2) \leq K_{(f,p)}$. If $\min(t_1, t_2) = t_1$, then using Equation 12, we get $t_1 f'(t) - p'(t) \geq t_1 f'(u) - p'(u)$. Else, $\min(t_1, t_2) = t_2$, in which case again, we get $t_2 f'(t) - p'(t) \geq t_2 f'(u) - p'(u)$. So, one of the above constraints must hold, which ensures that the $t \to u$ holds in (f', p').

(4) $s \to u$. Using the fact that $p'(u) > p'(t)$, $f'(u) > f'(t)$ and $s_1 \leq K_{(f,p)}$; rearranging terms in Equation 12 we get $s_1 f'(t) - p'(t) \geq s_1 f'(u) - p'(u)$. This combined with Equation 10 results in $s_1 f'(s) - p'(s) \geq s_1 f'(u) - p'(u)$, which is enough for $s \to u$ as $p'(s) = p(s) \leq B$.

(5) $u \to s$. Since $p'(u) > B$ and $p'(s) = p(s) \leq B$, we will need to show that

$$u_1 f'(u) - p'(u) \geq u_1 f'(s) - p'(s) \quad \text{and} \quad u_2 f'(u) - p'(u) \geq u_2 f'(s) - p'(s).$$

Combining Equations 11 and 12, we get $K_{(f,p)} \Big[f'(u) - f'(s) \Big] \geq p'(u) - p'(s)$. Using the fact that $p'(u) > p'(s)$, $f'(u) > f'(s)$ and $u_1, u_2 > K_{(f,p)}$ we arrive at the desired result.

Since $p(0,0) = p'(0,0)$ and (f,p) is IR and IC, Lemma 1 implies that (f', p') is IR. ■

A.2.3 Ironing Lemmas

The final Lemma before we start ironing, further simplifies the class of mechanisms that we need to consider for optimal mechanism design.

Lemma **13** *Suppose $(f, p) \in M^+$ is an IC and IR mechanism. Then, there exists another mechanism $(\hat{f}, \hat{p})$ such that*

1. $(\hat{f}(v), \hat{p}(v)) = (f(v), p(v))$ *for all* v *with* $v_1 \geq K_{(f,p)}$,
2. $(\hat{f}(v), \hat{p}(v)) = (\hat{f}(u), \hat{p}(u))$ *for all* u, v *with* $u_1 = v_1 < K_{(f,p)}$,
3. $\hat{p}(u) \geq p(u)$ *for all* u,
4. $\hat{p}(0, 0) = p(0, 0)$,
5. $u \to v$ *for every* u, v *with* $\hat{p}(u), \hat{p}(v) \leq B$ *holds in* $(\hat{f}, \hat{p})$.

Proof: Consider an IC and IR mechanism (f, p), and let $K_{(f,p)}$ be as defined in Lemma 8. We complete the proof in two steps.

Step 1. In this step, we show some implications of $u \to v$, where $u_1, v_1 < K_{(f,p)}$. Consider any $(u_1, u_2), (u_1, u_2')$ such that $u_1 < K_{(f,p)}$. Then, by Lemma 8, we have $p(u_1, u_2) \leq B$ and $p(u_1, u_2') \leq B$. Hence, the relevant pair of incentive constraints give us:

$$u_1 f(u_1, u_2) - p(u_1, u_2) = u_1 f(u_1, u_2') - p(u_1, u_2'). \tag{13}$$

Also, notice that Equation 13 implies that for all $u_2 \in [0, \beta]$,

$$p(0, u_2) = p(0, 0) \tag{14}$$

Finally, since only incentive constraints corresponding to agent's value are relevant in this region, revenue equivalence formula implies that for every $u_1 < K_{(f,p)}$ and $u_2, u_2' \in [0, \beta]$, we have

$$u_1 f(u_1, u_2) - p(u_1, u_2) = \int_0^{u_1} f(x, u_2) dx - p(0, u_2) = \int_0^{u_1} f(x, u_2) dx - p(0, 0)$$

