Game Theory Mid-term – Sept 2016

Q1.

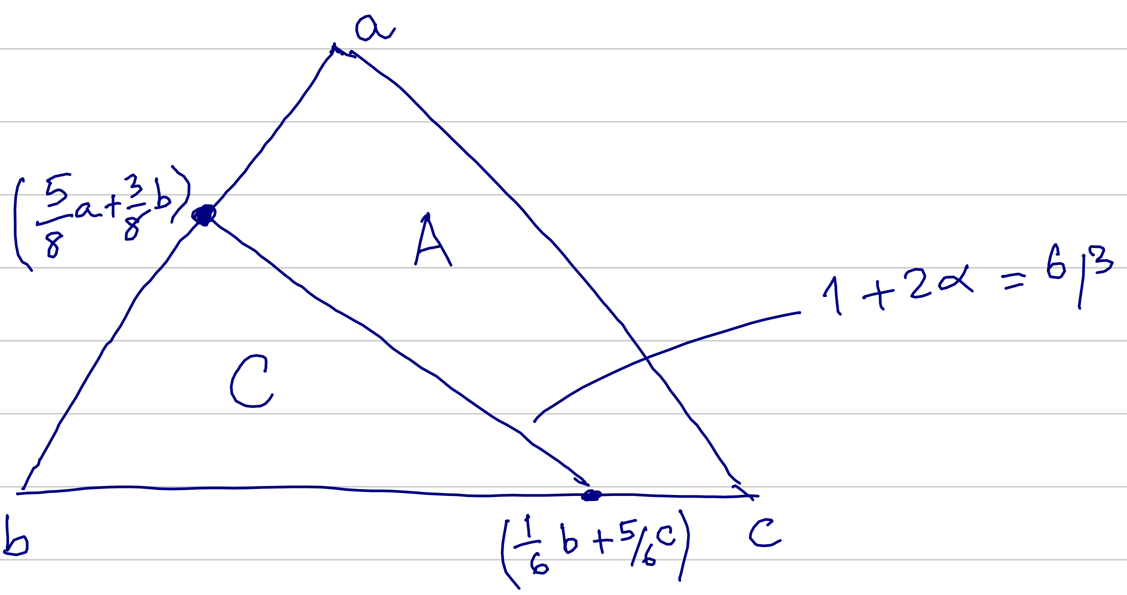
→ Strategy B is strictly dominated by $\frac{1}{2}A + \frac{1}{2}C$.

→ Player 1 never best-responds with B. It best-responds with A (for $\alpha a + \beta b + (1-\alpha-\beta)c$) if

