Roberts' Theorem with Neutrality: A Social Welfare Ordering Approach

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Introduction

Objectives of this research

- Characterize (dominant strategy) implementable social choice functions in quasi-linear environments when agents have multidimensional types.
- Restricted domains.

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- Characterize (dominant strategy) implementable social choice functions in quasi-linear environments when agents have multidimensional types.
- Restricted domains.

Motivations

- Auctioning multiple objects.
- Choosing a transport system for a city - metro rail, mono rail, highways etc.

Basic Notation

- Finite set of m alternatives: $A = \{a, b, c, \ldots\}$. Assume $m \geq 3$.
- Set of agents $N = \{1, 2, \ldots, n\}$.
- Type of agent i: $t_i = (t_i^a, t_i^b, \ldots,)$ - a vector in $\mathbb{R}^m$.
- Type profile of agents: t ($n \times m$ matrix) - n vectors in $\mathbb{R}^m$.
- The column vector t^a is called the **utility vector** for alternative a.

The Domain

- We assume that for all allocations, the set of all possible utility vectors is $\mathbb{D} \subseteq \mathbb{R}^n$.
- So, the space of all type profiles is $\mathbb{D}^m$.
- $\mathbb{D}$ is the domain of our problem - if $\mathbb{D} = \mathbb{R}^n$, it is the unrestricted domain.

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- $\mathbb{D}$ is the domain of our problem - if $\mathbb{D} = \mathbb{R}^n$, it is the unrestricted domain.
- $T_i \subseteq \mathbb{R}^m$: the set of all possible type vectors of agent i
- $\mathbb{T}^n = T_1 \times T_2 \times \ldots T_n$: the set of all possible type profiles.

Dominant Strategy Implementation

- A **social choice function (SCF)** is a mapping $f : \mathbb{T}^n \rightarrow A$.
- A **payment function** is a mapping $p : \mathbb{T}^n \rightarrow \mathbb{R}^n$, where $p_i(t)$ denotes the payment of agent i at $t \in \mathbb{T}^n$.

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Definition

A social choice function f is **implementable** *if there exists a payment function p such that for all $i \in N$ and for all $t_{-i} \in \mathbb{T}_{-i}$*

$$t_i^{f(t)} - p_i(t) \geq t_i^{f(s_i, t_{-i})} - p_i(s_i, t_{-i}) \qquad \forall\ s_i, t_i \in T_i.$$

What social choice functions are implementable?

Cycle Monotonicity Characterization

- For all $i \in N$ and $t_{-i} \in \mathbb{T}_{-i}$ we require for all $s_i, t_i \in T_i$

$$p_i(t_i, t_{-i}) - p_i(s_i, t_{-i}) \leq t_i^{f(t_i,t_{-i})} - t_i^{f(s_i,t_{-i})} = l^f_{t_{-i}}(s_i, t_i).$$

- A SCF f satisfies **cycle monotonicity** if for every $i \in N$, every $t_{-i} \in \mathbb{T}_{-i}$, and every $\{t^1, \ldots, t^k\} \subseteq T_i$

$$l^f_{t_{-i}}(t^1, t^2) + \ldots + l^f_{t_{-i}}(t^{k-1}, t^k) + l^f_{t_{-i}}(t^k, t^1) \geq 0.$$

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Theorem (Rochet, 1987, J. Math. Econ.)

A social choice function is implementable if and only if it satisfies cycle monotonicity.

Weak (2-cycle) Monotonicity Characterization

- A SCF satisfies **weak monotonicity** if for every $i \in N$, every $t_{-i} \in \mathbb{T}_{-i}$, and every $t^1, t^2 \in T_i$

$$l^f_{t_{-i}}(t^1, t^2) + l^f_{t_{-i}}(t^2, t^1) \geq 0.$$

Theorem (Bikhchandani et al., 2006, Econometrica)

In "auction domains" (severely restricted domains), a social choice function is implementable if and only if it satisfies weak monotonicity.

- Both monotonicity characterizations are difficult to verify - alternate characterizations?

A Property of Implementable SCFs

Definition

A social choice function f satisfies **positive association of differences (PAD)** *if for every $s, t \in \mathbb{T}^n$ such that $f(t) = a$ with $s^a - t^a \gg s^b - t^b$ for all $b \neq a$, we have $f(s) = a$.*

Lemma (Roberts, 1979, Book Chapter)

Every implementable social choice function satisfies PAD.

