Multi-Item Vickrey-Dutch Auctions[*]

Debasis Mishra [†]   David C. Parkes [‡]

January 30, 2008

Abstract

Descending price auctions are adopted for goods that must be sold quickly and in private values environments, for instance in flower, fish, and tobacco auctions. In this paper, we introduce ex post efficient descending auctions for two environments: multiple non-identical items and buyers with unit-demand valuations; and multiple identical items and buyers with non-increasing marginal values. Our auctions are designed using the notion of universal competitive equilibrium (UCE) prices and they terminate with UCE prices, from which the Vickrey payments can be determined. For the unit-demand setting, our auction maintains linear and anonymous prices. For the homogeneous items setting, our auction maintains a single price and adopts Ausubel's notion of "clinching" to compute the final payments dynamically. The auctions support truthful bidding in an ex post Nash equilibrium and terminate with an ex post efficient allocation. In simulation, we illustrate the speed and elicitation advantages of these auctions over their ascending price counterparts.

1 INTRODUCTION

An iterative auction can be described as a monotonic price adjustment procedure that takes bids from buyers in each iteration. Iterative auctions are often preferred over sealed-bid auctions, even in private value environments. The most important reasons are those of transparency, speed, and cost of participation (by avoiding the revelation of unnecessary valuation information through dynamic price discovery) (Cramton, 1998; Perry and Reny, 2005; Compte and Jehiel, 2005). Most of the iterative auction design literature has focused on ascending price auctions (Demange et al., 1986; Gul and Stacchetti, 2000; Parkes and Ungar, 2000; Bikhchandani and Ostroy, 2006; Ausubel, 2004; Ausubel and Milgrom, 2002; de Vries et al., 2007; Mishra and Parkes, 2007).In comparison, there are few results on the design of descending price auctions.[1]

Traditionally, in the descending price auction for a single item (called a *Dutch auction*), the seller starts the auction from a very high price and iteratively lowers the price. The first buyer to accept the price wins the auction at that price. The use of such auctions is popular because of its speed. Dutch auctions are used in selling flowers in Netherlands (thus the name Dutch auction) (van den Berg et al., 2001), fish in Israel, and tobacco in Canada. This type of descending price auction is strategically equivalent to a first-price sealed-bid auction and we can expect demand reduction and inefficiency for asymmetric settings (Krishna, 2002).

Contrast this with the simple equilibrium bidding strategies and efficiency in ascending price auctions such as the English auction, and its generalizations for multiple items (Demange et al., 1986; Parkes and Ungar, 2000; Ausubel, 2004; Ausubel and Milgrom, 2002; Mishra and Parkes, 2007).In these auctions, under appropriate assumptions on valuations, it is an *ex post* Nash equilibrium for a buyer to report his true demand set in every iteration; i.e., straightforward bidding is an equilibrium strategy whatever the private valuations of agents. The auctions terminate with the Vickrey-Clarke-Groves (VCG) (Vickrey, 1961; Clarke, 1971; Groves, 1973) outcome. Not only is this bidding strategy simple and robust to incorrect agent beliefs, but it is an ex post efficient equilibrium. One might ask whether there exists a descending price auction counterpart of the English auction with such simple bidding as an equilibrium strategy. The answer is "yes", as noted by Vickrey in his seminal paper (Vickrey, 1961). For the single item setting, Vickrey points out that the Dutch auction can be modified to run until a second buyer accepts an offer and the first buyer to accept an offer wins but pays the price at which the second offer is accepted. Quoting Vickrey (1961):

> "On the other hand, the Dutch auction scheme is capable of being modified
> with advantage to a second-bid price basis, making it logically equivalent to the

---

[1]One exception is in the work of Mishra and Garg (2006), who propose a generalized Dutch auction for one-to-one assignment setting, the setting in Demange et al. (1986), which terminates at the maximum competitive equilibrium price (approximately) if buyers bid honestly. But, this work provides no game theoretic equilibrium analysis.

second-price sealed-bid procedure ..."

Then, he goes on to describe an apparatus that is commonly used to implement the Dutch auction and how the same apparatus can be modified to implement this second-price auction. Quoting Vickrey (1961) again:

> *"There would be no particular difficulty in modifying the apparatus so that the first button pushed would merely preselect the signal to be flashed, but there would be no overt indication until the second button is pushed, whereupon the register would stop, indicating the price, and the signal would flash, indicating the purchaser."*

The appropriate method to extend Vickrey's idea to more general environments appears to be a puzzle in the current literature. For instance, in their work on the design of an ascending price auction for the homogeneous items case, Bikhchandani and Ostroy (2006) observe the following while interpreting their auction as a primal-dual algorithm:

> *"The primal-dual algorithm we describe starts at a low price where there is excess demand. One could start the primal-dual algorithm at a high price at which there would be excess supply, but it is unlikely that this would converge to a marginal pricing equilibrium.* [2]*"*

In this paper we present generalized "Vickrey-Dutch" auctions for multi-unit and multi-item private value environments which retain the speed and elicitation advantages that descending price auctions can enjoy over ascending price auctions, while inheriting the robust and simple equilibrium properties that come from termination at the Vickrey prices. The design of these auctions follows the methodology of *universal competitive equilibrium* (UCE) prices (Mishra and Parkes, 2007) to achieve the VCG outcome. Here we demonstrate, for the first time, the role of UCE prices in the design of auctions with *simple prices*.[3]

Our Vickrey-Dutch auctions maintain a single price trajectory which can pass "through" a part of the competitive equilibrium price space, before terminating at UCE prices. This dynamics provides for enough demand revelation to realize the VCG outcome. Truthful bidding is an ex post Nash equilibrium strategy, providing robustness to the particular distribution of agent valuations. Our auctions reduce to Vickrey's descending price auction for the single-item environment, while generalizing Vickrey's apparatus to the non-identical item and multiple identical items environments.

For the unit demand environment, the auction maintains a single set of item prices and can be considered to provide the descending analog to the ascending price auctions of

---

[2]A marginal pricing equilibrium is a competitive equilibrium price where all buyers get their respective payoffs in the VCG mechanism.

[3]Mishra and Parkes (2007) use UCE prices to design ascending-price combinatorial auctions and Lahaie et al. (2005) use UCE prices to design elicitation protocols for combinatorial auctions. Both approaches adopt non-linear and non-anonymous prices.

Demange et al. (1986). [4] For multiple identical items and non-increasing marginal valuations we design a "clinching" auction, which provides a descending price analog to the ascending price clinching auction of Ausubel (2004). Just as in Ausubel (2004), our auction maintains a single price in each iteration, with the allocation and payments determined dynamically across iteration. The analysis of the auction establishes that the price in any iteration when coupled with the history of clinching decisions up to that iteration actually provides a concise representation of a non-linear and non-anonymous price vector that terminates at UCE prices.

After the first version of this work, a Vickrey-Dutch auction has been described for the case of multiple heterogeneous items with buyers having combinatorial values in Mishra and Veeramani (2007) (henceforth, M&V). Just like in our auctions, the auction in M&V achieves the VCG outcome using descending prices. But their auction does not reduce to any of our auctions when applied to the unit demand or the homogeneous items settings. This anamoly can be explained as follows. The auction in M&V searches for a *non-linear and non-anonymous* UCE price vector and then adjusts from there to the VCG outcome while our auctions maintain simple prices and terminate directly or via clinching with VCG payments. The price dynamics of the auction in M&V are therefore very different from our auctions and do not produce the same final price vector and price adjustment as our auctions when applied in the unit demand and homogeneous items settings.

2   THE MODEL AND PRELIMINARIES

To begin we introduce a general model with n heterogeneous indivisible items. In a later section we specialize this model to consider identical items. We define competitive equilibrium and universal competitive equilibrium prices in this model and provide a general framework for the design of iterative VCG auctions.

The set of items is denoted by $A = \{1, \ldots, n\}$. There are $m \ (\geq 2)$ buyers, denoted by $B = \{1, \ldots, m\}$. The set of all bundles of items is denoted by $\Omega = \{S \subseteq A\}$. Naturally, $\emptyset \in \Omega$. For every buyer $i \in B$ and every bundle $S \in \Omega$, the valuation of i on bundle S is denoted by $v_i(S) \geq 0$, assumed to be a non-negative integer. If S is a singleton, we write $v_i(j)$ instead of $v_i(\{j\})$ for simplicity.

We assume a private values setting where each buyer knows his own valuation function and this does not depend on the valuations or allocations of other buyers. The payoff of any buyer $i \in B$ on any bundle $S \in \Omega$ is quasi-linear and given by $v_i(S) - p$, where p is the price paid by buyer i on bundle S. Also, if a buyer gets nothing and pays nothing, then his utility is zero: $v_i(\emptyset) := 0 \ \forall \ i \in B$. We also assume that $v_i(S) \leq v_i(T) \ \forall \ i \in B, \ \forall \ S, T \in \Omega$ with $S \subseteq T$. The seller values the items at zero. His payoff (or revenue) is the total payment he receives from buyers.

---

[4]To be precise, our auction is a descending price analog of the variation on the auction in Demange et al. (1986) presented in Sankaran (1994).

Let $B_{-i} = B \setminus \{i\}$ be the set of buyers without buyer i. Let $\mathbb{B} = \{B, B_{-1}, \ldots, B_{-m}\}$. We will denote the economy with buyers only from set $M \subseteq B$ as $E(M)$. Whenever, $M \neq B$ and $M \in \mathbb{B}$, we call economy $E(M)$ a *marginal economy*. $E(B)$ denotes the *main economy*.

Let x denote a feasible allocation in economy $E(M)$ ($M \in \mathbb{B}$). Allocation x is both a partitioning of the set of items and an assignment of the elements of the partition to buyers in M. Allocation x assigns bundle x_i to buyer i for every $i \in M$ and for every $i \neq j$, $x_i \cap x_j = \emptyset$. The possibility of $x_i = \emptyset$ is allowed. We will denote the set of all *feasible* allocations of economy $E(M)$ as $\mathbb{F}(M)$.

We now provide some basic definitions that are used in the design of our auction. An allocation X is **efficient** in economy $E(M)$ if $\sum_{i \in M} v_i(x_i) = \max_{y \in \mathbb{F}(M)} \sum_{i \in M} v_i(y_i)$. Consider general prices, that can be both non-linear and non-anonymous, and define the **demand set** of buyer $i \in M$ (for some $M \in \mathbb{B}$) at price vector $p \in \mathbb{R}_+^{|M| \times |\Omega|}$ as

$$D_i(p) := \big\{ S \in \Omega : v_i(S) - p_i(S) \geq v_i(T) - p_i(T) \ \forall \ T \in \Omega \big\}$$

and the **supply set** of the seller at price vector $p \in \mathbb{R}_+^{|M| \times |\Omega|}$ in economy $E(M)$ as

$$L(p) := \Big\{ x \in \mathbb{F}(M) : \sum_{i \in M} p_i(x_i) \geq \sum_{i \in M} p_i(y_i) \ \forall \ y \in \mathbb{F}(M) \Big\}.$$

Define $\pi^s(p) := \sum_{i \in M} p_i(x_i)$, where $x \in L(p)$, as the revenue of the seller at price vector $p \in \mathbb{R}_+^{|M| \times |\Omega|}$ in economy $E(M)$.

DEFINITION 1 *Price vector $p \in \mathbb{R}_+^{|M| \times |\Omega|}$ and allocation x are a **competitive equilibrium** (CE) of economy $E(M)$ for some $M \subseteq B$ if $x \in L(p)$, and $x_i \in D_i(p)$ for every buyer $i \in M$. Price p is called a CE price vector of economy $E(M)$.*

If $p \in \mathbb{R}_+^{|B| \times |\Omega|}$, then the components of p corresponding to a set of buyers $M \subset B$ will be denoted as p_M (or, p_{-i} if $M = B_{-i}$). A component of p_M to buyer $i \in M$ will still be denoted as p_i.

