Supplement to "Balanced Ranking Mechanisms"

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June 17, 2017

This supplementary material contains some missing proofs of Long et al. (2017). The numbering of Propositions and Lemmas are same as that in Long et al. (2017).

Proof of Proposition 2

Proof: We only focus on $n > 8$. Notice that value of ℓ in Theorem 1 in Long et al. (2017) is obtained by choosing the value of i for which i is even and $\frac{(i-1)}{C(n-2,i-1)+i}$ is minimized. But minimizing $\frac{(i-1)}{C(n-2,i-1)+i}$ is equivalent to maximizing

$$\frac{C(n-2,i-1)+1}{(i-1)}.$$

We now prove an elementary fact from combinatorics.

FACT **1** *If $n \geq 8$ and $4 \leq k \leq \frac{n-1}{2}$, then*

$$\frac{C(n-2,k-1)+1}{k-1} \geq \frac{C(n-2,k-2)+1}{k-2}.$$

Proof:

$$\begin{aligned}&\frac{C(n-2,k-1)+1}{k-1} - \frac{C(n-2,k-2)+1}{k-2}\\ &= \frac{1}{(k-1)(k-2)}\Big((k-2)C(n-2,k-1) - (k-1)C(n-2,k-2) - 1\Big)\end{aligned}$$

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Hence, to show the above expression is non-negative, we need to show that the expression below is no less than 1:

$$\begin{aligned}
&(k-2)C(n-2,k-1)-(k-1)C(n-2,k-2)\\
&=\frac{(k-2)(n-k)}{(k-1)}C(n-2,k-2)-(k-1)C(n-2,k-2)\\
&=\frac{1}{(k-1)}C(n-2,k-2)\Big((n-k)(k-2)-(k-1)^2\Big).
\end{aligned}$$

Since $k \le \frac{(n-1)}{2}$, we have $(n-k) \ge (k+1)$. Then the above expression is greater than or equal to

$$\frac{1}{(k-1)}C(n-2,k-2)\Big((k+1)(k-2)-(k-1)^2\Big).$$

But $(k+1)(k-2)-(k-1)^2 = k^2-k-2-k^2+2k-1 = k-3 \ge 1$ since $k \ge 4$. This means that

$$\begin{aligned}
(k-2)C(n-2,k-1)-(k-1)C(n-2,k-2) &\ge \frac{1}{(k-1)}C(n-2,k-2)\\
&= \frac{1}{n-1}C(n-1,k-1)\\
&\ge 1,
\end{aligned}$$

as desired. ■

Fact 1 implies that if $n > 8$, then $\ell \ge \lfloor\frac{(n-1)}{2}\rfloor_e$. Next we show that the maximum of the expression $\frac{C(n-2,i-1)+1}{(i-1)}$ is achieved for $i \le \lfloor\frac{(n+1)}{2}\rfloor_e$. To see this, pick an even number $k > \lfloor\frac{(n+1)}{2}\rfloor_e$. Note that since k is even, we get that $2k > (n+1)$. We consider two cases.

Case 1. n is even. But $2k > n+1$ implies $n-k-1 < n-k < k-1$. Then, $\frac{C(n-2,k-1)+1}{k-1} = \frac{C(n-2,n-k-1)+1}{(k-1)} < \frac{C(n-2,n-k-1)+1}{(n-k-1)}$. Since $(n-k)$ is even, we see that the expression $\frac{C(n-2,i-1)+i}{i-1}$ cannot be maximized at k.

Case 2. n is odd. The maximum of the expression $C(n-2,i-1)$ is found at two values: $i^*-1 = \frac{n-1}{2}$ and $i^*-1 = \frac{n-1}{2}-1$. Since $k > \frac{n+1}{2}$, we get $k-1 > \frac{n-1}{2}$. This implies that $C(n-2,k-1) < C(n-2,k-2) = C(n-2,n-k)$. But then, $k-1 > n-k$ implies that $\frac{C(n-2,k-1)+1}{k-1} < \frac{C(n-2,n-k)+1}{(n-k)}$. Since $n-k+1$ is even, this implies that k does not maximize the required expression. ■