$$6\alpha + 4(1-\alpha-\beta) \geq 10\beta + 2(1-\alpha-\beta),$$

where Player 2 is playing $\alpha a + \beta b + (1-\alpha-\beta)c$

$$\Longleftrightarrow \quad 1 + 2\alpha \geq 6\beta$$



$BR_1(\alpha a + \beta b + (1-\alpha-\beta)c)$

Suppose Player 1 plays $pA + qB + (1-p-q)C$ -

Player 2 best-responds with a if

$$2p + 12q + 6(1-p-q) \geq 6p + 3q$$

$$\Leftrightarrow \boxed{6 + 3q \geq 10p}$$

AND

$$2p + 12q + 6(1-p-q) \geq 4p + 5q + 2(1-p-q)$$

$$\Leftrightarrow \boxed{4 + 3q \geq 6p}$$

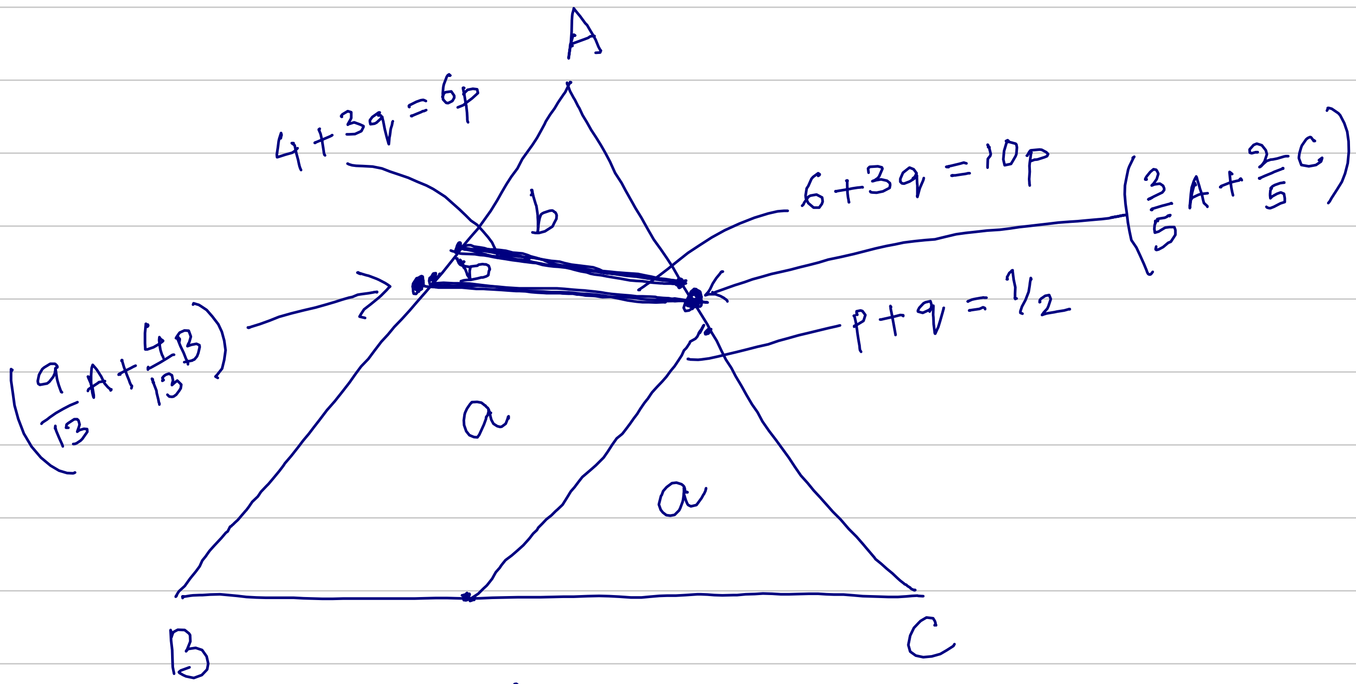
Player 2 best-responds with b if

$$\boxed{6 + 3q \leq 10p}$$

AND

$$6p + 3q \geq 4p + 5q + 2(1-p-q)$$

$$\Leftrightarrow \boxed{2p + 2q \geq 1}$$



$BR_2(\sigma_1)$

Nash Eqm: Player 1 never best responds with B in support.

Player 2 never best responds with c in support.

Pure strategy Nash:

$(A,a) \longrightarrow$ No. Since b is BR of Player 2 for A.
$(A,b) \longrightarrow$ No. Since C is BR of Player 1 for b.
$(C,a) \longrightarrow$ No. Since A is BR of Player 1 for a.
$(C,b) \longrightarrow$ No. Since a is BR of Player 2 for C.

Player 2 mixes a+b if Player 1 mixes A+C as: $\left(3/5\, A + \frac{2}{5} C\right.$

Player 1 mixes A+C if Player 2 mixes a+b as: $\left.\frac{5}{8}a + \frac{3}{8}b\right)$

Unique Nash:

NE can also be computed easily using iterated elimination procedure.

→ Eliminate B (strictly dominated by $\frac{1}{2}A + \frac{1}{2}C$)

→ Having eliminated B, c is strictly dominated by $\left(\frac{1}{2}-\varepsilon\right)a + \left(\frac{1}{2}+\varepsilon\right)b$, where $\varepsilon > 0$ but very close to 0.

So, we eliminate c.

Hence, the reduced game only has:

	a	b
A	6,2	0,6
C	0,6	10,0

No pure strategy NE. For mixing, if Player 2 plays $\alpha a + (1-\alpha)b$, Player 1 is indifferent between A and C if

$$6\alpha = 10(1-\alpha) \Rightarrow \alpha = 5/8$$

Similarly if Player 1 plays $\beta A + (1-\beta)C$, Player 2 is indifferent between a and b if

$$2\beta + 6(1-\beta) = 6\beta \Rightarrow \beta = 3/5.$$

Hence, a unique (completely) mixed NE is $\left(3/5\,A + 2/5\,C,\ 5/8\,a + 3/8\,b\right)$.