- Weak monotonicity implies PAD - but the converse is not true in general.
- In "auction domains", every SCF satisfies PAD - so PAD does not imply implementability in restricted domains.

Roberts' Theorem - Affine Maximizers

Definition

A social choice function f satisfies **non-imposition** *if for every $a \in A$, there exists $t \in \mathbb{T}^n$ such that $f(t) = a$.*

Theorem (Roberts, 1979, Book Chapter)

Suppose $\mathbb{D} = \mathbb{R}^n$ (unrestricted domain). If f is an implementable social choice function which satisfies non-imposition, then there exists weights $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$ and a deterministic real-valued function $\kappa : A \rightarrow \mathbb{R}$ such that for all $t \in \mathbb{T}^n$,

$$f(t) \in \arg\max_{a \in A} \Big[\sum_{i \in N} \lambda_i t_i^a + \kappa(a) \Big]$$

Payment Functions

- If f is an affine maximizer (λ, κ), then consider the payment rule p as follows. For every $i \in N$ and every $t \in \mathbb{T}^n$, if $\lambda_i = 0$ then $p_i(t) = 0$, else,

$$p_i(t) = h_i(t_{-i}) - \frac{1}{\lambda_i}\Big[\sum_{j \neq i} \lambda_j t_i^{f(t)} + \kappa(f(t))\Big],$$

where $h_i : \mathbb{T}_{-i} \to \mathbb{R}$ is any arbitrary function.

Main Open Question

Under what subdomains can one derive a functional form of implementable social choice functions?

Non-Affine Maximizers in Bounded Domains

Let $N = \{1,2\}$ and $A = \{a,b,c\}$. Suppose $T_1 = T_2 = (0,1)^3$ (alternatively, suppose $\mathbb{D} = (0,1)^2$). Consider the following allocation rule f. Let

$$\mathbb{T}^g = \{(t_1,t_2) \in \mathbb{T}^2 : t_1^c < t_1^b + 0.5\} \cup \{(t_1,t_2) \in \mathbb{T}^2 : t_2^c > t_2^b - 0.5\}.$$

Then,

$$f(t_1,t_2) = \begin{cases} \arg\max\{-1.5 + t_1^a + t_2^a, t_1^b + t_2^b, t_1^c + t_2^c\} & \forall\ (t_1,t_2) \in \mathbb{T}^g \\ c & \text{otherwise.} \end{cases}$$

- f satisfies non-imposition and is implementable but not an affine maximizer.

The Single Agent Case

- The example does not work for a single agent.

The Single Agent Case

- The example does not work for a single agent.
- Roberts' affine maximizer theorem is true for single agent in any domain (with any set of alternatives - finite/infinite).

Neutrality and Weighted Welfare Maximizers

Definition

A social choice function f is **neutral** *if for every type profile $t \in \mathbb{T}^n$ and for all permutations ϱ on A such that $t \neq s$, where s is the type profile due to permutation ϱ, we have $\varrho(f(t)) = f(s)$.*

- Neutrality implies non-imposition.

Theorem (Roberts, 1979, Book Chapter)

Suppose $\mathbb{D} = \mathbb{R}^n$ (unrestricted domain). If f is an implementable and neutral social choice function, then there exists weights $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$ such that for all $t \in \mathbb{T}^n$,

$$f(t) \in \arg\max_{a \in A} \Big[\sum_{i \in N} \lambda_i t_i^a\Big]$$

Payment Functions

- If f is a weighted welfare maximizer (λ), then consider the payment rule p as follows. For every $i \in N$ and every $t \in \mathbb{T}^n$, if $\lambda_i = 0$ then $p_i(t) = 0$, else,

$$p_i(t) = h_i(t_{-i}) - \frac{1}{\lambda_i}[\sum_{j \neq i} \lambda_j t_i^{f(t)}],$$

where $h_i : \mathbb{T}_{-i} \rightarrow \mathbb{R}$ is any arbitrary function.

New Open Question

Under what subdomains does Roberts' theorem with neutrality hold?