Proof of Proposition 3

Proof: Consider n which is even such that $\frac{n}{2}$ is odd. Then, by Proposition 2 in Long et al. (2017), $\ell = \frac{n}{2} - 1$. As a result,

$$h(n) = \frac{(n-4)}{2(C(n-2, \frac{n}{2}-2) + \frac{(n-2)}{2})} = \frac{(n-4)}{(2C(n-2, \frac{n}{2}-2) + (n-2))}.$$

But observe that

$$C(n-2, \frac{n}{2}-2) = \frac{(n-2)!}{(\frac{n}{2})!(\frac{n}{2}-2)!} = \frac{(\frac{n}{2}-1)(\frac{n}{2})}{(n-1)n} C(n, \frac{n}{2}) = \frac{(n-2)}{4(n-1)} C(n, \frac{n}{2}).$$

Hence, we can write

$$\begin{aligned} h(n) &= \frac{(n-4)}{\frac{(n-2)}{2(n-1)} C(n, \frac{n}{2}) + (n-2)} \\ &= \big(1 - \frac{2}{n-2}\big) \frac{1}{\frac{1}{2(n-1)} C(n, \frac{n}{2}) + 1}. \end{aligned}$$

Now, define $\rho(n) = \frac{1}{\sqrt{2\pi n}} 2^{n+1}$. Note that by Stirling's approximation of central binomial coefficient (Eger, 2014), we have

$$\lim_{n\to\infty} \frac{C(n, \frac{n}{2})}{\rho(n)} = 1. \tag{1}$$

Now, using the previous equation, we can write

$$h(n) = \big(1 - \frac{2}{n-2}\big) \frac{1}{\frac{2^n}{\sqrt{2\pi n}(n-1)} \frac{C(n, \frac{n}{2})}{\rho(n)} + 1}$$

Define $\sigma(n) = \frac{\sqrt{2\pi n}(n-1)}{2^n}$, and note that

$$\lim_{n\to\infty} \sigma(n) = 0. \tag{2}$$

Now, we can rewrite the expression of $h(n)$ as

$$\frac{h(n)}{\sigma(n)} = \big(1 - \frac{2}{n-2}\big) \frac{1}{\frac{C(n, \frac{n}{2})}{\rho(n)} + \sigma(n)}$$

So, as $n \to \infty$ (by considering sequence where n is even $\frac{n}{2}$ is odd), we see that the first term of RHS is 1 and the denominator of the second term in the RHS is 1 because of Equations (1) and (2). Hence, we get,

$$\lim_{n\to\infty} \frac{h(n)}{\sigma(n)} = 1.$$

∎

Proofs of Lemma 4 and Lemma 5

Both the proofs use the following simple lemma.

Lemma 1 *Suppose f is a satisfactorily implementable two-step ranking allocation rule defined by (π_1, ℓ). Then, for every 0-generic valuation profile $\mathbf{v}$, we have*

$$R^f(\mathbf{v}) = (\pi_1 - \pi_2)v_{(2)} + \ell\pi_2 v_{(\ell+1)},$$

where $\pi_2 = \frac{1}{\ell-1}(1-\pi_1)$.

Proof: The proof of the formula for R^f follows from the formula derived for any satisfactorily implementable ranking allocation rule in Lemma 3 in Long et al. (2017). ■

Proof of Lemma 4

Proof: Pick a satisfactory mechanism $(f, \mathbf{p})$, where f is a two-step ranking allocation rule defined by (π_1, ℓ). Suppose $\mathbf{v}$ is such that $|N^0_{\mathbf{v}}| = n-K$, $K \leq \ell$. If $K = 0$, then by symmetry and budget-balance, we get $p_i(\mathbf{v}) = 0$ for all $i \in N$. Else, suppose $v_1 > \ldots > v_K > 0$. If $K = 1$, then, by budget-balance and symmetry we get $p_1(\mathbf{v}) + (n-1)p_i(\mathbf{v}) = 0$ for any $i \in N^0_{\mathbf{v}}$. But $p_1(\mathbf{v}) = p_1(0, v_{-1}) + v_1\pi_1 - v_1\pi_1 = p_1(0, v_{-1}) = 0$, where we used revenue equivalence formula for the first equality and $p_1(0, v_{-1}) = 0$ for the last equality. Hence, we get $p_1(\mathbf{v}) = 0$, and hence, $p_i(\mathbf{v}) = 0$ for all $i \neq 1$. Now, suppose $K = 2$. Then, budget-balance requires

$$p_1(\mathbf{v}) + p_2(\mathbf{v}) + \sum_{i \notin \{1,2\}} p_i(\mathbf{v}) = 0.$$

