GAME THEORY - FINAL EXAMINATION

Date: November 9, 2017
Total marks: **30**
Duration: 10:00 AM to 1:00 PM

Note: Answer all questions clearly using pen. Please avoid unnecessary discussions. In all questions, an incomplete description of a strategy or equilibrium will be considered an incorrect answer.

1. Consider a game Γ with two players $\{1, 2\}$. Player 2 can be one of the following types: type G with probability $\mu \in [0, 1]$ and type B with probability $1 - \mu$. The probability μ is common knowledge. Player 2 realizes his type (which Player 1 does not observe), and then each player simultaneously chooses one of the actions: Player 1 chooses from $\{X, Y\}$; Player 2 chooses from $\{x, y\}$.

   Consider Figure 1. If Player 2 is of type G, then the payoffs corresponding in G_1 is realized. If Player 2 is of type B, then the payoffs in G_2 is realized.

		Player 2	
		x	y
Player 1	X	$(0, -2)$	$(-10, 1)$
	Y	$(-1, -10)$	$(-5, -5)$

Game G_1

		Player 2	
		x	y
Player 1	X	$(0, -2)$	$(-10, -7)$
	Y	$(-1, -10)$	$(-5, -11)$

Game G_2

Figure 1: Bayesian game

   (a) What is a strategy of Player 1 in the game Γ? What is a strategy of Player 2 in the game Γ? **(2 marks)**

   ANSWER. Player 1's strategy is an element from the set $\{X, Y\}$. Player 2's strategy is a map: $s_2 : \{G, B\} \to \{x, y\}$.

   (b) Describe a Bayesian equilibrium of the game Γ. **(5 marks)**

   ANSWER. If Player 2 is of type G: then y strictly dominates x. If Player 2 is of type B: then x strictly dominates y. Hence, any Bayesian equilibrium (s_1, s_2) of Γ must have $s_2(G) = y, s_2(B) = x$.

   Given this s_2, playing X gives Player 1 a payoff of -10μ and playing Y gives a payoff of $-5\mu - (1 - \mu) = -4\mu - 1$. So, playing X is a best response if $-10\mu \geq -4\mu - 1$ or $\mu \leq \frac{1}{6}$. Similarly, playing Y is a best response if $\mu \geq \frac{1}{6}$.

So, the Bayesian equilibrium if $(s_1 = X, s_2(G) = y, s_2(B) = x)$ if $\mu < \frac{1}{6}$, $(s_1 = Y, s_2(G) = y, s_2(B) = x)$ if $\mu > \frac{1}{6}$ and Player 1 mixing X and Y in any way, $s_2(G) = y, s_2(B) = x$ if $\mu = \frac{1}{6}$.

(c) Describe game Γ as an extensive form game of incomplete information - a game tree representation will suffice. **(3 marks)**

ANSWER. See Figure 2 and 3.