DEFINITION 2 *A price vector p is a **universal competitive equilibrium** (UCE) price vector if p_M is a CE price vector of economy $E(M)$ for **every** $M \in \mathbb{B}$.*

A UCE price vector always exists since $p := v$ is a (trivial) UCE price vector. UCE prices are necessary and sufficient to realize the VCG outcome from a CE of the main economy, as shown in the following theorem:[5]

THEOREM 1 ((**Mishra and Parkes, 2007**)) *Let (p, x) be a CE of the main economy with $p \in \mathbb{R}_+^{|B| \times |\Omega|}$. The VCG payments of every buyer can be calculated from (p, x) if and only if p is a UCE price vector. Moreover, if p is a UCE price vector, then for every buyer $i \in B$, the VCG payment of every buyer $i \in B$ is $p_i^{vcg} = p_i(x_i) - [\pi^s(p) - \pi^s(p_{-i})]$.*

---

[5]Lahaie et al. (2005) provide further support for UCE prices by establishing a strong equivalence between the informational requirements of UCE prices and the VCG outcome for a general class of protocols.

We illustrate the idea of discounts discussed in Theorem 1 by an example. Consider an economy with two buyers and two items with valuations: $v_1(\{1\}) = 8, v_1(\{2\}) = 9, v_1(\{1,2\}) = 12$ and $v_2(\{1\}) = 6, v_2(\{2\}) = 8, v_2(\{1,2\}) = 14$. Consider the following price vector: $p_1(\{1\}) = 7, p_1(\{2\}) = 9, p_1(\{1,2\}) = 11$ and $p_2(\{1\}) = 2, p_2(\{2\}) = 4, p_2(\{1,2\}) = 10$. Verify that $D_1(p) = \{\{1\}, \{1,2\}\}$ and $D_2(\{p\}) = \{\{1\}, \{2\}, \{1,2\}\}$. Further, a revenue maximizing allocation in the main economy is to allocate item 1 to buyer 1 and item 2 to buyer 2. Since these allocations are in the demand sets of the respective buyers, p is a CE price vector of the main economy. Similarly, it can be checked that p is a UCE price vector. The maximum revenue from the main economy is $p_1(\{1\}) + p_2(\{2\}) = 11$, from the economy with only buyer 1 (i.e., without buyer 2) is $p_1(\{1,2\}) = 11$, and from the economy with only buyer 2 (i.e., without buyer 1) is $p_2(\{1,2\}) = 10$. Hence, discount to buyer 1 is $11 - 10 = 1$ and that to buyer 2 is $11 - 11 = 0$. So, the final payments using the formula in Theorem 1 are: $p_1(\{1\}) - 1 = 7 - 1 = 6$ for buyer 1 and $p_2(\{2\}) - 0 = 4$ for buyer 2, which are also their respective VCG payments.

Using the idea of UCE prices, we will design our descending price auctions. It is well known that any iterative auction that achieves the VCG outcome under truthful bidding (submission of demand sets) of the buyers has the property that truthful bidding is an ex post equilibrium for the buyers when coupled with appropriate activity rules (see de Vries et al. (2007); Mishra and Parkes (2007)). Therefore, we will only show that our auctions achieve the VCG outcome under truthful bidding, and the equilibrium properties of our auctions follow from this. We omit discussions about specific activity rules in the interest of space.

3 THE UNIT DEMAND ENVIRONMENT

In this section we introduce a Vickrey-Dutch auction for the environment with heterogeneous, indivisible items and unit-demand valuations so that each buyer is interested in buying at most one item. This is the standard *assignment problem*. Our auction maintains an individual price on each item and decreases prices until supply balances demand in the main economy and also in all marginal economies.

For convenience, we will assume that there is a *dummy* item, indexed 0, available such that the value of the dummy item is zero for all buyers, and the dummy item can be allocated to any number of buyers. A feasible allocation x assigns to every buyer $i \in B$ either an item $j \in A$ or the dummy item. No item is assigned more than once (but an item may go unassigned). By a slight abuse of notation, let x_i denote the item assigned to buyer i in allocation x.

Let $v_i(j)$ denote buyer i's value for item $j \in A$. A price vector p in this section denotes *linear and anonymous prices*, with $p \in \mathbb{R}^{n+1}_+$ and $p(0) = 0$. The definition of demand set is modified to restrict to include only singleton bundles, with $D_i(p) = \{j \in A \cup \{0\} : v_i(j) - p(j) \geq \max_{j' \in A \cup \{0\}}[v_i(j') - p(j')]\}$, and the definition of CE is specialized as follows:

DEFINITION **3** *Price vector $p \in \mathbb{R}^{n+1}_+$ is a* **competitive equilibrium price** *vector if there*

exists an allocation x such that $x_i \in D_i(p)$ for every $i \in B$ and $p(j) = 0$ for every item j that is not assigned in x.

If (p, x) is a CE in the assignment problem, then x is an efficient allocation, and the set of CE price vectors form a complete lattice (Shapley and Shubik, 1972), i.e., there is a unique minimum and a unique maximum CE price vector. Moreover, the minimum CE price vector corresponds to the VCG payments (Leonard, 1983). We obtain the following simple but useful observation:

PROPOSITION 1 *There is a unique linear and anonymous UCE price vector in the unit-demand environment, and this is equal to the minimum CE price vector and defines the VCG payments.*

Proof: Leonard (1983) shows that the minimum CE price vector p^{min} corresponds to the VCG payments. By Theorem 1, it is a UCE price vector. We show that it is the unique linear and anonymous UCE price vector in the unit-demand environment.

Assume for contradiction $p \neq p^{min}$ is a linear and anonymous UCE price vector. Define $S = \{j \in N : p(j) > p^{min}(j)\}$. Since p^{min} is the unique minimum CE price vector, $p \geq p^{min}$ and S is non-empty. By definition, $p(j) > 0$ for all $j \in S$, and must be assigned to a buyer in a CE of the main and marginal economies. Since p is a UCE price vector, at least $|S| + 1$ buyers must demand items from S at p (these are the buyers who are assigned items from S in main and marginal economies at p). By going from p to p^{min}, prices of items in S fall, whereas prices other items remain the same. So, buyers who were demanding items from S at p will demand items *only* from S at p^{min} (this is because their payoffs from items outside S remain the same but payoffs from items in S strictly increase). So, at least $|S| + 1$ buyers will demand items *only* from S at p^{min}. Since p^{min} is a CE, we should be able to assign every buyer an item from his demand set at p^{min}. But at least $|S| + 1$ buyers demand items from only S at p^{min}, and we cannot possibly assign these buyers items from their demand sets. This gives us a contradiction. ∎

This observation means that searching for a linear and anonymous UCE price vector will directly (without any discount) give the VCG outcome.

	1	2
v_1	8	4
v_2	6	3

Table 1: An Example in Unit Demand Environment

To illustrate Proposition 1, consider the example in Table 1 with two buyers $\{1, 2\}$ and two items $\{1, 2\}$. The minimum CE price vector is $(3, 0)$, which also provides the VCG payments for buyers. It can be easily verified that $(3, 0)$ is a UCE price vector (item 1 is allocated to the remaining buyer in both marginal economies) Any other CE price vector in

7

this example will reduce the payoff of at least one buyer below his VCG payoff, and hence it is not a UCE price vector.

3.1 The Vickrey-Dutch Auction (Unit-Demand Environment)

In every iteration of the auction, the seller announces (anonymous, linear) prices $p^t \in \mathbb{R}_+^{n+1}$ and receives demands from each buyer. Let $D(p^t) = \{D_i(p^t)\}_{i \in B}$ and $D_{-i}(p^t) = \{D_k(p^t)\}_{k \in B_{-i}}$ denote the vector of demand sets received in iteration t from buyers in B and in B_{-i} respectively.

Given an allocation x, the *revenue* is the sum of prices of all the items *allocated*. A buyer i is *satisfied* in an allocation x at price vector p if $x_i \in D_i(p)$. An *admissible allocation* x at a price vector p is an allocation such that $x_i \in D_i(p) \cup \{0\}$. A *provisional allocation* is an admissible allocation that generates the maximum revenue across all admissible allocations, breaking ties in favor of satisfying the maximum number of buyers and then at random.

Consider the price vector $(4, 1)$ for the example in Table 1. The demand sets of the buyers are: $D_1(p) = \{1\}$ and $D_2(p) = \{1, 2\}$. There is an admissible allocation in which buyer 1 gets item 1 and buyer 2 gets item 2. Since this allocates both the items and both the buyers are satisfied in this allocation, this is also a provisional allocation. Another admissible allocation is to allocate item 1 to either of the buyers and the other buyer is assigned the dummy item (item 2 is unassigned). Clearly, this is not a provisional allocation.

Let $X(D(p^t))$ denote the set of provisional allocations at price vector p^t and let $X(D_{-i}(p^t))$ denote the set of provisional allocations for economy $E(B_{-i})$. Given an allocation x, let $S(x) \subseteq A$ denote the *set of allocated items with positive prices*. The following concept plays a central role in defining the auction.

DEFINITION 4 *Item $j \in A$ is* **universally allocated***, written $j \in U(p, D(p), x)$ given a price vector p, demand sets $D(p)$, and provisional allocation $x \in X(D(p))$, if $p(j) = 0$ or item j is provisionally allocated to some buyer i with $S(x) = S(y)$ for some $y \in X(D_{-i}(p))$.*

A universally allocated item j should either have a price of zero or be provisionally allocated to some buyer i such that all items with positive prices (including item j) that are allocated can also be allocated in the marginal economy without buyer i given the current demand. In the example in Table 2, consider the price vector $(4, 4)$. Here, consider the provisional allocation in which buyer 1 is assigned item 1 and buyer 2 is assigned the dummy item (item 2 is unassigned). In the absence of buyer 1, we can assign buyer 2 to item 1 to get a provisional allocation in which item 1 is assigned. Hence, item 1 is universally allocated at price $(4, 4)$ given the demand sets and this provisional allocation, but item 2 is not universally allocated. We give more examples in Section 3.2 to illustrate the idea of universally allocated item.

The Vickrey-Dutch auction in this environment seeks prices for which all items are universally allocated and reduces prices on items that are not universally allocated until this is achieved. We refer to the auction as the *linear-price Vickrey-Dutch* (LVD) auction:

DEFINITION 5 *The **linear-price Vickrey-Dutch** (LVD) auction for the unit demand environment is defined as follows:*

(S0) Start from a high price p^0 where no buyer demands any item from A. Set $t := 0$.

(S1) In iteration t of the auction, with price vector p^t:

   (S1.1) Collect the demand sets $D(p^t)$ of all the buyers at p^t.

   (S1.2) Based on the demand sets of buyers at p^t, calculate a provisional allocation $x^t \in X(D(p^t))$.

   (S1.3) Find the universally allocated set of items, $U(p^t, D(p^t), x^t)$.

   (S1.4) If $U(p^t, D(p^t), x^t) = A$ (the set of all items), go to Step (S2). Else, set $p^{t+1}(j) := p^t(j) - 1 \ \forall \ j \in (A \setminus U(p^t, D(p^t), x^t))$. Set $t := t + 1$ and repeat from Step (S1).

(S2) The auction terminates in current iteration T with price vector p^T and the provisional allocation x^T. If $x_i^T = j$ for buyer i, then he pays an amount $p^T(j)$ and gets item j.

We illustrate the steps of the LVD auction for the example in Table 1. The starting price of the LVD auction is $(9, 9)$.