But using revenue equivalence and the fact that $p_1(0, v_{-1}) = 0$, we get that

$$p_1(\mathbf{v}) = p_1(0, v_{-1}) + v_1\pi_1 - (v_1 - v_2)\pi_1 - v_2\pi_2 = v_2(\pi_1 - \pi_2).$$

Similarly, we get $p_2(\mathbf{v}) = p_2(0, v_{-2}) + v_2\pi_2 - v_2\pi_2 = 0$. Hence, by choosing some $i \notin \{1, 2\}$, we can simplify the budget-balance equation as $v_2(\pi_1 - \pi_2) + (n-2)p_i(\mathbf{v}) = 0$. This implies that

$$p_i(\mathbf{v}) = -\frac{(\pi_1 - \pi_2)}{(n-2)}v_2,$$

which is the required expression.

Next, suppose $K > 2$ and use induction. Suppose the claim is true for all $k < K$. Then, by revenue equivalence and symmetry we get

$$\sum_{j \in N} p_j(\mathbf{v}) = \sum_{j \in N} p_j(0, v_{-j}) + R^f(\mathbf{v}) = (n-K)p_i(\mathbf{v}) + \sum_{j=1}^{K} p_j(0, v_{-j}) + R^f(\mathbf{v}),$$

where i is some agent in $N^0_{\mathbf{v}}$. By budget-balance, the above summation is zero, and $R^f(\mathbf{v}) = (\pi_1 - \pi_2)v_2$ since $K \leq \ell$ (by Lemma 1). Using this, we get

$$0 = (n-K)p_i(\mathbf{v}) + \sum_{j=1}^{K} p_j(0, v_{-j}) + (\pi_1 - \pi_2)v_2. \tag{3}$$

Now, for every $j \in \{1, \ldots, K\}$, the profile $(0, v_{-j})$ has one more zero-valued agent than the profile $\mathbf{v}$, and hence, we can apply our induction hypothesis. We refer to $(0, v_{-j})$ for each $j \in \{1, \ldots, K\}$ as a **marginal** profile having an additional zero-valuation agent than $\mathbf{v}$, and denote this as $\mathbf{v}^j$ with the valuation of the k-th ranked agent in this valuation profile denoted as $v^j_{(k)}$. Note that a marginal profile contains $(K-1)$ non-zero valuation agents. Thus, using our induction hypothesis, Equation 3 can be rewritten as

$$\begin{aligned}
&(n-K)p_i(\mathbf{v}) \\
&= \sum_{j=1}^{K} \frac{(\pi_1 - \pi_2)}{\psi(n-K+1, n-2)} \Big[\sum_{k=2}^{K-2} (-1)^k (k-1)! \psi(n-K+1, n-k-1) v^j_{(k)} + (-1)^{K-1}(K-2)! v^j_{(K-1)} \Big] \\
&- (\pi_1 - \pi_2)v_2 \\
&= \frac{(\pi_1 - \pi_2)}{\psi(n-K+1, n-2)} \sum_{j=1}^{K} \Big[\sum_{k=2}^{K-2} (-1)^k (k-1)! \psi(n-K+1, n-k-1) v^j_{(k)} + (-1)^{K-1}(K-2)! v^j_{(K-1)} \Big] \\
&- (\pi_1 - \pi_2)v_2
\end{aligned}$$

We write this equivalently as

$$\begin{aligned}
\frac{\psi(n-K, n-2)}{\pi_1 - \pi_2} p_i(\mathbf{v}) = \sum_{j=1}^{K} \Big[\sum_{k=2}^{K-2} (-1)^k (k-1)! \psi(n-K+1, n-k-1) v^j_{(k)} + (-1)^{K-1}(K-2)! v^j_{(K-1)} \Big] \\
- \psi(n-K+1, n-2)v_2. \qquad (4)
\end{aligned}$$

Now, we remind that $\mathbf{v}$ is a valuation profile of the form $v_1 > v_2 > \ldots > v_K > 0$ and $v_j = 0$ for all $j > K$. We now simplify the RHS of Equation 4 in terms of $v_1, \ldots, v_K$. To do so, we explicitly compute the coefficients of v_k for each $k \in \{1, \ldots, K\}$ in the RHS of Equation 4.