$$u_1 f(u_1, u_2') - p(u_1, u_2') = \int_0^{u_1} f(x, u_2')dx - p(0, u_2) = \int_0^{u_1} f(x, u_2')dx - p(0, 0)$$

where the second equalities in each of the equations above are implied by Equation 14. Using Equation 13, we get

$$\int_0^{u_1} f(x, u_2)dx = \int_0^{u_1} f(x, u_2')dx.$$

Hence, we can write for every $u_1 < K_{(f,p)}$ and every $u_2 \in [0, \beta]$,

$$u_1 f(u_1, u_2) - p(u_1, u_2) = \int_0^{u_1} f(x, 0)dx - p(0, 0). \tag{15}$$

Notice that the RHS of the above equation is independent of u_2. Denoting the RHS of the above equation as $\mathcal{U}^{(f,p)}(u_1)$, we see that

$$u_1 \sup_{u_2 \in [0,\beta]} f(u_1, u_2) = \sup_{u_2 \in [0,\beta]} p(u_1, u_2) + \mathcal{U}^{(f,p)}(u_1). \tag{16}$$

Notice that f and p are bounded from above (p is bounded from above because $p(u_1, u_2) \leq B$ for each $u_2 \in [0, \beta]$ due to Lemma 8). As a result, the supremums in the above equation exist. We denote this supremums as follows:

$$\alpha(u_1) := \sup_{u_2 \in [0,\beta]} f(u_1, u_2) \ \forall \ u_1 < K_{(f,p)} \tag{17}$$

$$\pi(u_1) := \sup_{u_2 \in [0,\beta]} p(u_1, u_2) \ \forall \ u_1 < K_{(f,p)}. \tag{18}$$

We use these to define our new mechanism in the next step.

Step 2. Now, we define the following mechanism $(\hat{f}, \hat{p})$. For every v with $v_1 \geq K_{(f,p)}$, we have $(\hat{f}(v), \hat{p}(v)) = (f(v), p(v))$. For all v with $v_1 < K_{(f,p)}$, we define

$$\hat{f}(v) := \alpha(v_1); \hat{p}(v) := \pi(v_1).$$

By definition of $\hat{p}$, it is clear that $\hat{p}(v) \geq p(v)$ for all v. Also, Equation 14 ensures that $\hat{p}(0, 0) = \pi(0) = p(0, 0)$. Hence, (1), (2), (3), (4) hold for $(\hat{f}, \hat{p})$.

For (5), assume for contradiction that $u \to v$ in $(\hat{f}, \hat{p})$ does not hold for some u, v with $\hat{p}(u), \hat{p}(v) \leq B$. By definition of $\hat{p}$, we must have $p(u) \leq B$ and $p(v) \leq B$. Also, incentive constraints cannot be violated if $u_1, v_1 \geq K_{(f,p)}$ since (f, p) is IC and $(\hat{f}(u), \hat{p}(u)) = (f(u), p(u))$

and $(\hat{f}(v), \hat{p}(v)) = (f(v), p(v))$. The other possibilities are analyzed below.

CASE 1. $u_1, v_1 < K_{(f,p)}$. In that case, violation of $u \to v$ implies

$$u_1\alpha(u_1) - \pi(u_1) < u_1\alpha(v_1) - \pi(v_1) = (u_1 - v_1)\alpha(v_1) + v_1\alpha(v_1) - \pi(v_1).$$

Using Equation (16), we get that $\mathcal{U}^{(f,p)}(u_1) < \mathcal{U}^{(f,p)}(v_1) + (u_1 - v_1)\alpha(v_1)$. By definition, there exists, $y \in [0, \beta]$ such that $\alpha(v_1)$ is arbitrarily close to $f(v_1, y)$. Using Equation (15) gives us

$$u_1 f(u_1, y) - p(u_1, y) < v_1 f(v_1, y) - p(v_1, y) + (u_1 - v_1) f(v_1, y) = u_1 f(v_1, y) - p(v_1, y).$$

This means $u \to v$ is violated for (f, p), a contradiction.

