GAME THEORY I - FINAL EXAMINATION

Date: 17 November, 2023, Total marks: **50**, Time: **3 hours**

Please start each question on a fresh page and all subquestions of a question together

1. Two agents $\{m, w\}$ play the following game in Table 1. But they are uncertain if they are the ROW player or the COLUMN player. There are two states: $\{M, W\}$. State M corresponds to m agent being the ROW player and w agent being the COLUMN player. State W corresponds to w agent being the ROW player and m agent being the COLUMN player. Probability of State M is $(1-\epsilon)$ and State W is $\epsilon \in (0, 1)$. These probabilities are common knowledge.

   The game goes as follows. Agent w observes the state but agent m does not. Then, both the agents simultaneously take one of the actions in $\{a, b\}$. Depending on who is the ROW player and who is the COLUMN player, players realize their payoffs according to Table 1, where ROW player payoff is written first, followed by COLUMN player payoff.

	a	b
a	$(2,1)$	$(0,0)$
b	$(0,0)$	$(1,2)$

Table 1: Normal-form game

   (a) Describe this as a Bayesian game. **(2 marks)**

   Answer. Agent m has no type. It has two actions available $\{a, b\}$. But by playing any of these actions, agent m does not know for sure what payoffs will be obtained.

   Agent w has two types (corresponding the two states): $\{M, W\}$. For each type same actions are available: $\{a, b\}$. The probabilities of M and W types are $1-\epsilon$ and ϵ respectively.

   If agent w is of type M, then agent m gets the row payoff; else agent w gets row payoff in Table 1.

   (b) Is there a Bayesian equilibrium where agent w uses the following strategy s_w: $s_w(M) = b, s_w(W) = a$ and agent m uses a pure strategy? **(6 marks)**

   Answer. Consider an equilibrium where agent w uses the following strategy: $s_w(M) = b, s_w(W) = a$

Suppose agent m plays a in this equilibrium. Then, payoff of agent w is zero when type/state is M. This can be improved if agent w chooses a. So, in no such equilibrium agent m can play a.

Similarly, if agent m plays b, then payoff of agent w is zero when type is W. This can be improved if agent w chooses b.

Hence, there is no Bayesian equilibrium where agent m plays a pure strategy.

(c) Is there a Bayesian equilibrium where agent w uses the following strategy s_w: $s_w(M) = b, s_w(W) = a$ and agent m mixes? **(8 marks)**

Answer. Suppose agent m mixes: a with probability p and b with probability $(1-p)$. For agent m to mix, he should be indifferent between a and b. His payoff by playing a is calculated as follows:

- With probability $(1-\epsilon)$, state is M, in which case agent w (COLUMN player) plays b. This gives zero payoff to both the players.
- With probability ϵ, state is W, in which case agent w plays a. This gives a payoff 1 to the m (COLUMN player).

Hence, expected payoff is ϵ. Similarly, payoff by playing b is calculated as follows:

- With probability $(1-\epsilon)$, state is M, in which case agent w (COLUMN player) plays b. This gives $(1-\epsilon)$ payoff to m.
- With probability ϵ, state is W, in which case agent m plays a. This gives a payoff 0 to both the players.

Hence, expected payoff is $1-\epsilon$. So, for agent m to be randomize, we need $1-\epsilon = \epsilon$, which is $\epsilon = 0.5$. So, there is no Bayesian equilibrium if $\epsilon \neq 0.5$.

Now, assume $\epsilon = \frac{1}{2}$. Then, agent m can randomize $pa + (1-p)b$. Next, we compute the value of p that makes agent w play s_w.

- If state/type is M, then expected payoff by playing s_w (choosing action b) is $2(1-p)$. But by choosing action a, expected payoff is p. Hence, we need $2(1-p) \geq p$ or $p \leq \frac{2}{3}$.
- If state/type is W, then expected payoff by playing s_w (choosing action a) is $2p$. But by choosing action b, expected payoff is $(1-p)$. Hence, we need $2p \geq 1-p$ or $p \geq \frac{1}{3}$.

Hence, s_w is optimal if $p \in [\frac{1}{3}, \frac{2}{3}]$. So, if $\epsilon = \frac{1}{2}$, there is a unique Bayesian equilibrium where agent w plays s_w and agent m randomizes $pa + (1-p)b$, where $p \in [\frac{1}{3}, \frac{2}{3}]$.