Our Answer

Theorem

Suppose $\mathbb{D}$ is open and connected. If f is an implementable and neutral social choice function, then there exists weights $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$ such that for all $t \in \mathbb{T}^n$,

$$f(t) \in \arg\max_{a \in A} \Big[\sum_{i \in N} \lambda_i t_i^a \Big]$$

Auction Domains not Covered

- Our results do not apply to auction domains because of many reasons:
  - Auction domains are not open.
  - The set of allocations are (partially) ordered in auction domains.
  - Neutrality is not appropriate in auction domains.

Where can the Results be Applied?

- A city plans to choose a transport system. Citizens have value for each alternative and the planner treats all systems symmetrically.

Disconnected Domains

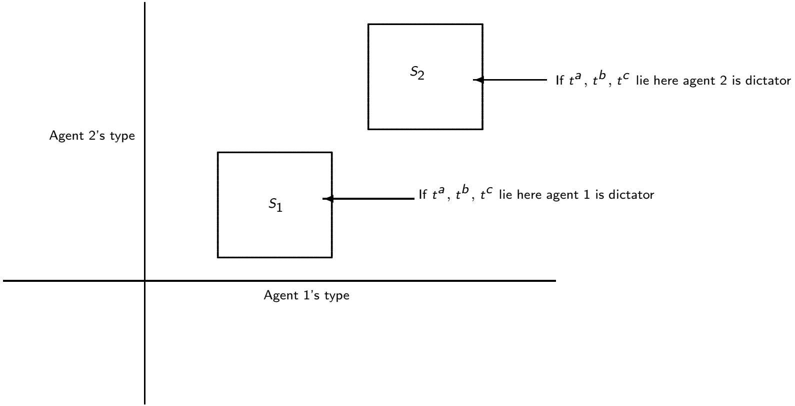

Figure: A disconnected but open $\mathbb{D}$

Many Proofs of Roberts' Theorem (for Unrestricted Domains)

- Roberts' original proof (Roberts, 1979, Book Chapter).
- Lavi, Mualem, Nisan (2009, Soc. Cho. Welfare; 2003 Conference Paper) - similar to but simpler than Roberts original proof.
- Dobzinski and Nisan (2009 Conference Paper) - "in the spirit of Barbera and Peleg's proof of Gibbard-Satterthwaite Theorem".
- Vohra (2008, Mimeo) - similar to Roberts original proof.
- Jehiel, Meyer-ter-Vehn, and Moldovanu (2008, Econ. Theory) - for interdependent values model.
- Carbajal and Tourky (2009, Mimeo) - for arbitrary continuous value functions.

Mystery of Roberts' Theorem

Extending Roberts' theorem to other domains has remained elusive. While Roberts' proof itself is not very difficult or long, it is quite mysterious (to us, at least). . . . The second author has already been involved in efforts to extend [7] and simplify the proof of [8] Roberts' theorem, but still finds it mysterious.

- Shahar Dobzinski and Noam Nisan

A Social Welfare Ordering Approach

Step 1 Every implementable and neutral SCF induces an ordering on $\mathbb{D}$.

Step 2 This ordering satisfies three axioms: weak Pareto, invariance, and continuity.

Step 3 Every ordering which satisfies these axioms can be represented as weighted welfare maximizers.

Explaining Step 1

Step 1 **Every implementable and neutral SCF induces an ordering on $\mathbb{D}$.**

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A Choice Set

Definition

The **choice set** *of an SCF f at every type profile t is*

$$C^f(t) = \{a \in A : \ \forall\, \varepsilon \gg 0, f(t^a + \varepsilon, t^{-a}) = a\}.$$

Lemma

If f is implementable and $\mathbb{D}$ is open from above, then for all type profiles t, $f(t) \in C^f(t)$.

Social Welfare Ordering

Definition

A **social welfare ordering (SWO)** *R^f induced by a social choice function f is a binary relation on $\mathbb{D}$ defined as follows. The symmetric component of R^f is denoted by I^f and the antisymmetric component of R^f is denoted by P^f. Pick $x, y \in \mathbb{D}$.*

- *We say $xP^f y$ if and only if there exists a profile t with $t^a = x$ and $t^b = y$ for some $a, b \in A$ such that $a \in C^f(t)$ but $b \notin C^f(t)$.*
- *We say $xI^f y$ if and only if there exists a profile t with $t^a = x$ and $t^b = y$ for some $a, b \in A$ such that $a, b \in C^f(t)$.*

R^f is an Ordering

Proposition

Suppose f is an implementable and neutral social choice function and $\mathbb{D}$ is open from above and a meet semi-lattice. Then, R^f induced by f on $\mathbb{D}$ is an ordering.