CASE 2. $u_1 < K_{(f,p)}$ and $v_1 \geq K_{(f,p)}$. In that case, we must have $u_1\alpha(u_1) - \pi(u_1) < u_1 f(v) - p(v)$. But using Equations (15) and (16), we see that there is some y such that $u_1 f(u_1, y) - p(u_1, y) < u_1 f(v) - p(v)$ which contradicts IC of (f, p).

CASE 3. $u_1 \geq K_{(f,p)}$ and $v_1 < K_{(f,p)}$. In that case, we must have

$$u_1 f(u) - p(u) < u_1\alpha(v_1) - \pi(v_1) = (u_1 - v_1)\alpha(v_1) + \mathcal{U}^{(f,p)}(v_1).$$

Now, pick y such that $\alpha(v_1)$ is arbitrarily close to $f(v_1, y)$. By Equations (15) and (16):

$$u_1 f(u) - p(u) < (u_1 - v_1) f(v_1, y) + v_1 f(v_1, y) - p(v_1, y) = u_1 f(v_1, y) - p(v_1, y).$$

This contradicts IC of (f, p) and completes the proof. ■

DEFINITION **5** *We call a mechanism (f, p)* **simple** *if there exists $K, A, \hat{A}, P$ with $K \in (0, B)$, $P \in (B, \beta]$, $A, \hat{A} \in [0, 1]$, $A > \hat{A}$ such that*

1. $p(0, 0) \leq 0$.

2. $K(A - \hat{A}) = P - B$ *with* $KA - P \geq 0$.

3. $(f(v), p(v)) = (A, P)$ *for all* v *with* $\min(v_1, v_2) > K$*,*

4. $p(v) \leq B$ *for all* v *with* $v_1 < K$*.*

5. $(f(v), p(v)) = (\hat{A}, B)$ *for all* v *with* $\min(v_1, v_2) \leq K$ *and* $v_1 \geq K$*.*

6. $(f(v), p(v)) = (f(v'), p(v'))$ *for all* v, v' *with* $v_1 = v'_1 < K$*.*

7. $v \to v'$ *hold for all types with* $p(v), p(v') \leq B$*.*

Based on Lemmas 12 and 13, the following is a simple corollary.

Corollary **1** *If* (f, p) *is an optimal mechanism in* M^+*, then there is a simple mechanism* $(\hat{f}, \hat{p})$ *such that* $\text{Rev}(f, p) \leq \text{Rev}(\hat{f}, \hat{p})$.

Proof: Suppose (f, p) is an optimal mechanism in M^+, then Lemma 12 says that there is another IC and IR mechanism (f', p') such that $\text{Rev}(f', p') \geq \text{Rev}(f, p)$. Using $K = K_{(f,p)}$, Lemma 13 shows that (f', p') satisfies all the properties of a simple mechanism. ■

Because of property (6), for any simple mechanism (f, p), we denote the allocation probability at any type v with $v_1 < K$ as simply $\alpha^f(v_1)$ and the payment as $\pi^p(v_1)$. We also denote by $\alpha^f(K) \equiv \hat{A}$ and $\pi^p(K) \equiv B$, where $\hat{A}$ is the parameter specified in the simple mechanism (f, p).

Lemma **14** *Suppose* (f, p) *is a simple mechanism with parameters* $(K, A, \hat{A}, P)$*. Then, the revenue from* (f, p) *is*

$$\begin{aligned} \text{Rev}(f, p) = G_1(K)\Big[B - K\alpha^f(K)\Big] + \int_0^K h(x)\alpha^f(x)dx \\ + B(1 - G_1(K)) + K(A - \alpha^f(K))(1 - G_1(K) - G_2(K) + G(K, K)), \end{aligned}$$

where $h(x) = xg_1(x) + G_1(x)$ *for all* $x \in [0, K]$*.*

Proof: Fix a simple mechanism with parameters $(K, A, \hat{A}, P)$. We divide the proof into two parts, where we compute revenue from two disjoint regions of the type space.