Case 1. Note that v_1 does not appear in the summation, and hence, its coefficient is always zero. Next, $v_2 = v^j_{(2)}$ for all $j \neq \{1, 2\}$. Hence, it has a rank 2 in $(K-2)$ marginal profiles, and in each such case, its coefficient in the first summation is

$$(-1)^2 (1)! \psi(n-K+1, n-3).$$

Adding this with $-\psi(n-K+1,n-2)v_2$, we get the coefficient of v_2 as

$$(K-2)\psi(n-K+1,n-3)-\psi(n-K+1,n-2)=-\psi(n-K,n-3)=-(-1)^2(1!)\psi(n-K,n-3).$$

Case 2. Now, consider $K>k>2$. Note that $v_k=v^j_{(k')}$ where $k'\in\{k,k-1\}$. In particular, $k'=k$ if $j\in\{k+1,\dots,K\}$ and $k'=k-1$ if $j\in\{1,\dots,k-1\}$. Hence, it has rank k in $(K-k)$ marginal profiles and rank $(k-1)$ in $(k-1)$ marginal profiles. When it has rank k in a marginal profiles, its coefficient in the RHS of Equation 4 is

$$(-1)^k(k-1)!\psi(n-K+1,n-k-1),$$

and when it has rank $(k-1)$, its coefficient is

$$(-1)^{k-1}(k-2)!\psi(n-K+1,n-k).$$

Hence, collecting the coefficient of v_k, we get

$$\begin{aligned}&(-1)^k(K-k)(k-1)!\psi(n-K+1,n-k-1)+(-1)^{k-1}(k-1)(k-2)!\psi(n-K+1,n-k)\\&=(-1)^k(k-1)!\psi(n-K+1,n-k-1)\Big((K-k)-(n-k)\Big)\\&=-(-1)^k(k-1)!\psi(n-K,n-k-1).\end{aligned}$$

Case 3. Finally, $v_K=v^j(k')$ where $k'=K-1$ when $j\in\{1,\dots,K-1\}$. Hence, v_K has rank $(K-1)$ in $(K-1)$ marginal profiles. Whenever it has rank $(K-1)$ its coefficient in the RHS of Equation 4 is $(-1)^{K-1}(K-2)!$. Hence, the coefficient of v_K in the RHS of Equation 4 is

$$-(-1)^K(K-1)(K-2)!=-(-1)^K(K-1)!$$

Aggregating the findings from all the three cases, we can rewrite Equation 4 as

$$\frac{\psi(n-K,n-2)}{\pi_1-\pi_2}p_i(\mathbf{v})=\Big[\sum_{k=2}^{K-1}(-1)^k(k-1)!\psi(n-K,n-k-1)v_k+(-1)^K(K-1)!v_K\Big]. \tag{5}$$

This simplifies to the desire expression:

$$p_i(\mathbf{v})=-\frac{(\pi_1-\pi_2)}{\psi(n-K,n-2)}\Big[\sum_{k=2}^{K-1}(-1)^k(k-1)!\psi(n-K,n-k-1)v_k+(-1)^K(K-1)!v_K\Big]$$

■

Proof of Lemma 5

Proof: We follow a similar line of proof as Lemma 4. Consider a valuation profile $\mathbf{v}$ with $|N^0_{\mathbf{v}}| = n - K$, $K \geq \ell + 1$, $v_1 > \ldots > v_K > 0$ and $v_j = 0$ for all $j > K$.