Iteration	Price	$D_1()$	$D_2()$	Provisional Allocation	Universally Allocated Items
0	$(9,9)$	$\{0\}$	$\{0\}$	-	$\emptyset$
1	$(8,8)$	$\{0,1\}$	$\{0\}$	$1 \to 1$	$\emptyset$
2	$(7,7)$	$\{1\}$	$\{0\}$	$1 \to 1$	$\emptyset$
3	$(6,6)$	$\{1\}$	$\{0,1\}$	$1 \to 1$	$\{1\}$
4	$(6,5)$	$\{1\}$	$\{0,1\}$	$1 \to 1$	$\{1\}$
5	$(6,4)$	$\{1\}$	$\{0,1\}$	$1 \to 1$	$\{1\}$
6	$(6,3)$	$\{1\}$	$\{0,1,2\}$	$1 \to 1, 2 \to 2$	$\emptyset$
7	$(5,2)$	$\{1\}$	$\{1,2\}$	$1 \to 1, 2 \to 2$	$\emptyset$
8	$(4,1)$	$\{1\}$	$\{1,2\}$	$1 \to 1, 2 \to 2$	$\emptyset$
9	$(3,0)$	$\{1\}$	$\{1,2\}$	$1 \to 1, 2 \to 2$	$\{1,2\}$

A CE of the main economy is achieved in iteration 6 but this is not a UCE price vector. In the final iteration, item 2 is universally allocated since its price is zero and item 1 is universally allocated since it can be allocated to buyer 2 in the absence of buyer 1. The final price vector $(3, 0)$ is the UCE price vector identified in Proposition 1. One can also observe that both buyers are satisfied in every iteration and that the demand set of buyer 2 monotonically increases while he is not allocated. Note, though, that the set of universally allocated items is not monotonically increasing in this example.

We plot the CE price vector space, price trajectory of the LVD auction, and price trajectory of the Demange et al. (1986) (DGS) auction in Figure 1. It is interesting to note that

9

the price trajectory never touches the maximum CE price vector ($p^{max} = (7,3)$). We can also see that the LVD auction price path travels through a significant portion of the entire CE price vector space before reaching the minimum CE price vector, whereas the first CE price vector in the DGS auction is the minimum CE price vector.
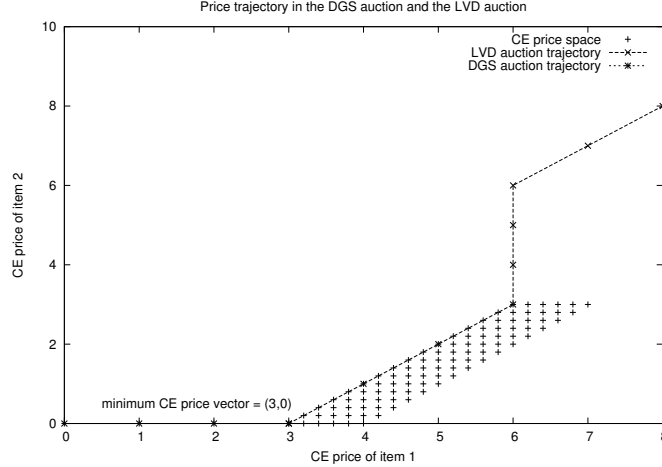

Figure 1: Plot of price trajectory of the LVD auction and the DGS auction on the Example in Table 1

3.2 IDENTIFYING UNIVERSALLY ALLOCATED ITEMS

The problem of finding a provisional allocation $x^t \in X(D(p^t))$ is a variant of the standard assignment problem and can be solved efficiently.[6] A computationally efficient procedure is also required to determine the set of universally allocated items. This is used both to adjust prices and also to check for termination. Consider the examples in Figures 2(a) and 2(b), which illustrate demand sets and an allocation at two different price vectors. Suppose all prices are positive. A solid line between a buyer and an item means that the buyer is provisionally allocated to the item. A dashed line between a buyer and an item means that the buyer has this item in his demand set but is not provisionally allocated to the item. Each figure represents all the information required to determine the set of universally allocated items.

In Figure 2(a), item 3 is universally allocated: remove buyer 3 (provisionally allocated to item 3), then allocate buyer 4 to item 3 without changing the total set of allocated items. But, no other items are universally allocated. In case of item 1, if we remove buyer 1, the only buyer that demands item 1 is buyer 2 and thus we cannot allocate item 1 without reducing

---

[6]It can be solved with two linear programs (LPs). The first LP computes an admissible allocation that maximizes total revenue given demand sets $D(p^t)$. A second LP is then formulated to break ties in favor of maximizing the number of satisfied buyers, with the objective defined as such and a constraint included to ensure that the revenue from the allocation is equal to that obtained in solving the first LP.
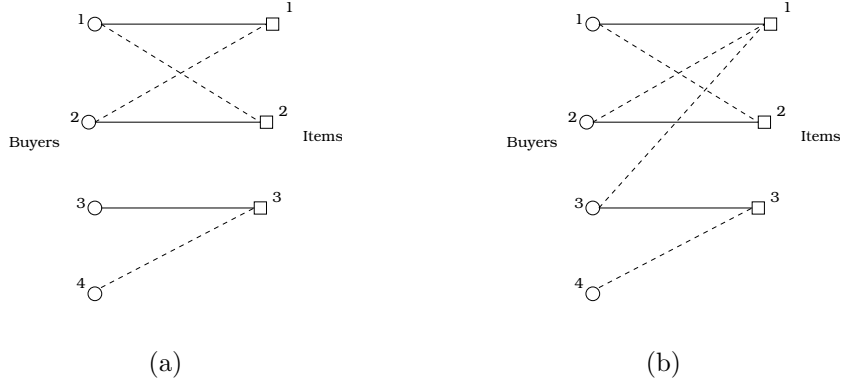
Figure 2: Identifying universally allocated items. Dashed line: item in demand set but not provisionally allocated. Solid line: provisional allocation. All prices are positive.

the total set of provisionally allocated items. Item 2 is also not universally allocated, by symmetry. In Figure 2(b), all 3 items are universally allocated. Buyer 4 will take buyer 3's item. Without buyer 1 (allocated to item 1), we can allocate buyer 4 to item 3 and buyer 3 to item 1 without changing the total set of allocated items. Without buyer 2, we can allocate buyer 4 to item 3, buyer 3 to item 1 and buyer 1 to item 2.

It is instructive that item 3, which is demanded by an unallocated buyer (4), is a universally allocated item and the starting point for finding other universally allocated items. We use this idea to develop a procedure to determine the set of universally allocated items. We first handle items with zero price. Given $(p, D(p), x)$, let $(p', D'(p'), x')$ denote the restriction to items with positive price, with $p' = (p(j) : j \in A, p(j) > 0)$, $D'_i(p') = \{j : j \in D_i(p), p(j) > 0\}$, and $x'_i = x_i$ when $p(x_i) > 0$ and $x'_i = 0$ otherwise. Let $A_0(p) = \{j \in A, p(j) = 0\}$ denote the items with zero price. The proof of the following and some subsequent results are delayed until the Appendix.

LEMMA 1 $U(p, D(p), x) = U(p', D'(p'), x') \cup A_0(p)$ where $(p', D'(p'), x')$ is the restriction of $(p, D(p), x)$ to items with positive price and $A_0(p)$ is the set of items with zero price.

We now formalize the intuition in the example, in which we identified a sequence of reassignments of items, starting with a currently unallocated buyer. Given allocation x, let a *well-defined chain* with respect to buyer i, allocated to item j in x, be $z_{-i}(x, D(p)) = j_0 i_1 j_1 \ldots i_c j_c$, with $c \geq 1$ (this is a sequence of alternating buyers and items) and with the property that:

(i) item $j_0 = 0$ and item $j_c = j$
(ii) buyer i_r, for $1 \leq r \leq c$ is assigned item j_{r-1} in allocation x
(iii) buyer $i \notin \{i_1, \ldots, i_c\}$
(iv) item $j_r \in D_{i_r}(p)$ for all $1 \leq r \leq c$

Such a chain defines a reassignment of items, with a modified allocation x' defined with

11

$x'_{i_r} = j_r$ for all $r \in \{1, \ldots, c\}$, $x'_k = x_k$ for all $k \notin \{i_1, \ldots, i_c\} \cup \{i\}$, and $x'_i = 0$.

LEMMA **2** *Given a price vector p with non-zero prices for all the items, item $j \in U(p, D(p), x)$ if and only if there is a well-defined chain $z_{-i}(x, D(p)) = j_0 i_1 j_1 i_2 j_2 \ldots i_c j$ where buyer i is allocated j in allocation x.*

In Figure 2(a) the well-defined chain that explains item 3 is $j_0 i_1 j_1 = 043$. In Figure 2(b) the well-defined chain that explains item 1 is 04331 and the chain that explains item 2 is 0433112. Let $\hat{U}(p, D(p), x) = \{j' \in A : j' \in D'_i(p')$ for some $i \in B$ with $x'_i = 0\}$, where $(p', D'(p'), x')$ is the restriction of $(p, D(p), x)$ to positive prices. We can determine the set of universally allocated items $U(p, D(p), x)$ as follows.

Procedure:	FINDUNIVALLOCITEMS
Step 0:	Initialize $U^{(0)}(p, D(p), x) = \hat{U}(p, D(p), x)$. Set $r := 1$.
Step 1:	Let $T^{(r)}$ denote the set of buyers allocated to items in $U^{(r-1)}(p, D(p), x)$
Step 2:	Let $W^{(r)} = \bigcup\limits_{i \in T^{(r)}} D_i(p) \cap S(x)$ denote the set of provisionally allocated items with positive price demanded by buyers in $T^{(r)}$.
Step 3:	If $W^{(r)} \subseteq U^{(r-1)}(p, D(p), x)$, output $U^{(r-1)}(p, D(p), x) \cup A_0(p)$ and **STOP**. Else, $U^{(r)}(p, D(p), x) := U^{(r-1)}(p, D(p), x) \cup W^{(r)}$. $r := r + 1$. Repeat from **Step 1**.

As an illustration, we apply the FINDUNIVALLOCITEMS algorithm to the example in Figure 2(b). In the first round of the FINDUNIVALLOCITEMS algorithm, set $U(p, D(p), x) = \{3\}$. This gives $T = \{3\}$ and $W = \{1, 3\}$. We update $U(p, D(p), x) = \{1, 3\}$. Now, $T = \{1, 3\}$ and $W = \{1, 2, 3\}$. We update $U(p, D(p), x) = \{1, 2, 3\}$. Now, $T = \{1, 2, 3\}$ and $W = \{1, 2, 3\}$ and we stop because $W = U(p, D(p), x)$.

PROPOSITION **2** *The FINDUNIVALLOCITEMS procedure determines all universally allocated items.*

Every round of the FINDUNIVALLOCITEMS procedure either finds new universally allocated items or stops, and thus the maximum number of rounds of the FINDUNIVALLOCITEMS algorithm is n, the total number of items. The computations in each round of LVD can therefore be done in polynomial time.

3.3 THEORETICAL ANALYSIS

When designing ascending price auctions it is common to construct a price trajectory such that the seller is satisfied in every iteration and the buyers are all satisfied upon termination. In the LVD auction we establish the reverse: every buyer is satisfied with the provisional allocation in every iteration and the seller is satisfied upon termination.

PROPOSITION **3** *In the provisional allocation in every iteration of the LVD auction, every buyer is satisfied under truthful bidding.*

Next, we show that the LVD auction terminates at the unique UCE price vector and consequently the minimum CE price vector, and therefore collects the VCG payment.