Region 1. Here, we consider all v such that $v_1 \le K$. By properties (4) and (5) of the simple mechanism, payments in this region of type space is not more than B and by property (7), all the incentive constraints in this region hold. Using standard Myersonian techniques, it is easy to see that

$$\alpha^f(v_1) \ge \alpha^f(v_1') \qquad \forall\ v_1' < v_1 \le K \tag{19}$$

$$\pi^p(v_1) = \pi^p(0) + v_1\alpha^f(v_1) - \int_0^{v_1} \alpha^f(x)dx \qquad \forall\ v_1 \le K \tag{20}$$

Hence, the expected payment from this region is

$$\begin{aligned}
\int_0^K \pi^p(v_1)g_1(v_1)dv_1 &= \int_0^K \pi^p(0)g_1(v_1)dv_1 + \int_0^K v_1\alpha^f(v_1)g_1(v_1)dv_1 - \int_0^K \big(\int_0^{v_1} \alpha^f(x)dx\big)g_1(v_1)dv_1 \\
&= G_1(K)\pi^p(0) + \int_0^K v_1\alpha^f(v_1)g_1(v_1)dv_1 - \int_0^K \big((G_1(K) - G_1(v_1)\big)\alpha^f(v_1)dv_1 \\
&= G_1(K)\big[\pi^p(0) - \int_0^K \alpha^f(x)dx\big] + \int_0^K h(x)\alpha^f(x)dx \\
&= G_1(K)\big[\pi^p(K) - K\alpha^f(K)\big] + \int_0^K h(x)\alpha^f(x)dx \\
&= G_1(K)\big[B - K\alpha^f(K)\big] + \int_0^K h(x)\alpha^f(x)dx,
\end{aligned}$$

where the last but one equality follows from Equation 20 at $v_1 = K$ and the last equality follows from the fact $\pi^p(K) = B$.

Region 2. Finally, we consider all v such that $v_1 > K$. By definition, the expected revenue from this region is

$$\begin{aligned}
&B(1 - G_1(K)) + (P - B)(1 - G_1(K) - G_2(K) + G(K,K)) \\
&= B(1 - G_1(K)) + K(A - \alpha^f(K))(1 - G_1(K) - G_2(K) + G(K,K)),
\end{aligned}$$

where the equality follows from property (2) of simple mechanism.

Putting together the revenues from both the regions, we get the desired expression of the expected revenue from the simple mechanism. ■

We now prove that for every simple mechanism, there is a post-2 mechanism that generates as much expected revenue.

LEMMA **15** *For every simple mechanism* (f,p)*, there is a* POST-2 *mechanism* $(\bar{f},\bar{p})$ *such that* $\text{REV}(\bar{f},\bar{p}) \geq \text{REV}(f,p)$.

Proof: Suppose (f,p) is a simple mechanism with parameters $(K, A, \hat{A}, P)$. Now, by property (5) of the simple mechanism, Equation 20 along with property (1) imply that

$$\pi^f(K) = B \leq K\alpha^f(K) - \int_0^K \alpha^f(x)dx. \tag{21}$$

Now, define a POST-2 mechanism by parameters: $K_1 := \frac{B}{\hat{A}} = \frac{B}{\alpha^f(K)}$, $K_2 := K$. By property (1) of simple mechanism, we get that $K_1 = \frac{B}{\alpha^f(K)} \leq K_2 = K$. Also, $K_1 > B$. This means that the new mechanism is a well-defined POST-2 mechanism. Denote this mechanism as (f', p').

It is also easily verified that it is a simple mechanism: the parameters are $K' := K_2 = K; A' = 1; \hat{A}' := \hat{A} = \alpha^f(K); P' := B + K_2(1 - \frac{B}{K_1}) = B + K(1 - \alpha^f(K))$, and also note that every POST-2 mechanism is IC (Proposition 1). Note here that $\alpha^{f'}(K) = \alpha^f(K)$. Also, $\alpha^{f'}(x) = 0$ for all $x \leq K_1$ and $\alpha^{f'}(x) = \frac{B}{K_1} = \alpha^f(K)$ for all $x \in (K_1, K]$. Using these observations and Lemma 14,