We now modify Equation 3 by using $R^f(\mathbf{v}) = (\pi_1 - \pi_2)v_2 + \ell\pi_2 v_{\ell+1}$ (by Lemma 1) as follows:

$$0 = (n-K)p_i(\mathbf{v}) + \sum_{j=1}^{K} p_j(0, v_{-j}) + (\pi_1 - \pi_2)v_2 + \ell\pi_2 v_{\ell+1}. \tag{6}$$

Now, for every $j \in \{1, \ldots, K\}$, the profile $\mathbf{v}^j$ has one more zero-valued agent than the profile $\mathbf{v}$, and hence, we can apply our induction argument - the base case of $K = \ell$ is solved in Lemma 4, where we computed $p_i(\mathbf{v})$ with $K \leq \ell$ agents having non-zero valuations. Using induction hypothesis, we simplify Equation 6 as follows:

$$\begin{aligned}-(n-K)p_i(\mathbf{v}) = \sum_{j=1}^{K} -\frac{(\pi_1-\pi_2)}{\psi(n-\ell, n-2)}\Big[\sum_{k=2}^{\ell-1}(-1)^k(k-1)!\psi(n-\ell, n-k-1)v^j_{(k)} + (-1)^\ell(\ell-1)!v^j_{(\ell)}\Big] \\ + (\pi_1 - \pi_2)v_2 + \ell\pi_2 v_{\ell+1}.\end{aligned}$$

This can be rewritten as follows:

$$\begin{aligned}\frac{(n-K)\psi(n-\ell,n-2)}{\pi_1-\pi_2}p_i(\mathbf{v}) &= \sum_{j=1}^{K}\Big[\sum_{k=2}^{\ell-1}(-1)^k(k-1)!\psi(n-\ell,n-k-1)v^j_{(k)} + (-1)^\ell(\ell-1)!v^j_{(\ell)}\Big] \\ &\quad - \psi(n-\ell,n-2)v_2 - \frac{\ell\pi_2\psi(n-\ell,n-2)}{\pi_1-\pi_2}v_{\ell+1}.\end{aligned} \tag{7}$$

By Proposition 8 in Long et al. (2017),

$$\begin{aligned}\pi_1 - \pi_2 &= 1 - \frac{(\ell-1)}{C(n-2,\ell-1)+\ell} - \frac{1}{C(n-2,\ell-1)+\ell} \\ &= \frac{C(n-2,\ell-1)}{C(n-2,\ell-1)+\ell} \\ &= C(n-2,\ell-1)\pi_2 \\ &= \frac{\psi(n-\ell,n-2)}{(\ell-1)!}\pi_2.\end{aligned} \tag{8}$$

Hence, Equation 7 can be rewritten as

$$\begin{aligned}\frac{(n-K)\psi(n-\ell,n-2)}{\pi_1-\pi_2}p_i(\mathbf{v}) &= \sum_{j=1}^{K}\Big[\sum_{k=2}^{\ell-1}(-1)^k(k-1)!\psi(n-\ell,n-k-1)v^j_{(k)} + (-1)^\ell(\ell-1)!v^j_{(\ell)}\Big] \\ &\quad - \psi(n-\ell,n-2)v_2 - \ell!v_{\ell+1}\end{aligned} \tag{9}$$

Like in Lemma 4 proof, we will rewrite the RHS of Equation 10 in terms of $v_1, \ldots, v_K$. For this, observe that for any k, v_k will appear on the RHS of Equation 10 if there is some $j \in \{1, \ldots, K\}$ and some $k' \in \{2, \ldots, \ell\}$ such that $v^j_{(k')} = v_k$. Hence, v_1 and $v_{\ell+2}, \ldots, v_n$ do not appear on the RHS of Equation 10. We compute the coefficients of v_k for $k \in \{2, \ldots, \ell+1\}$. We consider three cases.