2. Consider the normal form game in Table 2. Suppose this game is infinitely repeated and players use discounted criterion (with common discount factor $\delta \in (0, 1)$).

	C	D
C	$(2, 2)$	$(0, 3)$
D	$(3, 0)$	$(1, 1)$

Table 2: Normal-form game

(a) Consider the following strategy $s^\sharp$, which we call *imitate*. In the first period, play C. At any any history h^t, if in the previous period history (which leads to h^t) the other player has taken action $a \in \{C, D\}$, then play a; i.e., you match the action of the other player in the previous period. Represent $s^\sharp$ using a finite automaton. **(4 marks)**

Answer. There are four states: $\{cc, cd, dc, dd\}$ corresponding to action profiles in the previous period. Initial state is cc. In state cc both players take action C and in state dd both take action D. In state cd, the Row player takes action D and the Column player takes action C. In state dc, the Row player takes action C and the column player takes action D.

In all the states, if action profile XY is played then state becomes xy, where $X, Y \in \{C, D\}$ and $x, y \in \{c, d\}$.[1]

(b) Assume that the Column player uses $s^\sharp$. At every possible history of this game, argue whether $s^\sharp$ is one-shot deviation optimal for the Row player. If your answer depends on the value of δ, describe the values of δ for which your answer to this question is YES and NO. **(8 marks)**

Answer. We verify one shot deviation at all four kinds of histories.

[1] Having only two states to represent the action of the opponent in the previous history should be enough for describing the finite automaton, and the strategy. I will give full credit if someone has done so.

cc. By deviating Row player plays D in the first period and sticks to imitation from there on. As a result, the action profile stream is

$$(D, C), (C, D), (D, C), (C, D), \ldots$$

This gives the following payoff to the Row player:

$$(1-\delta)[3 + 0 + 3\delta^2 + 0 + 3\delta^4 + \ldots] = \frac{3}{1+\delta}$$

By playing the imitation strategy, (C, C) is played in every period. Hence, payoff is 2. So deviation payoff is more if

$$\frac{3}{1+\delta} - 2 > 0 \quad \text{or} \quad \delta < \frac{1}{2}$$

Deviation payoff is less if $\delta > \frac{1}{2}$.

dd. By deviating Row player plays C in the first period and sticks to imitation after that. As a result the action profile stream is

$$(C, D), (D, C), (C, D), (D, C), \ldots$$

This gives the following payoff to the Row player:

$$(1-\delta)[0 + 3\delta + 0 + 3\delta^3 + 0 + \ldots] = \frac{3\delta}{1+\delta}$$

By following the imitation strategy (D, D) played in every period. As a result, the payoff is 1. So deviation payoff is more if

$$\frac{3\delta}{1+\delta} - 1 > 0 \quad \text{or} \quad \delta > \frac{1}{2}$$

Deviation payoff is less if $\delta < \frac{1}{2}$.

dc. By deviating Row player plays D in the first period and sticks to imitation after that. As a result, the action profile stream is

$$(D, D), (D, D), (D, D), (D, D), \ldots$$

This gives a payoff of 1. By playing imitation strategy the action profile would be

$$(C, D), (D, C), (C, D), (D, C), \ldots$$

As we saw earlier, this gives a payoff of $\frac{3\delta}{1+\delta}$ to Row player. Hence, deviation is profitable for Row player if $\delta < \frac{1}{2}$ and not profitable if $\delta > \frac{1}{2}$.

cd. By deviating Row player plays C in the first period and sticks to imitation after that. As a result, the action profile stream is

$$(C, C), (C, C), (C, C), (C, C), \ldots$$

This gives a payoff of 2. By playing imitation strategy the action profile would be

$$(D, C), (C, D), (D, C), (C, D), \ldots$$

As we saw earlier, this gives a payoff of $\frac{3}{1+\delta}$ to Row player. Hence, deviation is profitable for Row player if $\delta > \frac{1}{2}$ and not profitable if $\delta < \frac{1}{2}$.

This exhausts all cases. So, we see that only for $\delta = \frac{1}{2}$. One-shot deviation is not profitable for all histories.