$$\begin{aligned}
&\text{REV}(f',p') - \text{REV}(f,p) \\
&= \Bigg(G_1(K)\Big[B - K\alpha^f(K)\Big] + \int_0^K h(x)\alpha^{f'}(x)dx + B(1 - G_1(K)) + \\
&K(1 - \alpha^f(K))(1 - G_1(K) - G_2(K) + G(K,K)) \Bigg) \\
&- \Bigg(G_1(K)\Big[B - K\alpha^f(K)\Big] + \int_0^K h(x)\alpha^f(x)dx + B(1 - G_1(K)) + \\
&K(A - \alpha^f(K))(1 - G_1(K) - G_2(K) + G(K,K)) \Bigg)
\end{aligned}$$

$$\geq \int_0^K h(x)\alpha^{f'}(x)dx - \int_0^K h(x)\alpha^f(x)dx$$

$$\geq \int_{K_1}^K h(x)\big(\alpha^f(K) - \alpha^f(x)\big)dx - \int_0^{K_1} h(x)\alpha^f(x)dx.$$

$$\geq (K - K_1)h(K_1)\alpha^f(K) - h(K_1)\int_{K_1}^K \alpha^f(x)dx - h(K_1)\int_0^{K_1} \alpha^f(x)dx$$

(using h and α^f to be increasing functions)

$$= (K - K_1)h(K_1)\alpha^f(K) - h(K_1)\int_0^K \alpha^f(x)dx$$

$$\geq h(K_1)(K - K_1)\alpha^f(K) - h(K_1)(K - K_1)\alpha^f(K)$$

(using Equation (21) and definition of K_1)

$$= 0.$$

■

A.2.4 Proof of Proposition 4

The proof of (2) in Proposition 4 now follows from Corollary 1 and Lemma 15. Proof of (1) in Proposition 4 is given below.

This requires to show that the optimal mechanism in M^- is a POST-1 mechanism. Every mechanism $(f, p) \in M^-$ satisfies the property that types satisfying $p(v) > B$ have zero measure. We first argue that it is without loss of generality to assume that $p(v) \leq B$ for *all* v. To see this, note that by (1) in Lemma 8 and the fact that $V^+(f, p)$ has zero measure, it must be that $K_{(f,p)} = \beta$. Let $\pi^p(\beta) := \sup_{v_2<\beta} p(\beta, v_2)$ and $\alpha^f(\beta) := \sup_{v_2<\beta} f(\beta, v_2)$. Observe that $\alpha^p(\beta) \leq B$. Hence, we consider the following mechanism (f', p'): $(f'(v), p'(v)) = (f(v), p(v))$ if $v \notin V^+(f, p)$ and $(f'(v), p'(v)) = (\alpha^f(\beta), \pi^p(\beta))$ otherwise. By construction, the expected revenue of (f', p') is the same as (f, p) and $p'(v) \leq B$ for all v. Further, (f', p') is IC (we only need to worry about incentive constraints of types $v \in V^+(f, p)$, and they hold because for all v, $p'(v) \leq B$ implies we only need to check incentive constraints for value of agent, which holds due to an argument similar to that in Lemma 13(5)). IR of (f', p') follows from Lemma 1.

Now, we state an analogue of Lemma 13 for M^- class of mechanisms - the proof of this lemma is identical to that of Lemma 13, and is skipped.

LEMMA **16** *Suppose $(f,p) \in M^-$ is an IC and IR mechanism. Then, there exists another mechanism $(\hat{f},\hat{p})$ such that*

1. $(\hat{f}(v),\hat{p}(v)) = (\hat{f}(u),\hat{p}(u))$ *for all u,v with $u_1 = v_1$,*

2. $\hat{p}(u) \geq p(u)$ *for all u,*

3. $\hat{p}(0,0) = p(0,0)$,

4. $(\hat{f},\hat{p})$ *is IC and IR.*

Using Lemma 16, we only focus on mechanisms satisfying the properties stated in Lemma 16. Let (f,p) be such a mechanism and define α^f and π^p as before, i.e., $\alpha^f(v_1) = f(v_1,v_2)$ and $\pi^p(v_1) = p(v_1,v_2)$ for all v with $v_1 < \beta$.