THEOREM **2** *The LVD auction achieves the VCG outcome when every buyer follows the truthful bidding strategy, and is ex post efficient.*

Proof: The price is reduced on at least one item with positive price in each iteration and thus the auction must terminate since any item with zero price is universally allocated. Upon termination, every item is universally allocated and thus every item with positive price is allocated and the allocation is in the supply set of the seller. Taken together with every buyer being satisfied (Proposition 3) we see that the final prices and the final provisional allocation are a CE. We will next show that the final price vector, p^T, is a UCE price vector. Consider a buyer i. Let $x_i^T = j$. If $j = 0$, then clearly (p^T, x_{-i}^T) is a CE of the marginal economy without buyer i. If $j \neq 0$, then because this item is universally allocated there exists an allocation $y^T \in X(D_{-i}(p^T))$ such that all items with positive price are allocated (and so it is in the supply set), and all the buyers except i are satisfied. This is because of Lemma 2, wherein the well-defined chain shows that upon moving from allocation x^T to allocation y^T, that any agent $\neq i$ allocated a non-dummy item in x^T is still allocated a non-dummy item in y^T. One additional buyer, unallocated (but satisfied) in x^T is now also allocated a non-dummy item in y^T. Hence, (p^T, y^T) is a CE in the marginal economy without buyer i. This shows that p^T is a UCE price vector. By Proposition 1 and the payment rule of the LVD auction, the LVD auction achieves the VCG outcome. ■

The LVD auction achieves ex post efficiency in an ex post Nash equilibrium when coupled with appropriate activity rules. See the discussion at the end of Section 2. The LVD auction may achieve a CE price vector before termination and has a price trajectory that traverses *through* the CE price vector space to reach the minimum CE price vector. This was illustrated in Figure 1. Losing buyers must continue to bid even after the first CE price vector has been identified, and thus after it is known to the auction that the buyer is not a winner. We can keep buyers ignorant of this by disclosing only the price information in each iteration and withholding information about the provisional allocation or demand sets. When there is exactly one item for sale it is universally allocated when the number of buyers demanding the item becomes more than one. Hence, the auction will allocate the item to the first buyer who demands it but terminate at a price for which the item is also demanded by the buyer with the second-highest value. Clearly, this translates to the flash-button apparatus implementation described by Vickrey, who also points to similar obfuscation.

4 The Homogeneous Items Environment with Decreasing Marginal Values

In this section we introduce a Vickrey-Dutch auction for selling multiple units of a homogeneous item for buyers with marginal-decreasing values for each additional unit.

The auction maintains a single price rather than a vector of prices and provides a descending-price analog to the ascending price "clinching" auction of Ausubel (2004): buyers also clinch items in our auction and the final payments are determined from demand information revealed during the auction. The underlying philosophy of the design of this auction remains the UCE price concept: all buyers are satisfied in the provisional allocation in every iteration, and the auction eventually terminates with supply equal to demand in the main economy as well as in every marginal economy.

In introducing the auction we first describe the auction with a non-linear and non-anonymous price dynamic. This version of the auction serves to make the UCE framework clear and facilitate our analysis. In Section 4.3 we show that the auction can be equivalently implemented as an auction that maintains a single price for the item in each round, best thought of as defining the current ask price for a marginal unit over and above the units already clinched.

4.1 The Clinching Auction: Non-linear and Non-anonymous Prices

In this section, we adopt n to denote the number of units for sale. Let $v_i(j)$ denote the value of buyer i for j units of the item, assumed to be a non-negative integer. Assume that $v_i(0) = 0$ for every buyer and consider only *non-increasing marginal values*, so that $v_i(j) - v_i(j-1) \geq v_i(j+1) - v_i(j)$ for every buyer i and every unit $1 \leq j < n$. An example is given in Table 2.

j	0	1	2	3	4
$v_1(j)$	0	**7**	9	10	10
$v_2(j)$	0	8	**13**	15	15
$v_3(j)$	0	**4**	8	10	10

Table 2: An example with 4 units of a homogeneous item and 3 buyers, each with non-increasing marginal values. The values in the table provide the *total* value of each buyer for some number of units. The efficient allocation is depicted in bold.

By a slight abuse of notation, let $p \in \mathbb{R}_+^{m \times (n+1)}$ denote the non-anonymous and non-linear price maintained in each iteration. Let $p_i(j)$ denote the price to buyer i for j units. Let demand set $D_i(p) = \{j \in \{0, 1, \ldots, n\} : v_i(j) - p_i(j) \geq \max_{0 \leq j' \leq n} v_i(j') - p_i(j')\}$. Let the *maximal demand* of buyer i at price vector p be $\overline{D}_i(p)$, defined as the maximum number of units demanded. Allocation x, where x_i denotes the number of units allocated to buyer i, is a

feasible allocation of economy $E(M)$ for $M \in \mathbb{B}$ if $\sum_{i \in M} x_i \leq n$. Admissible and provisional allocations are defined as before. A feasible allocation is also admissinle if $x_i \in D_i(p) \cup \{0\}$ for every $i \in M$, and a provisional allocation maximizes the revenue over all admissible allocations, breaking ties in favor of satisfying as many buyers as possible. As before, we denote the set of provisional allocations in economy $E(M)$ at a price vector p as $X(D_M(p))$.

For the example in Table 2, consider the price vector in Table 3. The prices in parentheses indicate that the buyer is demanding a bundle with that number of units at the current prices. As an example, the demand set of buyer 1 at this price vector is $\{1, 2\}$, and his maximal demand is 2. An admissible allocation in this example is to allocate two units to buyer 1, two units to buyer 2, and zero units to buyer 3. This generates a total revenue of $p_1(2) + p_2(2) + p_3(0) = 16$. Another admissible allocation in this example is to allocate zero units to buyer 1, two units to buyer 2, and two units to buyer 3, which generates a total revenue of 18. Notice that buyer 1 is not satisfied in this allocation since he is allocated zero units, which he does not demand. A provisional allocation for this example is to allocate one unit each to buyers 1 and 3 and allocate two units to buyer 2. This generates the same revenue as the previous admissible allocation but satisfies all the buyers.

j	0	1	2	3	4
$p_1(j)$	0	(4)	(6)	9	9
$p_2(j)$	0	6	(10)	13	14
$p_3(j)$	(0)	(4)	(8)	(10)	11

Table 3: A non-linear and non-anonymous price vector for the example in Table 2

Given prices p, define $\alpha(M, p) := \max(0, n - \sum_{i \in M} \overline{D}_i(p))$ for all $M \in \mathbb{B}$. Quantity $\alpha(M, p)$ denotes the *under-demand* in economy $E(M)$. For the example in Table 2 and the price vector in Table 3, $\alpha(M, p) = 0$ for all $M \in \mathbb{B}$.

DEFINITION **6** *The **clinching auction** is defined as:*

(S0) *Initialize prices as $p_i^0(j) = q^0 j$ for all $i \in B$ and for all $j \leq n$, where q^0 is a large integer. Set $t := 0$.*

(S1) *In iteration t of the auction with price vector p^t:*

   (S1.1) *Collect the demand sets $D(p^t)$ of all the buyers at p^t.*

   (S1.2) *Based on the demand sets of buyers at p^t, calculate the under-demand $\alpha(M, p^t)$ for every $M \in \mathbb{B}$.*

   (S1.3) *If $\alpha(M, p^t) = 0$ for every $M \in \mathbb{B}$ or $q^t = 0$ then go to Step (S2). Else, $q^{t+1} := q^t - 1$ and for every $i \in B$, set*

$$p_i^{t+1}(j) := \begin{cases} p_i^t(j) & \text{for all } j \leq \overline{D}_i(p^t) \\ p_i^t(\overline{D}_i(p^t)) + q^{t+1}(j - \overline{D}_i(p^t)) & \text{, otherwise.} \end{cases}$$

Set $t := t + 1$ and repeat from Step (S1).

(S2) *The auction terminates in current iteration T with price vector p^T. Final allocation is $x^T \in X(D(p^T))$, and for each buyer $i \in B$ with $x_i^T > 0$, his payment is $p_i^T(x_i^T) - \left[\pi^s(p^T) - \pi^s(p_{-i}^T)\right]$.*

The initial price is set high enough that no buyer has maximal demand more than zero in the first iteration. The only feedback provided to each buyer in each iteration is the current price vector p_i^t. The parameter q^t represents the ask price in round t on each marginal unit over and above the current number of items demanded by the buyer. The auction terminates when this marginal price is zero or when there is no under-demand. The price vector faced by buyer i in round t is defined recursively in terms of the price in the previous round but adjusted downwards on quantities greater than its current demand. The payment is finally determined as a discount from the prices at the end of the auction.

We illustrate the clinching auction for the example in Table 4. The first column indicates the iteration number, the next three columns indicate the personalized prices of each buyer for each number of units, and the last column indicates the excess supply in economy $E(M)$ for every $M \in \mathbb{B}$. Parentheses around prices for a buyer indicate the quantity of units in his demand set in each iteration. The final price vector is a UCE price vector. Also, $\pi^s(B) = 24$, $\pi^s(B_{-1}) = 21$, $\pi^s(B_{-2}) = 17$, and $\pi^s(B_{-3}) = 22$. Using discounts from UCE price as in Eq. (1), the payment of buyer 1 can be calculated as $p_1^{vcg} = p_1(1) - [\pi^s(B) - \pi^s(B_{-1})] = 7 - [24 - 21] = 4$. Similarly, $p_2^{vcg} = 13 - [24 - 17] = 6$, $p_3^{vcg} = 4 - [24 - 22] = 2$. It is a simple matter to check that these are indeed the VCG payments in this example.

#	q^t	Buyer 1				Buyer 2				Buyer 3				$\alpha(\cdot)$			
		1	2	3	4	1	2	3	4	1	2	3	4	B	B_{-1}	B_{-2}	B_{-3}
1	9	9	18	27	36	9	18	27	36	9	18	27	36	4	4	4	4
2	8	8	16	24	32	(8)	16	24	32	8	16	24	32	3	3	4	3
3	7	(7)	14	21	28	(8)	15	22	29	7	14	21	28	2	3	3	2
4	6	(7)	13	19	25	(8)	14	20	26	6	12	18	24	2	3	3	2
5	5	(7)	12	17	22	(8)	(13)	18	23	5	10	15	20	1	2	3	1
	From iteration 6 onwards, we have a CE of the main economy.																
6	4	(7)	11	15	19	(8)	(13)	17	21	(4)	(8)	12	16	0	0	1	0
7	3	(7)	10	13	16	(8)	(13)	17	20	(4)	(8)	11	14	0	0	1	0
8	2	(7)	(9)	11	13	(8)	(13)	17	19	(4)	(8)	(10)	12	0	0	0	0

Table 4: A non-linear and non-anonymous price trajectory in the clinching auction

4.2 THEORETICAL ANALYSIS: CLINCHING AUCTION

We first make the following observations about our clinching auction.

LEMMA **3** *In every iteration $t \geq 0$ of the clinching auction in the homogeneous items environment, we have*

(a) $\overline{D}_i(p^{t+1}) \geq \overline{D}_i(p^t)$
(b) $D_i(p^t) = \{0, 1, \ldots, \overline{D}_i(p^t)\}$, *and*
(c) $v_i(j) - v_i(j-1) \leq p_i^t(j) - p_i^t(j-1)$ *for all $0 < j \leq n$, for all buyers $i \in B$.*

A consequence of (b) is that every admissible allocation satisfies all buyers in the clinching auction under truthful bidding since 0 is always in the demand set of every buyer. So, under truthful bidding, a provisional allocation is an allocation that gives the maximum revenue over all admissible allocations (one need not worry about breaking ties in favor of number of satisfied buyers). A consequence of (c) is that when every buyer bids truthfully in the auction and $q^t = 0$ in some iteration t, then $\alpha(M, p^t) = 0$ for all $M \in \mathbb{B}$ because of non-negative marginal values. We can now define the *marginal price* of the $(j+1)^{\text{st}}$ unit $(0 \leq j < n)$ for buyer i, given price p^t in iteration t, as $p_i^t(j+1) - p_i^t(j)$. The marginal price on the $(j+1)^{\text{st}}$ unit for $j \geq \overline{D}_i(p^t)$ is exactly q^t, for every buyer and in every iteration:

LEMMA **4** *In every iteration t of the clinching auction and for every buyer, the marginal price of the jth unit is less than or equal to the marginal price of $(j+1)^{st}$ unit for $0 \leq j \leq n-1$. Moreover, all buyers face the same marginal price q^t for all units j greater than the maximal demand of the buyer in that iteration.*

Proof: By definition of the auction, for j greater than or equal to the maximal demand of a buyer, the marginal price of the $(j+1)^{\text{st}}$ unit for that buyer is q^t in iteration t. The prices of units less than maximal demand are not changed. By Lemma 3, the maximal demand of every buyer is non-decreasing from iteration to iteration (starting at zero in iteration 0). Hence, the marginal price of jth unit is less than or equal to the marginal price of $(j+1)^{\text{st}}$ unit for $0 \leq j \leq n-1$. ∎

It is useful to define the provisional allocation that is associated with each iteration of the auction, even though this is not explicit in the auction rules. For this purpose, let $x_M^t \in X(D_M(p^t))$ denote a provisional allocation for economy $E(M)$. We will show that provisional allocation satisfies every buyer in every economy $E(M)$ in every iteration, and that upon termination these provisional allocations are also in the supply set for the seller in every economy. Thus we have UCE prices. This resembles our analysis for the LVD auction.