(c) Use (b) to argue if $(s^\sharp, s^\sharp)$ is a subgame perfect equilibrium. **(4 marks)**

Answer. We use the fact that one-shot deviation optimality is the same as subgame perfect equilibrium. Hence, for $\delta \neq \frac{1}{2}$, the imitate strategy of both the players do not constitute a subgame perfect equilibrium. On the other hand, for $\delta = \frac{1}{2}$, this is indeed a subgame perfect equilibrium.

3. Consider the extensive form game in Figure 1.

(a) Show all subgames of this game (a pictorial illustration is sufficient)? **(2 marks)**

Answer. The two subgames are the subgames starting at the two decision vertices of Player 1.

(b) Find all pure subgame perfect equilibria of this game? **(4 marks)**

Answer. First, all Nash equilibria of the smaller subgame are: (D, ℓ) and (U, r). So, playing L at the first decision vertex gives Player 1 payoff 5 which is greater

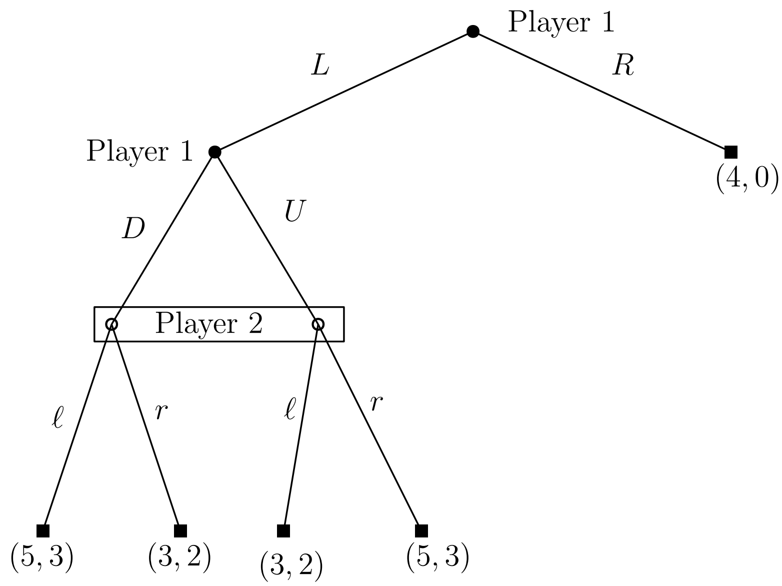

Figure 1: An extensive form game

than the payoff she gets by playing R. So, the SPNE: (L, D, ℓ) and (L, U, r).

(c) Is there a subgame perfect equilibrium of this game (where players may mix), where Player 1 plays R at the root vertex? If yes, give an example. **(4 marks)**

Answer. For Player 1 to play R, she should get at most 4 payoff by Playing L. The only mixed strategy Nash equilibrium of the smaller game involves both players randomizing: $\frac{1}{2}D + \frac{1}{2}U$ and $\frac{1}{2}\ell + \frac{1}{2}r$. The expected payoff from this is 4 to Player 1. So, $(R, \frac{1}{2}D + \frac{1}{2}U, \frac{1}{2}\ell + \frac{1}{2}r)$ is a SPNE.

(d) Is there a perfect Bayesian equilibrium where Player 1 plays R at root vertex and D at the other decision vertex and Player 2 plays a pure strategy? Will your answer change if Player 2 mixes? **(4+4=8 marks)**

Answer. There is no perfect Bayesian equilibrium where Player 2 plays ℓ or r (pure actions). To see this, if she plays r, Player 1 is better off playing U (gets payoff 5) to playing D (payoff 3). So, playing D by Player 1 is no longer sequentially rational.

Similarly, if she plays ℓ, Player 1 can play L to get a payoff 5 than playing R (gets payoff 4). So, again, sequential rationality of Player 1 is violated.

Consider the following assessment: Player 1 plays R at root and D at the next vertex; Player 2 randomizes $\frac{1}{2}\ell + \frac{1}{2}r$ and puts belief $\frac{1}{2}$ at each of her decision vertex in the information set. First, Player 2 is rational given her belief – she is indifferent between ℓ and r given beliefs. Second, her information set is not reached in this assessment. Hence, any belief is Bayesian rational. Third, Player 1 playing anything in the second decision vertex gives a payoff of 4. So, playing D is sequentially rational. Finally, Player 1 playing L or R gives a payoff of 4. So, playing R is sequentially rational.