Hence, the expected revenue from a mechanism (f,p) given in Lemma 16 is given by

$$\begin{aligned}
\textsc{Rev}(f,p) &= p(0,0) + \int_0^\beta u_1\alpha^f(u_1)g_1(u_1)du_1 - \int_0^\beta \Big(\int_0^{u_1}\alpha^f(x)dx\Big)g_1(u_1)du_1 \\
&= p(0,0) + \int_0^\beta x\alpha^f(x)g_1(x)dx - \int_0^\beta (1-G_1(x))\alpha^f(x)dx \\
&= p(0,0) + \int_0^\beta \Big[h(x)-1\Big]\alpha^f(x)dx.
\end{aligned}$$

We now construct another posted-price mechanism (f',p') that generates no less revenue than (f,p). The posted-price mechanism (f',p') is defined as follows. Let $K_1 := \frac{\pi^f(\beta)}{\alpha^f(\beta)}$. For all v with $v_1 \leq K_1$, we set: $f'(v) := 0, p'(v) := 0$ and for all v with $v_1 > K_1$, we set: $f'(v) := \alpha^f(\beta),\ \ p'(v) := K_1\alpha^f(\beta) = \pi^p(\beta)$.

It is not difficult to see that (f',p') is IR and IC. The expected revenue from (f',p') is given by $\textsc{Rev}(f',p') = K_1\alpha^f(\beta)(1-G_1(K_1))$. Now, note that

$$\alpha^f(\beta)\int_{K_1}^\beta \Big[h(x)-1\Big]dx = \alpha^f(\beta)\Big(K_1 - K_1G_1(K_1)\Big) = \textsc{Rev}(f',p').$$

So, we get

$$
\begin{aligned}
&\textsc{Rev}(f',p') - \textsc{Rev}(f,p) \\
&= \left(\alpha^f(\beta)\int_{K_1}^{\beta}\Big[h(x)-1\Big]dx\right) - \left(p(0,0) + \int_0^{\beta}\Big[h(x)-1\Big]\alpha^f(x)dx\right) \\
&= \alpha^f(\beta)\int_{K_1}^{\beta} h(x)dx - \int_0^{\beta} h(x)\alpha^f(x)dx + \int_0^{\beta}\alpha^f(x)dx - (\beta - K_1)\alpha^f(\beta) - p(0,0) \\
&= \alpha^f(\beta)\int_{K_1}^{\beta} h(x)dx - \int_0^{\beta} h(x)\alpha^f(x)dx + \int_0^{\beta}\alpha^f(x)dx - \beta\alpha^f(\beta) - \pi^p(\beta) - p(0,0) \\
&\text{(Using definition of } K_1\text{)} \\
&= \alpha^f(\beta)\int_{K_1}^{\beta} h(x)dx - \int_0^{\beta} h(x)\alpha^f(x)dx \\
&\text{(Using revenue equivalence formula (Equation 20) at } \beta\text{)} \\
&= \int_{K_1}^{\beta}\Big[\alpha^f(\beta)-\alpha^f(x)\Big]h(x)dx - \int_0^{K_1}\alpha^f(x)h(x)dx \\
&\geq h(K_1)\int_{K_1}^{\beta}\Big[\alpha^f(\beta)-\alpha^f(x)\Big]dx - h(K_1)\int_0^{K_1}\alpha^f(x)dx \\
&\text{(since } h \text{ is increasing and } \alpha^f \text{ is non-decreasing)} \\
&= h(K_1)(\beta-K_1)\alpha^f(\beta) - h(K_1)\int_0^{\beta}\alpha^f(x)dx \\
&\geq h(K_1)(\beta-K_1)\alpha^f(\beta) - h(K_1)(\beta-K_1)\alpha^f(\beta) \\
&= 0,
\end{aligned}
$$

where the last inequality follows from revenue equivalence formula (Equation 20) at β and $p(0,0) \leq 0$). Hence, every optimal mechanism in M^- is a posted-price mechanism described in (f',p'). It is characterized by a posted-price K_1 and an allocation probability α if the value of the agent is above the posted price. The optimization program can be written as follows.