Q4 Strategy set of each player is $[0,1]$, which is a lattice.

$$u_i(a_1, a_2) = f(a_1)\, f(a_2) - c(a_i)$$

Supermodularity of u_i is vacuously satisfied.

u_i is continuous in a_1 and a_2 since f is continuous and c is continuous.

For increasing differences, fix $i \in \{1,2\}$ and denote the other player as $j \neq i$.

Pick $a_i > a_i'$ and $a_j > a_j'$.

Then,

$$u_i(a_i, a_j) - u_i(a_i', a_j)$$

$$= f(a_i) f(a_j) - f(a_i') f(a_j) - c(a_i) + c(a_i')$$

$$= [f(a_i) - f(a_i')]\, f(a_j) - c(a_i) + c(a_i')$$

$$> [f(a_i) - f(a_i')]\, f(a_j') - c(a_i) + c(a_i')$$

[Since f is increasing
$f(a_i) \geq f(a_i')$ & $f(a_j) \geq f(a_j')$]

$$= f(a_i) f(a_j') - c(a_i) - f(a_i') f(a_j') + c(a_i')$$

$$= u_i(a_i, a_j') - u(a_i', a_j')$$

Hence, the given game is a supermodular game.

$\Rightarrow$ It has a pure strategy Nash equilibrium.

Q2. If s_i is part of every max-min strategy, then for any mixed strategy σ_i with the property

$$\min_{\sigma_{-i}} u_i(\sigma_i, \sigma_{-i}) \geq \min_{\sigma_{-i}} u_i(\sigma_i', \sigma_{-i}) \quad \forall \sigma_i',$$

we have $\sigma_i(s_i) > 0$.

Suppose s_i is weakly dominated by $\hat{\sigma}_i$.

$\Rightarrow u_i(s_i, \sigma_{-i}) \leq u_i(\hat{\sigma}_i, \sigma_{-i}) \quad \forall \sigma_{-i}$.

with strict inequality holding for some σ_{-i}.

First, we show that $\hat{\sigma}_i$ can be chosen such that $\hat{\sigma}_i(s_i) = 0$

To see this suppose $\hat{\sigma}_i(s_i) > 0$.

Then, we have $\forall\, \sigma_{-i}$,

$$u_i(s_i, \sigma_{-i}) \leq \hat{\sigma}_i(s_i)\, u_i(s_i, \sigma_{-i}) + \sum_{s_i' \neq s_i} u_i(s_i', \sigma_{-i})\, \hat{\sigma}_i(s_i').$$

$$\Rightarrow u_i(s_i, \sigma_{-i}) \leq \sum_{s_i' \neq s_i} \frac{\hat{\sigma}_i(s_i')}{[1 - \hat{\sigma}_i(s_i)]} u_i(s_i', \sigma_{-i})$$

Defining $\bar{\sigma}_i(s_i') = \dfrac{\hat{\sigma}_i(s_i')}{1 - \hat{\sigma}_i(s_i)} \quad \forall\, s_i' \neq s_i$

and $\bar{\sigma}_i(s_i) = 0$

defines a new mixed strategy $\bar{\sigma}_i$ with $\bar{\sigma}_i(s_i) = 0$ and

$u_i(s_i, \sigma_{-i}) \leq u_i(\bar{\sigma}_i, \sigma_{-i}) \quad \forall\, \sigma_{-i}$.

Hence, we assume WLOG that $\hat{\sigma}_i(s_i) = 0$.

Now, fix any maxmin strategy σ_i.

We know that $\sigma_i(s_i) > 0$.

For any σ_{-i},

$$u_i(\sigma_i, \sigma_{-i}) = u_i(s_i, \sigma_{-i})\,\sigma_i(s_i) + \sum_{s_i' \neq s_i} u_i(s_i', \sigma_{-i})\,\sigma_i(s_i')$$

$$\leq u_i(\hat{\sigma}_i, \sigma_{-i})\,\sigma_i(s_i) + \sum_{s_i' \neq s_i} u_i(s_i', \sigma_{-i})\,\sigma_i(s_i')$$

$$= u_i(s_i, \sigma_{-i})\,\hat{\sigma}_i(s_i)\,\sigma_i(s_i)$$
$$+ \sum_{s_i' \neq s_i} u_i(s_i', \sigma_{-i})\left(\sigma_i(s_i') + \hat{\sigma}_i(s_i')\,\sigma_i(s_i)\right)$$