Explaining Step 2

Step 1 Every implementable and neutral SCF induces an ordering on $\mathbb{D}$.

Step 2 **This ordering satisfies three axioms: weak Pareto, invariance, and continuity**.

Step 3 Every ordering which satisfies these axioms can be represented as weighted welfare maximizers.

Three Axioms for an Ordering

Definition

An ordering R on $\mathbb{D}$ satisfies **weak Pareto (WP)** *if for all $x, y \in \mathbb{D}$ with $x \gg y$ we have xPy.*

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Definition

An ordering R on $\mathbb{D}$ satisfies **invariance (INV)** *if for all $x, y \in \mathbb{D}$ and all $z \in \mathbb{R}^n$ such that $(x + z), (y + z) \in \mathbb{D}$ we have xPy implies $(x + z)P(y + z)$ and xIy implies $(x + z)I(y + z)$.*

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Definition

An ordering R on $\mathbb{D}$ satisfies **continuity (C)** *if for all $x \in \mathbb{D}$, the sets $U^x = \{y \in \mathbb{D} : yRx\}$ and $L^x = \{y \in \mathbb{D} : xRy\}$ are closed in $\mathbb{R}^n$.*

An SWO Satisfies these Axioms

Proposition

Suppose f is an implementable and neutral social choice function and $\mathbb{D}$ is open from above and a meet semi-lattice. Then the social welfare ordering R^f induced by f on $\mathbb{D}$ satisfies weak Pareto, invariance, and continuity.

Explaining Step 3

Step 1 Every implementable and neutral SCF induces an ordering on $\mathbb{D}$.

Step 2 This ordering satisfies three axioms: weak Pareto, invariance, and continuity.

Step 3 **Every ordering which satisfies these axioms can be represented as weighted welfare maximizers**.

Characterizing the Ordering

- Suppose an ordering on $\mathbb{D}$ satisfies WP, INV, and C. Can we say anything about the ordering?
- When $\mathbb{D} = \mathbb{R}^n$, there is ample literature - see d'Aspremont and Gevers (2002, Book Chapter).

Characterizing the Ordering

- Suppose an ordering on $\mathbb{D}$ satisfies WP, INV, and C. Can we say anything about the ordering?
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Proposition (Blackwell and Girshick (1954, Book))

Suppose an ordering R on $\mathbb{R}^n$ satisfies weak Pareto, invariance, and continuity. Then there exists weights $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$ and for all $x, y \in \mathbb{R}^n$

$$xRy \Leftrightarrow \sum_{i \in N} \lambda_i x_i \geq \sum_{i \in N} \lambda_i y_i.$$

Our Ordering Characterization in Restricted Domains

Proposition

Suppose $\mathbb{D}$ is open and convex and let R be an ordering on $\mathbb{D}$ which satisfies weak Pareto, invariance, and continuity. Then there exists weights $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$ and for all $x, y \in \mathbb{D}$

$$xRy \Leftrightarrow \sum_{i \in N} \lambda_i x_i \geq \sum_{i \in N} \lambda_i y_i.$$

Final Proof Sketch

- Note that in an open cube in $\mathbb{R}^n$, f induces an ordering which satisfies WP, INV, and C.
- Hence, in an open cube, f is a weighted welfare maximizer.
- Now, take any open and connected $\mathbb{D}$, and fill it up with intersecting open cubes. Since the cubes can be made to intersect, f must be a welfare maximizer over entire $\mathbb{D}$.

Final Comments

- If we impose anonymity (permuting rows), then λs become equal (i.e., f is efficient) in Roberts' theorem. An easy proof using our approach and an elegant result of Milnor (1954) exists for this case.
- We are able to show that in symmetric, open, and connected domains every neutral and implementable social choice function is a weighted welfare maximizer.

Open Questions

- Are there any interesting subdomains where the affine maximizer version of Roberts' theorem holds?
- Can we give a precise functional form of implementable social choice functions in auction domains (severely restricted domains)?
- On what subdomains PAD implies implementability (may be it does not imply affine maximization)?
- Other approaches? Other axioms for implementable social choice functions?