PROPOSITION **4** *Suppose $\alpha(M, p^t) = 0$ for economy $E(M)$ for some $M \in \mathbb{B}$ in iteration t of the clinching auction in the homogeneous items environment. Then every provisional allocation of economy $E(M)$ is in the supply set of the seller in that economy in iteration t.*

Proof: Assume for contradiction that a provisional allocation x is not in the supply set of the seller in economy $E(M)$ in iteration t. By definition, x maximizes revenue of the seller across all *admissible* allocations of economy $E(M)$ in iteration t. This means no allocation

17

in the supply set of the seller is an admissible allocation in economy $E(M)$ (if there are allocations in the supply set that are admissible then one of them must be selected, because they satisfy more buyers than an allocation that is not admissible). Consider some allocation y in the supply set of the seller in economy $E(M)$ in iteration t. Since y is not admissible, and by Lemma 3 (which says that zero is in demand set of every buyer throughout the auction), there is some buyer $i \in M$ such that $y_i > \overline{D}_i(p^t)$. Since $\alpha(M, p^t) = 0$, there exists $k \in M \setminus \{i\}$ such that $y_k < \overline{D}_k(p^t)$. Now, construct a new allocation z of economy $E(M)$ such that $z_i = y_i - 1$, $z_k = y_k + 1$, and $z_{i'} = y_{i'}$ for all $i' \in M \setminus \{i, k\}$. From Lemma 4, the revenue from z is greater than or equal to the revenue from y because the marginal price of a unit less than the maximal demand of a buyer is greater than the marginal price of a unit greater than maximal demand. This process of constructing a new allocation can be repeated till we get an allocation z such that $z_i \leq \overline{D}_i(p^t)$ for all $i \in M$. Thus, we get an admissible allocation of economy $E(M)$ with revenue greater than or equal to the revenue from y. This is a contradiction. ∎

Every provisional allocation satisfies all buyers. Further, if $\alpha(M, p^t) = 0$ then every provisional allocation of economy $E(M)$ is in the supply set of the seller in economy $E(M)$ by Proposition 4. This means once we have $\alpha(M, p^t) = 0$, we achieve a CE of economy $E(M)$. These results lead to the main result of this section.

THEOREM **3** *Suppose every buyer follows the truthful bidding strategy. Then the clinching auction terminates at a UCE price vector in the homogeneous items environment, achieves the VCG outcome, and is ex post efficient.*

Proof: Since buyers are truthful, the maximal demand of all the buyers will be n when the marginal price reaches zero. Thus, the zero marginal price provides a lower bound for the prices in the auction and it will terminate after a finite number of iterations.

We have already seen that by Proposition 4 once $\alpha(M, p^t) = 0$, we have a CE of economy $E(M)$. By (a) of Lemma 3, $\alpha(M, p^t) = 0$ implies $\alpha(M, p^{t+1}) = 0$ for all $M \in \mathbb{B}$. Hence, once a CE of an economy is achieved, the following iterations in the auction also produce a CE of that economy. Therefore, by the terminating condition of the auction, the final price vector is a UCE price vector.

Since the final price vector and any provisional allocation of the main economy is a CE of the main economy, any provisional allocation of the main economy is an efficient allocation. Hence, the final allocation of the clinchin auction is an efficient allocation. From the payment rule of the clinching auction and Theorem 1, the payments of buyers are their respective VCG payments. Thus, the clinching auction achieves the VCG outcome. ∎

4.3 THE SIMPLE CLINCHING AUCTION

We now present a simple representation of the clinching auction that maintains only the marginal price q^t in each iteration and makes the clinching behavior explicit. One challenge here is that a marginal price does not correspond to a unique non-linear and non-anonymous price vector, as seen in the prior analysis. The analysis re-parameterizes the demand sets and under-demand in terms of this marginal price.

By Lemma 3, a buyer can report his entire demand set by revealing only his maximal demand in each iteration. Since q^t is the marginal price of a unit, the maximal demand given q^t is:

$$\overline{D}_i(q^t) = \begin{cases} 0 & \text{, if } v_i(1) - v_i(0) < q^t \\ \max_{j \in \{0,1,\dots,n\}} j \text{ s.t. } v_i(j) - v_i(j-1) \geq q^t & \text{, otherwise.} \end{cases}$$

The final payments are determined from the demand sets reported by buyers across iterations by a clinching argument. Let t^m denote the first iteration in which $\sum_{i \in B} \overline{D}_i(q^t) \geq n$, and thus for which $\alpha(B, q^t) = 0$. Let x^t be the provisional allocation of the main economy in iteration t. Define the **residual demand without buyer** i as $r_{-i}(q^t) = \min(x_i^t, \sum_{j \neq i} [\overline{D}_j(q^t) - x_j^t])$ for all iterations $t \geq t^m$. For iterations $t < t^m$, define $r_{-i}(q^t) = 0$ for all $i \in B$. For iterations $t \geq t_m$, this gives the number of units from x_i^t that can be allocated to buyers in economy $E(B_{-i})$. The change in residual demand from iteration t^m until termination also provides enough information to define the final payments of each buyer (see Lemma 5), discounting from the final marginal ask price.

The simple clinching auction achieves the same outcome as the earlier auction but with a simpler representation of prices.

DEFINITION 7 *The* **simple clinching auction** *is an iterative procedure with the following steps:*

(S0) Start from a high price q^0. Set $t := 0$. Set the total number of units allocated to buyers, $(x_1, \dots, x_m)$, to zero. Set the total payments of buyers, $(s_1, \dots, s_m)$, to zero.

(S1) In iteration t of the auction with price q^t:

(S1.1) Collect maximal demand $\overline{D}_i(q^t)$ of every buyer i at price q^t. Impose $\overline{D}_i(q^t) \geq \overline{D}_i(q^{t-1})$ for every buyer for all $t > 0$.

(S1.2) If $\sum_{i \in B} \overline{D}_i(q^t) < n$ then $x_i = \overline{D}_i(q^t)$ for all $i \in B$. Set $q^{t+1} := q^t - 1$, $t := t + 1$, and repeat from Step (S1.1).

(S1.3) If $\sum_{i \in B} \overline{D}_i(q^t) \geq n$ and $\sum_{i \in B} \overline{D}_i(q^{t-1}) < n$, then set $t^m := t$ and set x to be the new provisional allocation.

(S1.4) Set $s_i := s_i + q^t(r_{-i}(q^t) - r_{-i}(q^{t-1}))$ for all $i \in B$.

19

(S1.5) If $r_{-i}(q^t) = x_i$ for all $i \in B$ or $q^t = 0$, then go to Step (S2). Else, set $q^{t+1} := q^t - 1$, $t := t + 1$, and repeat from Step (S1.1).

(S2) Final allocation is $(x_1, \ldots, x_m)$ and the final payment vector is $(s_1, \ldots, s_m)$.

The simple clinching auction maintains the marginal price of an additional unit over and above the "clinched units" and lowers it in every iteration. Buyers clinch all units they currently demand while the total demand is less than the total number of units. In the first period in which the total demand exceeds supply, then the history is used to determine the provisional allocation and thus also the number of clinched units. The clinched units of a buyer are priced by monitoring the residual demand without each buyer. The auction stops when all the clinched units of the buyers are priced, and this occurs exactly when the residual demand without each buyer equals the number of units clinched by that buyer, and thus exactly when supply equals demand in every economy. The computational requirements of this auction are very light: simple linear time operations are performed in every iteration.

The maximal demand reported by buyers across all iterations can be used to determine the provisional allocation. Consider the main economy and iteration t. If $\alpha(B, q^t) > 0$ then determine the provisional allocation by assigning $\overline{D}_i(q^t)$ to each buyer $i \in B$. In the first period t in which $\alpha(B, q^t) = 0$, allocate $\overline{D}_i(q^{t-1})$, i.e. the maximal demand in iteration $t - 1$, to each buyer $i \in B$ and complete the provisional allocation by assigning the $n - \sum_{i \in B} \overline{D}_i(q^{t-1})$ additional units to buyers at random such that no buyers gets more than his maximal demand. It is easy to see that this allocation is admissible, and one can use arguments similar to Proposition 4 to show that it maximizes revenue across all admissible allocations and is therefore a provisional allocation.

Consider the example in Table 2 and suppose there is a marginal price of 6 in some iteration. Then, the maximal demands of buyers 1 and 2 are one, and that of buyer 3 is zero. Hence, $\alpha(B, 6) > 0$, and the provisional allocation allocates every buyer his maximal demand. Consider marginal prices of 5 and 4 in two consecutive iterations. The maximal demand vector of buyers for marginal price 5 is $(1, 2, 0)$ and that for marginal price 4 is $(1, 2, 2)$. In the iteration when the marginal price is 4, we have $\alpha(B, 4) = 0$. The provisional allocation initially allocates one unit to buyer 1, two units to buyer 2, and zero units to buyer 3, with the remaining one unit given to buyer 3 since he is the only buyer whose maximal demand has increased over these two iterations. To illustrate the idea of residual demand, consider the same iteration in which the marginal price is 4 and the maximal demand vector is $(1, 2, 2)$. If we remove buyer 1, we can allocate a maximum of four units among buyers 2 and 3, whereas buyers 2 and 3 are allocated three units in the provisional allocation. Hence, residual demand without buyer 1 is $4 - 3 = 1$. Similarly, if we remove buyer 2, we can allocate a maximum of three units to buyers 1 and 3, whereas buyers 1 and 3 are allocated two units in the provisional allocation. Hence, residual demand without buyer 2 is also 1. Similarly, one can check that the residual demand without buyer 3 is zero.

By definition, in iterations $t \geq t^m$, $\alpha(B_{-i}, q^t) = 0$ if and only if $r_{-i}(q^t) = x_i^t$ for all buyers

$i \in B$, where x^t is the provisional allocation. Thus, the residual demand information also provides a *termination condition*: the auction terminates when $t \geq t^m$ and either $r_i(q^t) = x_i^t$ for all i or $q^t = 0$.