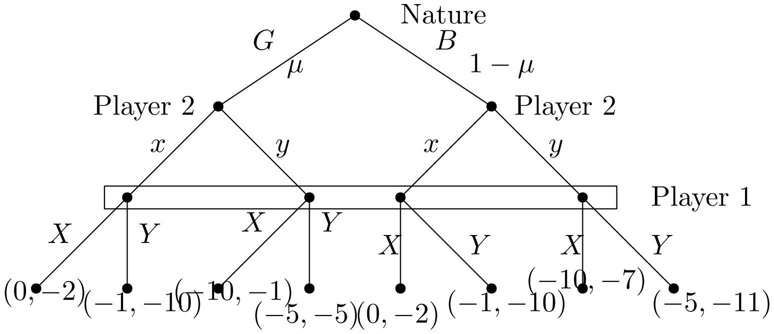

Figure 2: Extensive form game representation of Γ

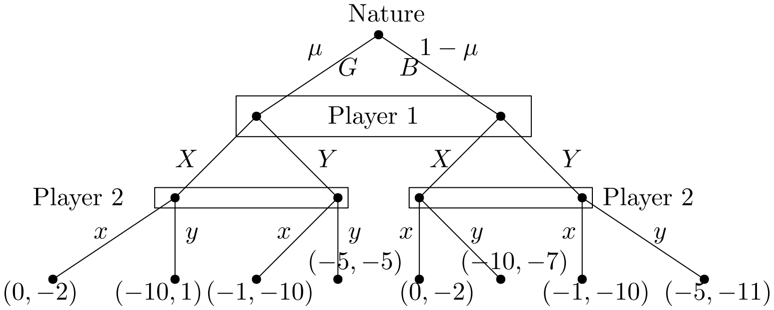

Figure 3: Extensive form game representation of Γ

2. Consider the extensive form game in Figure 4.

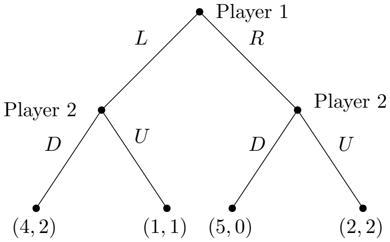

Figure 4: Extensive form game

(a) Find all pure strategy Nash equilibria and subgame perfect Nash equilibria of this game. **(3 marks)**

Answer. The reduced form of the game in Figure 4 is shown in Figure 5. It is easily verified that $(L, (DU))$ and $(R, (UU))$ are two pure strategy Nash equilibria of this game.

		Player 2			
		DD	DU	UD	UU
Player 1	L	$(4, 2)$	$(4, 2)$	$(1, 1)$	$(1, 1)$
	R	$(5, 0)$	$(2, 2)$	$(5, 0)$	$(2, 2)$

Figure 5: Reduced form of game in Figure 4

Unique subgame perfect equilibrium: Player 2 chooses D at the left decision vertex and U at the right decision vertex; Player 1 chooses L.

(b) Suppose Player 2 can observe the move of Player 1 with probability $p \in (0, 1)$, i.e., if Player 1 plays L, Player 2 will observe L with probability p and R with probability $(1-p)$ and if Player 1 plays R, Player 2 will observe R with probability p and L with probability $(1 - p)$. So, Player 2 observes L or R but does not know if it is the correct one.

i. Describe this as an extensive form game of incomplete information - a game tree representation with information sets is sufficient. **(4 marks)**

Answer. See Figure 6. First Player 1 moves followed by a move by Nature which alters the action of Player 1 (with probability p keeps it as it is and with probability $(1 - p)$ flips it).

Figure 7 gives another representation of the same game. Here, Nature moves to decide if it will be a correct observation for Player 2 or incorrect observation. Then, Player 1 moves L or R. Player 2 can observe L but that may come from a correct L or incorrect R. This gives rise to the information sets as shown.

ii. Find a perfect Bayesian equilibrium of this game for $p = \frac{1}{2}$. **(3 marks)**

Answer. We compute using Figure 6 representation - a similar analysis can be done using Figure 7 also. On the left information set, let the belief on the