$$
\max_{K_1,\alpha} K_1\alpha(1-G_1(K_1))
$$
$$
\text{subject to } \; K_1\alpha \leq B, \; \alpha \in [0,1].
$$

We argue that the optimal solution to this program must have $\alpha = 1$ and that will imply that the optimal solution is a POST-1 mechanism. To see this, let K^* be the unique solution to the following optimization $\max_{K_1 \in [0,B]} K_1(1 - G_1(K_1))$. The fact that this optimization program has a unique solution follows from the fact that $x - xG_1(x)$ is strictly concave (since $xG_1(x)$ is strictly convex). Hence, the revenue from the solution when $\alpha = 1$ is $K^*(1 - G_1(K^*))$. Now, suppose the optimal solution has $\hat{K}$ and $\hat{\alpha}$. Note that the $\hat{K}\hat{\alpha} \leq B$. So, define $\tilde{K} = \hat{K}\hat{\alpha} \leq B$. By definition,

$$K^*(1 - G_1(K^*)) \geq \tilde{K}(1 - G_1(\tilde{K})) = \hat{K}\hat{\alpha}(1 - G_1(\hat{K}\hat{\alpha})) \geq \hat{K}\hat{\alpha}(1 - G_1(\hat{K})),$$

where the final inequality used the fact that $G_1(\hat{K}\hat{\alpha}) \leq G_1(\hat{K})$. This implies that the optimal solution must have $\alpha = 1$.

A.2.5 Proof of Proposition 2

We now combine the optimal solutions in M^+ and M^- as follows. The optimal in M^- is a solution to

$$\max_{K_1 \in [0,B]} K_1(1 - G_1(K_1)).$$

The optimal in M^+ is a solution to

$$\max_{K_2 \in (B,\beta), K_1 \in [B,K_2]} B\Big[1 - G_1(K_1)\Big] + K_2\big(1 - \frac{B}{K_1}\big)\Big[1 - G_1(K_2) - G_2(K_2) + G(K_2, K_2)\Big].$$

Notice that the optimization for M^+ does not admit $K_2 = B$. But if $K_2 = B$ and $K_1 \in [B, K]$, we must have $K_1 = B$ and then the objective function value reduces to $B(1 - G_1(B))$. This is the same objective function value of the program for M^- when $K_1 = B$. Similarly, if $K_2 = \beta$ is allowed in the optimization for M^+, we see that the objective function is maximized at $K_1 = B$ giving a value of $B(1 - G_1(B))$ to the objective function. Again, this is the same objective function value of the program for M^- when $K_1 = B$.

Summarizing these findings, we get that the expected revenue from the optimal mechanism is $\max(R_1, R_2)$, where

$$R_1 = \max_{K_1 \in [0,B]} K_1(1 - G_1(K_1))$$

$$R_2 = \max_{K_2 \in [B,\beta], K_1 \in [B,K_2]} B\Big[1 - G_1(K_1)\Big] + K_2\big(1 - \frac{B}{K_1}\big)\Big[1 - G_1(K_2) - G_2(K_2) + G(K_2, K_2)\Big].$$

This proves Proposition 2.

A.3 Proof of Lemma 2

Proof: Suppose (K_1^*, K_2^*) are values of (K_1, K_2) in the optimal POST-2 mechanism. By definition $K_1^* \leq K_2^*$. Using the uniform distribution of G, we see that (K_1^*, K_2^*) are optimal solutions to the following optimization problem:

$$\max_{K_2 \in [B,1],\ K_1 \in [B,K_2]} \ B\Big[1 - K_1\Big] + \big(1 - \frac{B}{K_1}\big)K_2(1 - K_2)^2. \tag{22}$$

We consider the following optimization problem, where we fix the value of K_1^* and maximize over all K_2:

$$\max_{K_2 \in [0,1]} \ B\Big[1 - K_1^*\Big] + \big(1 - \frac{B}{K_1^*}\big)K_2(1 - K_2)^2.$$

Notice that the objective function is strictly concave in K_2, and the unique maximum occurs when $K_2 = \frac{1}{3}$.

Now, assume for contradiction $K_1^* < K_2^*$. We consider two cases and reach a contradiction in both the cases.