Case 1. For v_2, we note that $v_2 = v^j_{(2)}$ for all $j \neq \{1, 2\}$. Hence, it has a rank 2 in $(K-2)$ marginal profiles, and in each such case, its coefficient in the first summation is

$$(-1)^2(1)!\psi(n-\ell, n-3).$$

Adding this with $-\psi(n-\ell, n-2)$, we get the coefficient of v_2 in the RHS of Equation 10 as

$$\begin{aligned}&(K-2)\psi(n-\ell, n-3) - \psi(n-\ell, n-2) = -\psi(n-\ell, n-3)(n-K)\\ &= -(-1)^2(1!)\psi(n-\ell, n-3)(n-K).\end{aligned}$$

Case 2. Now, consider $2 < k < \ell$. For v_k, note that $v_k = v^j_{(k')}$ where $k' \in \{k, k-1\}$. In particular, $k' = k$ if $j \in \{k+1, \ldots, K\}$ and $k' = k-1$ if $j \in \{1, \ldots, k-1\}$. Hence, it has rank k in $(K-k)$ marginal profiles and rank $(k-1)$ in $(k-1)$ marginal profiles. In the RHS of Equation 10, the coefficient of v_k is $(-1)^{k-1}(k-2)!\psi(n-\ell, n-k)$ if its rank is $k-1$ and the coefficient is $(-1)^k(k-1)!\psi(n-\ell, n-k-1)$ if its rank is k. Adding them, we get the coefficient of v_k in the RHS of Equation 10 as

$$\begin{aligned}&(-1)^k(K-k)(k-1)!\psi(n-\ell, n-k-1) + (-1)^{k-1}(k-1)(k-2)!\psi(n-\ell, n-k)\\ &= (-1)^k(k-1)!\psi(n-\ell, n-k-1)\Big((K-k) - (n-k)\Big)\\ &= -(-1)^k(n-K)(k-1)!\psi(n-\ell, n-k-1).\end{aligned}$$

Case 3. For v_ℓ, note that $v_\ell = v^j_{(k')}$ where $k' \in \{\ell, \ell-1\}$. In particular, $k' = \ell$ if $j \in \{\ell+1, \ldots, K\}$ and $k' = \ell-1$ if $j \in \{1, \ldots, \ell-1\}$. Hence, it has rank ℓ in $(K-\ell)$ marginal profiles and rank $(\ell-1)$ in $(\ell-1)$ marginal profiles. In the RHS of Equation 10, the coefficient of v_ℓ is $(-1)^{\ell-1}(\ell-2)!\psi(n-\ell, n-\ell)$ if its rank is $\ell-1$ and the coefficient is $(-1)^\ell(\ell-1)!$ if its rank is ℓ. Adding them, we get the coefficient of v_ℓ in the RHS of Equation 10 as

$$\begin{aligned}&(-1)^{\ell-1}(\ell-1)(\ell-2)!\psi(n-\ell, n-\ell) + (-1)^\ell(K-\ell)(\ell-1)!\\ &= (-1)^\ell(\ell-1)!\big((K-\ell) - (n-\ell)\big)\\ &= -(-1)^\ell(n-K)(\ell-1)!\end{aligned}$$

Case 4. Now, consider $k = \ell + 1$. Note that $v_{\ell+1} = v^j_{(k')}$ if $k' = \ell$ and $j \in \{1, \ldots, \ell\}$. Hence, it has a rank ℓ in ℓ marginal economies, where its coefficient in the summation of the RHS of Equation 10 is

$$(-1)^\ell(\ell-1)! = (\ell-1)!,$$

since ℓ is even. Hence, the coefficient of $v_{\ell+1}$ in the RHS of Equation 10 is $\ell(\ell-1)! - \ell! = 0$.

Aggregating the findings from all the four cases, we can rewrite Equation 10 as

$$\frac{(n-K)\psi(n-\ell, n-2)}{\pi_1 - \pi_2} p_i(\mathbf{v}) = -\sum_{k=1}^{\ell-1}(-1)^k(n-K)(k-1)!\psi(n-\ell, n-k-1) - (-1)^\ell(n-K)(\ell-1)! \tag{10}$$

This simplifies to the desired expression:

$$p_i(\mathbf{v}) = -\frac{(\pi_1 - \pi_2)}{\psi(n-\ell, n-2)}\Big[\sum_{k=2}^{\ell-1}(-1)^k(k-1)!\psi(n-\ell, n-k-1)v_k + (-1)^\ell(\ell-1)!v_\ell\Big]$$