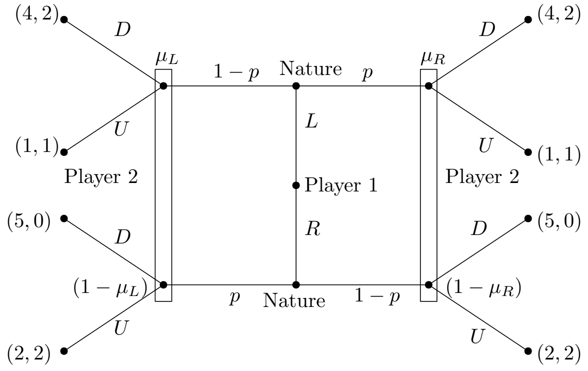

Figure 6: Extensive form game representation

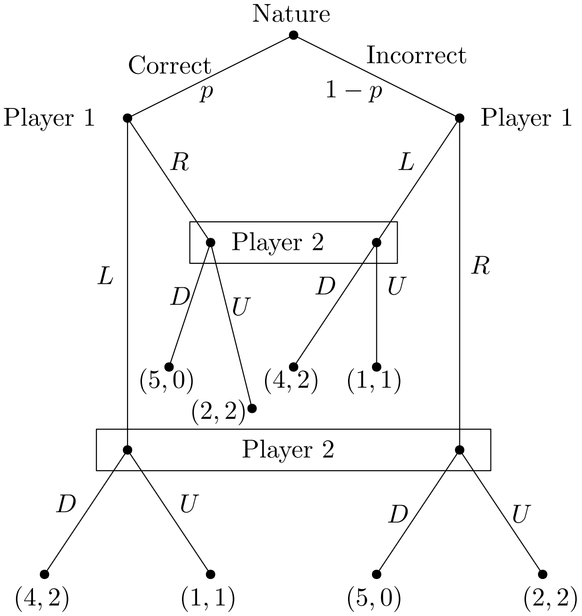

Figure 7: Extensive form game representation

upper decision vertex be μ_L. Similarly, on the right information set, let the belief on the upper decision vertex be μ_R. We consider two possibilities of perfect Bayesian equilibrium.

Case 1. Player 1 plays L. Then, $\mu_L = \mu_R = 1$ by Bayesian rationality. Then, sequential rationality of Player 2 says, he should play D in both the information sets. But, then sequential rationality of Player 1 says, he should play R. So, Player 1 playing L cannot be an equilibrium.

Case 2. Player 1 plays R. Then, $\mu_L = \mu_R = 0$ by Bayesian rationality. Then, sequential rationality of Player 2 says, he should play U in both the information sets. But, then sequential rationality of Player 1 says, he should play R.

Hence, the unique perfect Bayesian equilibrium is:

$$\Big(\mu_L = \mu_R = 0; (R, (U, U))\Big)$$

iii. Is this a sequential equilibrium? **(2 marks)**

Answer. Yes, this is also a sequential equilibrium because each information set is reached with positive probability.

3. Consider the stage game G shown in Figure 8.

Player 2

		x	y	z
Player 1	X	$(2,2)$	$(0,1)$	$(1,4)$
	Y	$(3,-1)$	$(1,1)$	$(1,0)$
	Z	$(0,0)$	$(0,3)$	$(3,1)$

Game G

Figure 8: Stage game

(a) Find the worst Nash equilibrium (pure action) for each player in G and the corresponding payoffs. **(2 marks)**

Answer. There is a unique Nash equilibrium (in pure actions) in G: (Y, y). The corresponding payoffs are $(1, 1)$.

(b) Consider the infinitely repeated game G^{∞} where players evaluate payoff using the discounted criteria. Describe a subgame perfect equilibrium strategy profile of G^{∞} where (X, x) is played on equilibrium path for sufficiently high values of discount factor. **(6 marks)**

Answer. There are two states: (1) Normal state (2) Punishment state. In the normal state, the strategy recommends actions (X, x). In the punishment state, the strategy recommends actions (Y, y). The initial state is normal. We go from initial state to punishment state if (X, x) is not played. Once we are in punishment state, we stay in punishment state.

To see that this is a subgame perfect equilibrium, any history where state is punishment, we see (Y, y) is the recommended action profile. Since it is a Nash equilibrium of G, it is a Nash equilibrium of this subgame. In any history, where state is normal state, we use one-shot deviation principle. Recommended strategy gives 2 payoff for each player. One-shot deviation for Player 1 gives

$$3 + 1 + 1 + \ldots = 3(1 - \delta_1) + \delta_1 = 3 - 2\delta_1.$$

So, if $\delta_1 \geq \frac{1}{2}$, then deviation is not profitable for Player 1. Similarly, one-shot deviation for Player 2 gives

$$4 + 1 + 1 + \ldots = 4(1 - \delta_2) + \delta_2 = 4 - 3\delta_2.$$

So, if $\delta_2 \geq \frac{2}{3}$ Player 2 does not deviate. So, for $\delta > \frac{2}{3}$ the given strategy is a subgame perfect equilibrium and sustains (X, x) on equilibrium path.