We illustrate the steps of the auction in Table 5. Buyers clinch units as soon as they demand them while there is available supply. For instance, in iteration 2, buyer 2 demands a unit and clinches it as there are 4 units of supply available in the main economy (and no other demand). In iteration 6 the total demand in economy $E(B_{-1})$ is 4 units and buyers in B_{-1} have clinched 3 units. This means there is a residual demand of 1 unit without buyer 1 and the single unit clinched by buyer 1 is priced at the current price, which is 4. Similarly, the total demand in economy $E(B_{-2})$ is 3 units and buyers in B_{-2} have clinched 2 units, creating a residual demand without buyer 2 of 1 unit. Thus, only one of the two units clinched by buyer 2 can be priced at the current price of 4. The next change in residual demand occurs in iteration 8, when the remaining units are priced and the auction terminates.

t	Price q^t	Demand κ_1^t	κ_2^t	κ_3^t	Units clinched x_1	x_2	x_3	Residual demand r_1^t	r_2^t	r_3^t	Payments s_1	s_2	s_3
1	9	0	0	0	0	0	0	0	0	0	0	0	0
2	8	0	1	0	0	1	0	0	0	0	0	0	0
3	7	1	1	0	1	1	0	0	0	0	0	0	0
4	6	1	1	0	1	1	0	0	0	0	0	0	0
5	5	1	2	0	1	2	0	0	0	0	0	0	0
From iteration 6 onwards, we have a CE of the main economy.													
6	4	1	2	2	1	2	1	1	1	0	4	4	0
7	3	1	2	2	1	2	1	1	1	0	4	4	0
8	2	2	3	3	1	2	1	1	2	1	4	4+2	2

Table 5: Marginal price trajectory in the simple clinching auction

LEMMA **5** *In the simple clinching auction in the homogeneous items environment, the final payment of buyer i is equal to $\sum_{t \geq t^m} \left[q^t(r_{-i}(q^t) - r_{-i}(q^{t-1})) \right]$.*

Proof: Let p^T be the final price vector in the clinching auction defined on non-linear and non-anonymous prices. Let x and y be efficient allocations of the main economy and the marginal economy corresponding to buyer i respectively. The VCG payment of buyer i when all buyers are truthful is:

$$p_i^{vcg} = \sum_{k \in B_{-i}} [v_k(y_k) - v_k(x_k)] = \sum_{k \in B_{-i}} [p_k^T(y_k) - p_k^T(x_k)] \qquad \text{(From Lemma 3)}$$

The above expression for the VCG payment of buyer i can be rewritten as

$$p_i^{vcg} = \sum_{k \neq i} \left[[p_k^T(y_k) - p_k^T(y_k - 1)] + \ldots + [p_k^T(x_k + 1) - p_k^T(x_k)] \right] \qquad (1)$$

21

For any $k \in B_{-i}$ and for any $j \in \{x_k, \ldots, y_k - 1\}$, the marginal price of $(j+1)$st unit in the final iteration, $p_k^T(j+1) - p_k^T(j)$, is the marginal price of $(j+1)$st unit in the iteration in which the maximal demand of k increased from j to $j+1$. Since every buyer sees the same marginal price in any iteration of our auction, we can monitor the change in maximal demand of all buyers in B_{-i} simultaneously in every iteration after iteration t^m to compute the VCG payment of buyer i. The change in residual demand without buyer i in an iteration $t \geq t^m$ reflects the total change in maximal demand of buyers in B_{-i}. The price of these units get fixed after this. Hence $q^t(r_{-i}(q^t) - r_{-i}(q^{t-1}))$ is the price component of buyer i for $r_{-i}(q^t) - r_{-i}(q^{t-1})$ units in Equation 1. Adding over all the iterations after iteration t^m gives the VCG payment of buyer i, i.e., $p_i^{vcg} = \sum_{t \geq t^m} q^t(r_{-i}(q^t) - r_{-i}(q^{t-1}))$. ∎

We have now established the following result.

THEOREM 4 *Suppose every buyer follows the truthful bidding strategy. Then the simple clinching auction achieves the VCG outcome and is ex post efficient.*

The simple clinching auction achieves ex post efficiency in an ex post Nash equilibrium when coupled with appropriate activity rules. See the discussion at the end of Section 2. Consider now the special case of unit-demand preferences, so that each buyer demands a single unit of the item. The maximal demand of a buyer never exceeds one except for the trivial case when q^t is zero and a buyer is awarded a unit as soon as it is demanded. So, the number of clinched units of a buyer does not exceed one as long as $q^t > 0$ and the residual demand without any buyer cannot exceed one. The residual demand without a buyer who has clinched a unit becomes equal to one exactly when the number of buyers demanding a unit is greater than the number of units. In that iteration, all the allocated buyers are priced– the payment of every allocated buyer is simply the marginal price in that iteration. Notice that the terminating condition in this case requires all units to be allocated in an economy without a buyer, which is equivalent to the universally allocated item condition of the LVD auction. Moreover, the simple clinching auction translates to the well known uniform second-price auction for multiple units, where buyers pay the (same) price of the highest losing buyer. When there is exactly one unit for auction (i.e., $n = 1$), the auction stops as soon as a second buyer demands the unit, translating again to Vickrey's flash-button apparatus.

5   CONCLUSIONS

We introduce ex post efficient Vickrey-Dutch auctions for environments with unit demand and with multiple homogeneous items and non-decreasing marginal values. These auctions are designed within the framework of universal competitive equilibrium prices. The auctions in both settings maintain simple prices and terminate with the VCG outcome. In the appendix, we demonstrate using simulations that their preference elicitation properties and

speed, especially in environments with high competition, are superior to their ascending auction counterparts. These properties make them very attractive for practical use in private value settings. Identifying additional valuation domains where "simple" prices can provide ex post efficient, descending price auctions provides an interesting direction for future work.

References

AUSUBEL, L. M. (2004): "An Efficient Ascending-Bid Auction for Multiple Objects," *American Economic Review*, 94, 1452–1475.

AUSUBEL, L. M. AND P. R. MILGROM (2002): "Ascending Auctions with Package Bidding," *Frontiers of Theoretical Economics*, 1, 1–42.

BIKHCHANDANI, S. AND J. OSTROY (2006): "Ascending Price Vickrey Auctions," *Games and Economic Behavior*, 55, 215–241.

CLARKE, E. (1971): "Multipart Pricing of Public Goods," *Public Choice*, 11, 17–33.

COMPTE, O. AND P. JEHIEL (2005): "Auctions and Information Acquisition: Sealed-bid or Dynamic Formats?" Tech. rep., CERAS, Ecole Nationale des Ponts et Chaussées, Forthcoming in Rand Journal of Economics.

CRAMTON, P. (1998): "Ascending Auctions," *European Economic Review*, 42, 745–756.

DE VRIES, S., J. SCHUMMER, AND R. V. VOHRA (2007): "On Ascending Vickrey Auctions for Heterogeneous Objects," *Journal of Economic Theory*, 132, 95–118.

DEMANGE, G., D. GALE, AND M. SOTOMAYOR (1986): "Multi-item Auctions," *Journal of Political Economy*, 94, 863–872.

GROVES, T. (1973): "Incentives in Teams," *Econometrica*, 41, 617–631.

GUL, F. AND E. STACCHETTI (2000): "The English Auction with Differentiated Commodities," *Journal of Economic Theory*, 92, 66–95.

KRISHNA, V. (2002): *Auction Theory*, Academic Press, San Diego, CA.

LAHAIE, S., F. CONSTANTIN, AND D. C. PARKES (2005): "More on the Power of Demand Queries in Combinatorial Auctions: Learning Atomic Languages and Handling Incentives," in *Proceedings of 19th International Joint Conference on Artificial Intelligence (IJCAI'05)*.

LEONARD, H. B. (1983): "Elicitation of Honest Preferences for the Assignment of Individuals to Positions," *Journal of Political Economy*, 91, 461–479.

MISHRA, D. AND R. GARG (2006): "Descending Price Multi-Item Auctions," *Journal of Mathematical Economics*, 42, 161–179.

MISHRA, D. AND D. C. PARKES (2007): "Ascending Price Vickrey Auctions for General Valuations," *Journal of Economic Theory*, 132, 335–366.

MISHRA, D. AND D. VEERAMANI (2007): "Vickrey-Dutch Procurement Auction for Multiple Items," *European Journal of Operational Research*, 180, 617–629.

PARKES, D. C. AND J. KALAGNANAM (2005): "Models for Iterative Multiattribute Vickrey Auctions," *Management Science*, 51, 435–451, special Issue on Electronic Markets.

PARKES, D. C. AND L. H. UNGAR (2000): "Iterative Combinatorial Auctions: Theory and Practice," in *Proceedings of the 17th National Conference on Artificial Intelligence (AAAI-00)*, 74–81.

PERRY, M. AND P. J. RENY (2005): "An Efficient Multi-Unit Ascending Auction," *Review of Economic Studies*, 72, 567–592.

SANKARAN, J. K. (1994): "On a Dynamic Auction Mechanism for a Bilateral Assignment Problem," *Mathematical Social Sciences*, 28, 143–150.

SHAPLEY, L. S. AND M. SHUBIK (1972): "The Assignment Game I: The Core," *International Journal of Game Theory*, 1, 111–130.

VAN DEN BERG, G. J., J. C. VAN OURS, AND M. P. PRADHAN (2001): "The Declining Price Anamoly in Dutch Rose Auctions," *American Economic Review*, 91, 1055–1062.

VICKREY, W. (1961): "Counterspeculation, Auctions, and Competitive Sealed Tenders," *Journal of Finance*, 16, 8–37.

APPENDIX 1: AN EXPERIMENTAL ANALYSIS OF SPEED AND PREFERENCE ELICITATION

In this section, we compare the speed and preference elicitation properties of our auctions with their ascending price counterparts. Our analysis is based on computer simulations. For speed, our metric for comparison is the number of iterations in the auction. For preference elicitation, we define a new metric (the *uncertainty index*), which provides a measure of the accuracy with which agent valuations are revealed to the auction.

In our computer simulations the input parameters are the number of buyers, the number of items (or units), and the *density*. In the unit-demand setting the density is a real number between 0 and 1, and reflects the probability with which a buyer will have positive value on an item: a higher density means that more buyers have positive value on items. In the homogeneous item setting, a higher density reflects a higher probability that a buyer has positive marginal value for a unit. All values are integers and the bid decrement in both

auctions is set to one throughout. All results are averaged over 100 trials. The results depend on the starting prices in the auctions. Our choice here is to simply set the starting price in the ascending auctions to the minimal possible value for an item and the starting price in the descending auctions to the maximal possible value.

5.1 THE LVD AUCTION: UNIT DEMAND

We simulate the LVD auction and the DGS auction[7] using a computer program and with agents following the truthful equilibrium. Valuations of buyers for items are drawn from a uniform distribution with range $[0, 100]$ and assigned according to the density. The starting price on every item in the LVD auction is 100 (the upper limit of valuation) and zero in the DGS auction (the lower limit of valuation). Thus, this is completely symmetric for the two different auctions. Before presenting the simulation results, we propose a novel way of measuring preference elicitation in iterative auctions.[8]

5.1.1 A Preference Elicitation Measure

We propose a metric that measures the amount of *revealed preference* information that is implied by the bids placed by buyers in response to increasing or decreasing prices. In the equilibrium of both the DGS and the LVD auctions, the demand set reported by a buyer in a round implies constraints on an agent's valuation. Let T be a finite set of iterations in an auction. A valuation function $\hat{v}_i$ is a **consistent valuation** in iteration t for buyer i if for all $t' \in T$ and for all $j \in D_i(p^{t'})$ we have that

$$\hat{v}_i(j) - p^{t'}(j) = \hat{v}_i(k) - p^{t'}(k) \qquad \forall\ k \in D_i(p^{t'}) \tag{2}$$

$$\hat{v}_i(j) - p^{t'}(j) = \hat{v}_i(k) - p^{t'}(k) + 1 \qquad \forall\ k \notin D_i(p^{t'}). \tag{3}$$

The size of the set of consistent valuations at the end of an auction indicates the amount of information revealed: a large set corresponds with considerable uncertainty about valuations and a small set corresponding to little uncertainty. The consistent valuations in the LVD and DGS auctions form a lattice, which leads to a natural metric for preference elicitation. The proof of the following appears in the second part of the appendix.

THEOREM **5** *Suppose the valuation of every buyer is drawn from a distribution with finite support. Then the set of consistent valuations of every buyer in the LVD auction and the DGS auction forms a complete lattice.*

---

[7]The ascending price auction in Demange et al. (1986) has been modified in Sankaran (1994). This modification has a computationally faster price adjustment procedure than the original ascending price auction of Demange et al. (1986), and is suitable for practical implementation. We refer to the auction of Demange et al. (1986), as modified by Sankaran (1994), as the **DGS auction**.

[8]See Parkes and Kalagnanam (2005) for a related, but different, approach in which the probability mass that is consistent with the revealed information is adopted as a metric of the remaining uncertainty about preferences.

The previous theorem says that for every buyer i and for all histories there exists a unique *greatest consistent valuation*, v_i^{max}, and a unique *least consistent valuation*, v_i^{min}. We define the *uncertainty index UI_i* for buyer i as:

$$UI_i = \sum_{j \in A_i} \frac{\left[v_i^{max}(j) - v_i^{min}(j)\right]}{100 \times |A_i|}, \tag{4}$$

where $A_i = \{j \in A : v_i(j) > 0\}$. This is the sum of the difference between the greatest and smallest consistent values on items for which the buyer has non-zero value. We normalize this by the size of the domain from which the valuations are drawn (here, 100-0=100) and the number of items on which it has non-zero value. The UI metric ignores the items for which a buyer's value is zero because we choose to model a buyer as finding it easy to identify items for which he has no value.[9]

The uncertainty index (UI) of an auction is the average uncertainty index across all buyers and then averaged over a number of instances.[10] Note that the index captures the amount of preference information not elicited and thus a large index is better than a small index.