CASE 1. Suppose $K_1^* \geq \frac{1}{3}$. Then, $K_2^* > \frac{1}{3}$. But $K_2 = K_1^*$ and K_1^* defines a feasible POST-2 mechanism, and generates more revenue. This is a contradiction.

CASE 2. Suppose $K_1^* < \frac{1}{3}$. Since $K_2^* \geq K_1^*$, we see that $K_2 = \frac{1}{3}$ and K_1^* defines a feasible POST-2 mechanism and generates more revenue. Hence, K_2^* must be equal to $\frac{1}{3}$. Now, fixing the value of K_2 at $\frac{1}{3}$, we optimize the Expression (22) with relaxed constraints on K_1:

$$\max_{K_1 \in [0,1]} \ B\Big[1 - K_1\Big] + \big(1 - \frac{B}{K_1}\big)\frac{4}{27}.$$

This objective function is strictly concave with a unique maxima at $K_1 = \frac{2}{3\sqrt{3}} > \frac{1}{3}$. Hence, the objective function of the Expression in (22) is higher at $K_1 = \frac{1}{3} = K_2^*$ than at (K_1^*, K_2^*) with $K_1^* < \frac{1}{3}$. Further, $K_1 = K_2 = \frac{1}{3}$ is a POST-2 mechanism since (K_1^*, K_2^*) with $K_2^* = \frac{1}{3}$ is a POST-2 mechanism. This is a contradiction.

Using this, we can conclude that the optimal POST-2 mechanism is a solution to the following single-variable constrained optimization problem.

$$\max_{K \in [B,1]} \; B\big(1-K\big) + \big(K-B\big)(1-K)^2. \tag{23}$$

We denote $J(K) := B\big(1-K\big) + \big(K-B\big)(1-K)^2$ for all K. Notice that

$$J'(K) = 3K^2 - K(2B+4) + (B+1) \quad \text{and} \quad J''(K) = 6K - (2B+4).$$

Hence, we get $J'(B) = B^2 - 3B + 1 = \Big(B - \frac{3-\sqrt{5}}{2}\Big)\Big(B - \frac{3+\sqrt{5}}{2}\Big)$. Thus, $J'(B) \leq 0$ if and only if $B \geq \frac{1}{2}\big(3-\sqrt{5}\big)$. But, $J''(K) = 0$ for $K = \frac{1}{3}(B+2)$. Hence, $J'(K)$ is decreasing in $[B, \frac{1}{3}(B+2)]$ and increasing in $[\frac{1}{3}(B+2), 1]$. Also, $J'(1) = -B < 0$. Hence, if $J'(B) \leq 0$, we must have $J'(K) < 0$ for all $K \in (B, 1]$.

Proof of (1). This implies that for $B \geq \frac{1}{2}\big(3-\sqrt{5}\big)$, we have $J'(K) < 0$ for all $K \in (B,1]$. This implies that J is decreasing in $[B,1]$, and hence, the optimal solution of Optimization (23) must have $K = B$. Then, the first part implies that the optimal POST-2 mechanism must have $K_1^* = K_2^* = B$.

Proof of (2). If $B < \frac{1}{2}\big(3-\sqrt{5}\big)$, then $J'(B) > 0$ and $J'(K) = 0$ at a unique point $K = \frac{1}{3}\big(B+2-\sqrt{(B^2+B+1)}\big)$. Denote this point of inflection as $\tilde{K}$. Notice that $J'(K) < 0$ for all $K > \tilde{K}$, and, hence, J is decreasing after $\tilde{K}$. Further, $\tilde{K} < \frac{1}{3}(B+2)$ and $J''(K) < 0$ for all $K < \tilde{K}$. This means J is strictly concave from B to $\frac{1}{3}(B+2)$. Combining these observations, we conclude that $K = \tilde{K}$ solves the Optimization in (23). The first part implies that the optimal POST-2 mechanism must have $K_1^* = K_2^* = \frac{1}{3}\big(B+2-\sqrt{(B^2+B+1)}\big)$, if $B < \frac{1}{2}\big(3-\sqrt{5}\big)$. ■

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