(The first term is zero since $\hat{\sigma}_i(s_i) = 0$)
Hence, ...

$$= \sum_{s_i' \neq s_i} u_i(s_i', \sigma_{-i})\,\overline{\sigma}_i(s_i'),$$

where $\overline{\sigma}_i(s_i') = \sigma_i(s_i') + \hat{\sigma}_i(s_i')\,\sigma_i(s_i)$
and $\overline{\sigma}_i(s_i) = 0$ is a new mixed strategy.

This means $\exists$ a mixed strategy $\bar{\sigma}_i$ with $\bar{\sigma}_i(s_i) > 0$ and

$$u_i(\sigma_i, \sigma_{-i}) \le u_i(\bar{\sigma}_i, \sigma_{-i}) \quad \forall \sigma_{-i}.$$

$$\Rightarrow \min_{\sigma_{-i}} u_i(\sigma_i, \sigma_{-i}) \le \min_{\sigma_{-i}} u_i(\bar{\sigma}_i, \sigma_{-i})$$

Since σ_i is a max-min strategy, $\bar{\sigma}_i$ is also a max-min strategy.

But $\bar{\sigma}_i(s_i) = 0$ contradicts the fact that <u>every</u> max-min strategy has s_i in its support.

For the other part, consider a 2×2 game.

For Row Player, both T and B (and any mixture of T+B) is a max-min strategy.

	L	R
T	$1+\varepsilon, 1$	$1, 1$
B	$1, 1$	$1, 1$

$\varepsilon > 0$

But B is <u>weakly</u> <u>dominated</u> by T.

Q3.

- Pure maxmin strategies
  Player 1 : A, B, C
  Player 2 : a, b, c

- This is a zero-sum game.
  Hence, a value exists.

  Value of this game is zero.
  To see this, consider Player 1.

If he plays $\frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$, then his payoff is zero if Player 2 plays any of the pure strategies.

Hence, by playing $\frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$, his payoff is zero if Player 2 plays any mixed strategy.

Hence $$\underline{v} = \max_{\sigma_1 \in \Delta S_1} \min_{\sigma_2 \in \Delta S_2} u(\sigma_1, \sigma_2) \geq 0$$

A similar argument shows that Player 2 can guarantee zero payoff by playing $\frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c$.

Hence, $-\bar{v} = \max_{\sigma_2 \in \Delta S_2} \min_{\sigma_1 \in \Delta S_1} (-u(\sigma_1, \sigma_2))$

≥ 0

$\Rightarrow \bar{v} \leq 0$

But $\bar{v} = \underline{v} \Rightarrow \underline{v} = \bar{v} = 0$

Since a value exists.

- Since the value of the game is zero every Nash eq^m (σ_1^*, σ_2^*) must give zero payoff to both the agents.

This means that every pure strategy in the support of σ_1^* (as a BR to σ_2^*) must give zero payoff.

Since no pure strategy NE exists, σ_1^* must have at least 2 pure strategies in its support. Without loss of generality let these two strategies be A, B.

Then, the payoff of Player 1 from A and B are zero:

Payoff from A: $0 \cdot \sigma_2^*(a) + (-1)\sigma_2^*(b) + 1 \cdot \sigma_2^*(c) = 0$

$$\Rightarrow \sigma_2^*(b) = \sigma_2^*(c)$$

Payoff from B: $1 \cdot \sigma_2^*(a) + 0 \cdot \sigma_2^*(b) + (-1)\sigma_2^*(c) = 0$

$$\Rightarrow \sigma_2^*(a) = \sigma_2^*(c)$$

$$\Rightarrow \sigma_2^*(a) = \sigma_2^*(b) = \sigma_2^*(c) = 1/3.$$

A similar argument shows

$$\sigma_1^*(A) = \sigma_1^*(B) = \sigma_1^*(C) = 1/3.$$

Hence the unique Nash eqlm of this zero-sum game is

$$\left(\tfrac{1}{3}A + \tfrac{1}{3}B + \tfrac{1}{3}C,\ \tfrac{1}{3}a + \tfrac{1}{3}b + \tfrac{1}{3}c\right)$$