5.1.2   Simulation Results: Unit Demand

We fix the number of items at 5, the density at 0.75 (i.e. on average, every buyer has has non-zero value for 75% of the items), and vary the number of buyers from 5 to 50. A comparison of the speed of the LVD auction and the DGS auction is shown in Figure 3(a), while a comparison of preference elicitation (using the uncertainty index) is shown in Figure 3(b). We also plot the *average market clearing price*, defined as the ratio between the total payments made by the bidders and the total number of items in the market (and normalized in Figure 3(a) (b).); this provides a measure of the overall level of competition in the market.

When the number of buyers is relatively low the DGS auction has better speed and preference elicitation properties. As the number of buyers increase, the LVD auction does better

---

[9]The results that we present are qualitatively unchanged if the uncertainty index is defined as the average residual value uncertainty across all items.

[10]To illustrate the idea of UI, we give an example of auctioning a single item. Suppose there are four buyers with values 10,8,6, and 4 respectively, drawn from a uniform distribution with range [0,10]. Suppose the starting price in the Vickrey-Dutch auction is 10 and that in the English auction is 0. Both the Vickrey-Dutch auction and the English auction will terminate at price 8. All buyers except the first buyer will reveal their valuations in the English auction, whereas the final price gives a lower bound of 9 on the valuation of the first buyer. So, the least consistent valuation profile and the greatest consistent valuation profile in the English auction are $(9, 8, 6, 4)$ and $(10, 8, 6, 4)$ respectively, and the UI is the average of UI of all the buyers, i.e., $\frac{1+0+0+0}{4 \times 10} = 0.025$. In the Vickrey-Dutch auction, the first and second buyers reveal their valuations. The final price gives an upper bound of 7 on the valuations of the third and fourth buyers. So, the greatest consistent valuation profile and the least consistent valuation profile in the Vickrey-Dutch auction are $(10, 8, 7, 7)$ and $(10, 8, 0, 0)$ respectively. So, the UI of the Vickrey-Dutch auction is $\frac{0+0+7+7}{4 \times 10} = 0.35$.
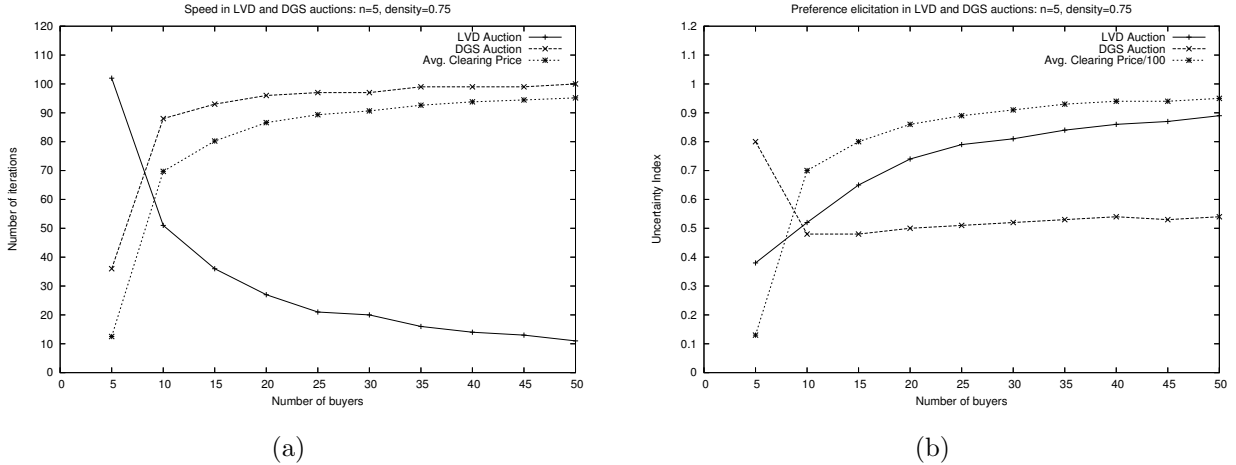
Figure 3: The speed and average uncertainty index in the LVD and the DGS auction.

in terms of both speed and preference elicitation. It is nice to observe that the cross-over point, beyond which the LVD auction dominates the DGS auction, occurs for prices around 50 which is the mean of the domain of values on items (0,100) in this environment (note that 0 and 100 also represent the initial prices in the DGS and LVD auctions respectively). Indeed, the number of iterations in the DGS auction tracks the clearing price while the number of iterations in the LVD auction tracks 100 minus the clearing price.

In terms of the average uncertainty index, the LVD auction is able to achieve approaching 90% UI for a large number of buyers. This is a significant improvement over the DGS auction for a large range of the number of buyers.[11][12]

5.2   The Clinching Vickrey Dutch Auction: Homogeneous Items

Turning to the environment with multiple units of a homogeneous item we define a distribution on valuations that satisfy the non-increasing marginal value requirement. The marginal value of the first unit is drawn uniformly from the integers on the range $[50, 100]$. For buyer i and $k > 1$, if $\hat{v}_i^{k-1}$ is the marginal value of the $(k-1)^{\text{st}}$ unit, then the marginal value of the

---

[11]One surprising observation from this simulation is that although the UI for the DGS auction initially decreases with the number of buyers, it is approximately steady (and even slightly improving) as the number of buyers increases from 10 towards 50. We expect that this results from two factors: (a) the number of iterations in DGS is fairly constant (and large) on this range; (b) as the clearing prices become higher on this range more buyers are "priced-out" and stop revealing information.

[12]We also run simulations with (a) a fixed density and number of buyers and varying the number of items and (b) a fixed number of buyers and items and varying the density. In both these experiments, we see very similar effects: at high average market clearing prices the LVD auction does better than the DGS auction but at low market clearing prices the DGS auction does better than the LVD auction. The cross-over point occurs consistently around an average clearing price of 50.

k^{th} unit is drawn uniformly from integers in the range $[\frac{\hat{v}_i^{k-1}}{2}, \hat{v}_i^{k-1}]$ with probability equal to density, where density is an input to our simulations. With probability equal to (1-density), we set the marginal value of kth ($k > 1$) unit of buyer i at zero (and zero value on future units as well).

Before presenting the simulation results, we describe how we measure preference elicitation in this setting. The revealed preference information implicit in the demand sets in the equilibrium of the clinching and AUS auctions in this environment provides information about the *marginal value* on each unit rather than the absolute value. This is because the prices in these auctions define the price on a marginal unit relative to the number of units already clinched. For this reason, the uncertainty index is defined here in terms of information about the *marginal valuations*, but is otherwise unchanged: we measure the difference between the greatest consistent *marginal* valuation and the least consistent *marginal* valuation, normalized by dividing it over 100, and considering only those units whose marginal value is non-zero.[13]

5.2.1   Simulation Results: Homogeneous Items

We fix the number of units at 20, the density at 0.75, and vary the number of buyers from 5 to 50. The plots for speed and for preference elicitation are shown in Figures 4(a) and 4(b). We continue to include the average market clearing price, defined here as the total payment made by the bidders divided by the number of units, and again normalized by dividing by 100 in the preference elicitation plot.
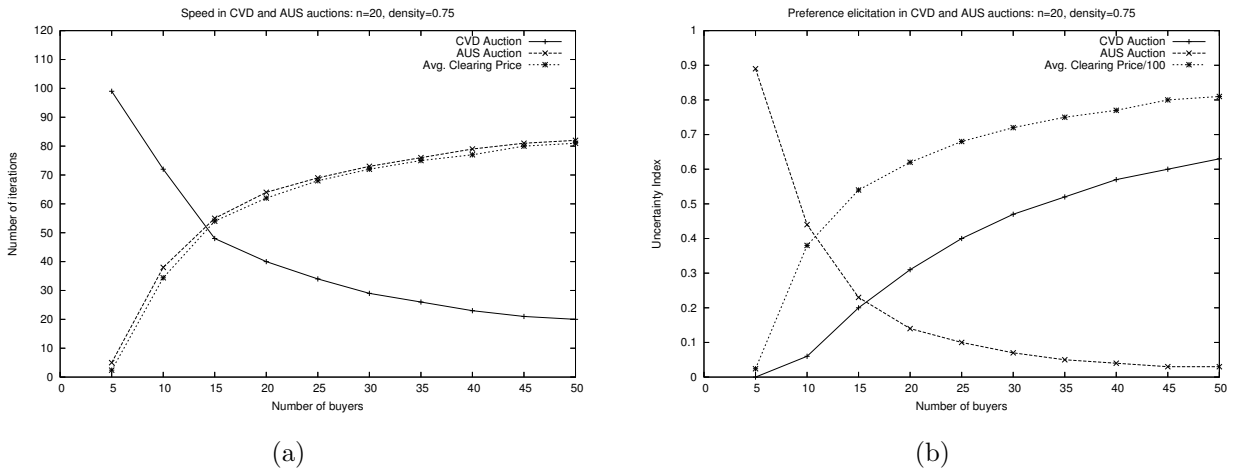

(a)   (b)

Figure 4: The average number of iterations and preference elicitation in the clinching (CVD) and the AUS auctions.

---

[13]The consistent marginal valuations form a lattice in both auctions. This can be verified along the lines of proof of Theorem 5, and is omitted in the interest of space.

We again see an excellent correspondence between the average clearing price and the number of iterations, with the clinching auction (labeled CVD) converging more quickly than AUS for a large number of buyers, with the crossover corresponding to an average clearing price of 50. The descending auction also enjoys better preference elicitation properties than the ascending auction for a large number of buyers. In this simulation the index of the AUS auction continues to decrease as the number of buyers increases, apparently trending towards 100% elicitation for a large number of buyers. The preference elicitation properties of the clinching auction are significantly better than the AUS auction for a moderate to large number of buyers.[14]

Appendix 2: Proofs

LEMMA **1**. $U(p, D(p), x) = U(p', D'(p'), x') \cup A_0(p)$ where $(p', D'(p'), x')$ is the restriction of $(p, D(p), x)$ to items with positive price and $A_0(p)$ is the set of items with zero price.
Proof: To show $U(p, D(p), x) \supseteq U(p', D'(p'), x') \cup A_0(p)$, if $j \in A_0(p)$ then $j \in U(p, D(p), x)$ by definition. If $j \in U(p', D'(p'), x')$ then j is allocated in x' and thus also in x, and moreover there is some $y' \in X(D'_{-i}(p'))$ (where j is allocated to i in x) with $S(y') = S(x')$. Now, consider $z \in X(D_{-i}(p))$. Clearly, $S(z) = S(y')$ because z must allocate as many items with positive price as y' for it to be a provisional allocation. Since $S(y') = S(x') = S(x)$, we have $S(z) = S(x)$. To show $U(p', D'(p'), x') \cup A_0(p) \supseteq U(p, D(p), x)$, consider $j \in U(p, D(p), x)$. If $p(j) = 0$ then $j \in A_0(p)$. If $p(j) \neq 0$ then j is allocated in x, to say i. Then, there is some $y \in X(D_{-i}(p))$ with $S(y) = S(x)$. Construct $y' \in X(D'_{-i}(p'))$ with $S(y') = S(x') = S(x)$ by assigning the dummy item to any agent $k \neq i$ allocated to an item with zero price in allocation y. Thus, item $j \in U(p', D_{-i}(p'), x')$. ∎