■

Proof of Proposition 4

Proof: Consider a valuation profile $\mathbf{v}$ with $v_1 > v_2 > \ldots > v_n > 0$. By Proposition 8 in Long et al. (2017),

$$\begin{aligned}
\pi_1 - \pi_2 &= 1 - \frac{(\ell-1)}{C(n-2, \ell-1) + \ell} - \frac{1}{C(n-2, \ell-1) + \ell} \\
&= \frac{C(n-2, \ell-1)}{C(n-2, \ell-1) + \ell} \\
&= C(n-2, \ell-1)\pi_2 \\
&= \frac{\psi(n-\ell, n-2)}{(\ell-1)!}\pi_2.
\end{aligned} \tag{11}$$

Then, the payments are computed using Lemma 5 in Long et al. (2017) as follows.

$$\begin{aligned}
p_1(\mathbf{v}) &= p_1(0, v_{-1}) + v_1\pi_1 - \int_0^{v_1} f_1(x_1, v_{-1})dx_1 \\
&= p_1(0, v_{-1}) + v_1\pi_1 - (v_1 - v_2)\pi_1 - (v_2 - v_{\ell+1})\pi_2 \\
&= p_1(0, v_{-1}) + v_2(\pi_1 - \pi_2) + v_{\ell+1}\pi_2 \\
&= -\frac{\pi_2}{(\ell-1)!}\Big[\sum_{k=2}^{\ell-1}(-1)^k(k-1)!\psi(n-\ell, n-k-1)v_{k+1}\Big] - v_{\ell+1}\pi_2 + v_2(\pi_1 - \pi_2) + v_{\ell+1}\pi_2
\end{aligned}$$

(The above simplification uses Lemma 5 in Long et al. (2017)

along with Equation 11 and the fact that ℓ is even.)

$$\begin{aligned}
&= -\frac{\pi_2}{(\ell-1)!}\Big[\sum_{k=2}^{\ell-1}(-1)^k(k-1)!\psi(n-\ell, n-k-1)v_{k+1}\Big] + \frac{\psi(n-\ell, n-2)}{(\ell-1)!}v_2\pi_2 \\
&= -\frac{\pi_2}{(\ell-1)!}\Big[\sum_{k=1}^{\ell-1}(-1)^k(k-1)!\psi(n-\ell, n-k-1)v_{k+1}\Big]
\end{aligned}$$

For every $i \in \{2, \dots, \ell\}$,

$$\begin{aligned}
p_i(\mathbf{v}) &= p_i(0, v_{-i}) + v_i\pi_2 - \int_0^{v_i} f_i(x_i, v_{-i})dx_i \\
&= p_i(0, v_{-i}) + v_i\pi_2 - (v_i - v_{\ell+1})\pi_2 \\
&= p_i(0, v_{-i}) + v_{\ell+1}\pi_2 \\
&= -\frac{\pi_2}{(\ell-1)!}\Big[\sum_{k=2}^{i-1}(-1)^k(k-1)!\psi(n-\ell, n-k-1)v_k + \sum_{k=i}^{\ell-1}(-1)^k(k-1)!\psi(n-\ell, n-k-1)v_{k+1}\Big] \\
&- v_{\ell+1}\pi_2 + v_{\ell+1}\pi_2
\end{aligned}$$

(The above simplification uses Lemma 5 in Long et al. (2017)

along with Equation 11 and the fact that ℓ is even.)

$$= -\frac{\pi_2}{(\ell-1)!}\Big[\sum_{k=2}^{i-1}(-1)^k(k-1)!\psi(n-\ell, n-k-1)v_k + \sum_{k=i}^{\ell-1}(-1)^k(k-1)!\psi(n-\ell, n-k-1)v_{k+1}\Big]$$

For every $i > \ell$, we directly use Lemma 5 in Long et al. (2017) along with Equation (11) to get

$$p_i(\mathbf{v}) = p_i(0, v_{-i}) = -\frac{\pi_2}{(\ell-1)!}\Big[\sum_{k=2}^{\ell-1}(-1)^k(k-1)!\psi(n-\ell, n-k-1)v_k + (-1)^\ell(\ell-1)!v_\ell\Big]$$

■

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