LEMMA **2**. Given a price vector p with non-zero prices for all the items, item $j \in U(p, D(p), x)$ if and only if there is a well-defined chain $z_{-i}(x, D(p)) = j_0 i_1 j_1 i_2 j_2 \ldots i_c j$ where buyer i is allocated j in allocation x.
Proof: Given such a chain $z_{-i}(x, D(p))$, the modified allocation x' induced by the chain when applied to x satisfies $x' \in X(D_{-i}(p))$ and $S(x') = S(x)$ by definition. Now, consider some $j \in U(p, D(p), x)$. We construct a well-defined chain $z_{-i}(x, D(p))$, where i is the buyer assigned j in x. Consider allocation $y \in X(D_{-i}(p))$ with $S(y) = S(x)$. Let $i^{(1)}$ denote the buyer assigned to j in y (such a buyer must exist). Let $j^{(1)}$ denote the item assigned to buyer $i^{(1)}$ in x. If $j^{(1)} = 0$ then the chain is $0 i^{(1)} j$. Otherwise, let $i^{(2)}$ denote the buyer assigned to $j^{(1)}$ in y and let $j^{(2)}$ denote the item assigned to this buyer in x. Now, if $j^{(2)} = 0$ then the chain is $0 i^{(2)} j^{(1)} i^{(1)} j$. Eventually, for some $q \leq n$ there must be a buyer $i^{(q)}$ assigned

---

[14]We also experimented with (a) fixing the density and the number of buyers and varying the number of units; (b) fixing the number of items and the number of buyers and varying the density. In both these experiments, we found similar conclusions: at higher market clearing prices the clinching auction performs better than the AUS auction in terms of both speed and preference elicitation, with the cross-over occurring at an average market price of around 50.

item $j^{(q-1)}$ in y but unallocated in x and this completes the chain $0i^{(q)}j^{(q-1)}\ldots i^{(1)}j$. Such a buyer must exist because y must allocate to at least one buyer not allocated in x since $S(y) = S(x)$. ∎

PROPOSITION **2** The FINDUNIVALLOCITEMS procedure determines all universally allocated items.

Proof: Each repetition of steps 1–3 in procedure FINDUNIVALLOCITEMS is referred to as a round, indexed $r = 1, 2, \ldots$. By Lemma 1 we can reduce the problem of determining the set of universally allocated items to that of finding the set $U(p', D'(p'), x')$ where $(p', D'(p'), x')$ is restricted to items with positive prices. For these, we know from Lemma 2 that an item j is universally allocated if and only if there is a well-defined chain, starting from an unallocated agent in x' and terminating with the item j. The FINDUNIVALLOCITEMS procedure completes the closure of all items reachable by any well-defined chain, first initializing $U(p, D(p), x)$ to the set of items demanded by an agent unallocated in x' (and thus reachable by a chain of length 1), and then in each round $r = 1, 2, \ldots$ identifying all allocated items W reachable by some chain of length $r + 1$. The items with zero price $A_0(p)$ are finally added to the set of universally allocated items. ∎

PROPOSITION **3** In the provisional allocation in every iteration of the LVD auction, every buyer is satisfied under truthful bidding.

Proof: The proof is by induction on the iteration $t \geq 0$ of the auction. The base case is easy because every buyer demands only the dummy item. Now, suppose the claim holds in iteration $t - 1$ and consider iteration $t > 0$. Let $x^{t-1} \in X(D(p^{t-1}))$. By assumption all buyers are satisfied in x^{t-1}.

We will first show that x^{t-1} remains *admissible*, so that every buyer is still satisfied with provisional allocation x^{t-1} in iteration t. Consider a buyer i and three possible cases.

Case 1: Buyer i is unallocated in iteration $t - 1$. Since i is satisfied, $0 \in D_i(p^{t-1})$ and for all $j \in D_i(p^{t-1}) \setminus \{0\}$ then j is universally allocated in iteration $t-1$ by Lemma 2. Thus, the price is unchanged in iteration t on all items in the demand set of i, while the price falls by at most 1 on all other items. This implies $D_i(p^t) \supseteq D_i(p^{t-1})$.

Case 2: Buyer i is allocated item $j \neq 0$, but j is not universally allocated in iteration $t - 1$. Thus, the price on this item falls in iteration t and since the price on no other item falls by more than this, $j \in D_i(p^t)$.

Case 3: Buyer i is allocated item $j \neq 0$ that is universally allocated. We show that for every $j' \in D_i(p^{t-1})$, we have $j' \in U(p^{t-1}, D(p^{t-1}), x^{t-1})$. Consider the interesting case that $p^{t-1}(j') > 0$ and suppose that j' is not allocated in x^{t-1}. But, the auction could have then achieved more revenue in iteration $t-1$ with allocation $y \in D_{-i}(p^{t-1})$ such that $S(y) = S(x)$ (which exists since j is universally allocated) and then augmenting allocation y to assign item j' to agent i. Thus, this would contradict with x^{t-1} being a provisional allocation. Now consider the chain $z_{-i}(x^{t-1}, D(p^{t-1}) = 0i_1j_1\ldots i_cj$ that establishes that item j is universally allocated. If buyer i', allocated to item j' in x^{t-1}, is not represented in the chain then the

30

well-defined chain $0i_1j_1 \ldots i_c j i j'$ establishes that j' is also universally allocated. On the other hand, if i' is in the chain, for instance $z_{-i}(x^{t-1}, D(p^{t-1})) = 0i_1j_1i_2j'i'j_3 \ldots i_c j$ then consider truncated chain $0i_1j_1i_2j'$ which is well-defined for $z_{-i'}(x^{t-1}, D(p^{t-1}))$ and establishes that j' is universally allocated. Since all items in $D_i(p^{t-1})$ are universally allocated, the price of every item remains unchanged in p^t while the price on other items falls by at most 1 on any other item. This implies $D_i(p^t) \supseteq D_i(p^{t-1})$.

Second, we show that a provisional allocation x^t exists in iteration t that satisfies every buyer. We provide a sequence of transformations to construct such a provisional allocation from any provisional allocation $\hat{x}^t$ in iteration t (i.e. $\hat{x}^t$ may assign the dummy item to some agents that do not have it in their demand sets). Initialize $B^{(0)}$ to the set of buyers not satisfied in $\hat{x}^t$ and initialize $A^{(0)} = \emptyset$, to denote the set of items whose assignment is fixed. Pick any $i \in B^{(0)}$ and consider item j allocated to i in x^{t-1}. By the above reasoning we know that $j \in D_i(p^t)$, and thus we construct admissible allocation $x^{(1)}$ by assigning item j to buyer i and assigning the buyer assigned to j in $\hat{x}^t$ to the dummy item. Set $A^{(1)} = \{j\}$ to indicate that the allocation of item j is now fixed. The revenue achieved by the seller in $x^{(1)}$ is the same as in $x^{(0)}$. Define $B^{(1)}$ as the buyers not satisfied in allocation $x^{(1)}$. Continue, picking a buyer $i \in B^{(1)}$, fixing the allocation of another item, and so on. Since this process assigns buyers to their assignments in x^{t-1}, eventually an admissible allocation is achieved with the same revenue as $\hat{x}^t$ that satisfies every buyer. ∎

LEMMA **3** In every iteration $t \geq 0$ of the clinching auction in the homogeneous items environment, we have
- (a) $\overline{D}_i(p^{t+1}) \geq \overline{D}_i(p^t)$
- (b) $D_i(p^t) = \{0, 1, \ldots, \overline{D}_i(p^t)\}$, and
- (c) $v_i(j) - v_i(j-1) \leq p_i^t(j) - p_i^t(j-1)$ for all $0 < j \leq n$, for all buyers $i \in B$.

Proof: Consider any buyer $i \in B$. The base case for $t = 0$ is easy because the prices are initially high and $\overline{D}_i(p^0) = 0$ (and thus $\overline{D}_i(p^1) \geq \overline{D}_i(p^0)$), $D_i(p^0) = \{0, \ldots, \overline{D}_i(p^0)\}$ and $v_i(j) - v_i(j-1) \leq q^0$ for all $0 < j \leq n$. For iteration $t > 0$, consider the interesting case that $\overline{D}_i(p^{t-1}) < n$. Otherwise the prices are unchanged. By the inductive hypothesis (b) and the price change rule, we have $v_i(j) - v_i(j-1) = p_i^{t-1}(j) - p_i^{t-1}(j-1) = p_i^t(j) - p_i^t(j-1)$ for all $0 < j \leq \overline{D}_i(p^{t-1})$, and $v_i(\overline{D}_i(p^{t-1})+1) - v_i(\overline{D}_i(p^{t-1})) < p_i^{t-1}(\overline{D}(p^{t-1})+1) - p_i^{t-1}(\overline{D}(p^{t-1})) = q^{t-1}$. By non-increasing marginal values, $v_i(j) - v_i(j-1) \leq v_i(\overline{D}_i(p^{t-1})+1) - v_i(\overline{D}_i(p^{t-1})) < p^{t-1}(j) - p^{t-1}(j-1) = q^{t-1} = q^t + 1$ for all $\overline{D}_i(p^{t-1}) < j \leq n$. This implies that $v_i(j) - v_i(j-1) \leq q^t = p_i^t(j) - p_i^t(j)$ for all $\overline{D}_i(p^{t-1}) < j \leq n$, where the last equality follows from the price change rule. This establishes inductive hypothesis (c) for iteration t. Property (a) is true since $\overline{D}_i(p^t) \geq \overline{D}_i(p^{t-1})$, and the price on marginal unit $\overline{D}_i(p^{t-1}) + 1$ falls while the price on all units $\leq \overline{D}_i(p^{t-1})$ is unchanged. Finally, for inductive hypothesis (b), assume $j \in D_i(p^t)$. If $j > 0$, then we will show that $j-1 \in D_i(p^t)$. Since $j \in D_i(p^t)$, $v_i(j) - v_i(j-1) \geq p_i^t(j) - p_i^t(j-1)$. From (c), which was proved earlier, we have $v_i(j) - v_i(j-1) \leq p_i^t(j) - p_i^t(j-1)$. Together, it gives $v_i(j) - v_i(j-1) = p_i^t(j) - p_i^t(j-1)$. This implies $j - 1 \in D_i(p^t)$. Hence

(b) holds. ∎

THEOREM **5** Suppose the valuation of every buyer is drawn from a distribution with finite support. Then the set of consistent valuations of every buyer in the LVD auction and the DGS auction forms a complete lattice.

Proof: Let v_i^1 and v_i^2 be two consistent valuations of buyer i in the LVD auction. Define $v_i^3 := \min(v_i^1, v_i^2)$ and $v_i^4 := \max(v_i^1, v_i^2)$. Consider an iteration t of the LVD auction. We will show that v_i^3 satisfies Equations (2) and (3) in iteration t. Consider $j \in D_i(p^t)$. Suppose $v_i^3(j) = v_i^1(j)$. Consider the case when $k \in D_i(p^t)$. If $v_i^3(k) = v_i^1(k)$, then $v_i^3(j) - v_i^3(k) = v_i^1(j) - v_i^1(k) = p_j^t - p_k^t$. Else, $v_i^1(k) > v_i^2(k)$. This gives, $v_i^1(j) - v_i^1(k) < v_i^2(j) - v_i^2(k)$. But, $v_i^1(j) - v_i^1(k) = v_i^2(j) - v_i^2(k) = p_j^t - p_k^t$. This gives a contradiction, implying that $v_i^3(k) = v_i^1(k)$ and $v_i^3(j) - v_i^3(k) = v_i^1(j) - v_i^1(k) = p_j^t - p_k^t$. Now, consider the case when $k \notin D_i(p^t)$. Suppose $v_i^3(j) = v_i^1(j)$. $v_i^3(j) - v_i^3(k) = v_i^1(j) - v_i^3(k) \geq v_i^1(j) - v_i^1(k) = p_j^t - p_k^t$. Thus, Equations (2) and (3) are satisfied by valuation function v_i^3. So, v_i^3 is a consistent valuation. Similar arguments show that v_i^4 is a consistent valuation. This shows that the set of consistent valuations is a lattice in the LVD auction. A similar proof follows for the DGS auction. Since the valuations are drawn from a distribution with finite support, this lattice is complete. ∎