GAME THEORY - MIDTERM EXAMINATION, 2

Date: October 14, 2017

Total marks: **30**

Duration: 10:00 AM to 1:00 PM

Note: Answer all questions clearly using pen. Please avoid unnecessary discussions. In all questions, an incomplete description of a strategy or equilibrium will be considered an incorrect answer.

1. There are two agents who want to complete a task. Each agent can either work or shirk. So, the possible set of actions for each agent is $\{0, 1\}$, where 0 corresponds to shirking and 1 corresponds to working. The task can be completed if *any* agent works. Working is costly - each agent $i \in \{1, 2\}$ incurs a cost c_i if he works. If the two agents choose actions (x_1, x_2), where $x_i \in \{0, 1\}$ for each $i \in \{1, 2\}$, then the utility of each agent $i \in \{1, 2\}$ is given by

$$u_i(x_1, x_2) = \begin{cases} 1 - x_i c_i & \text{if } x_1 + x_2 > 0 \\ 0 & \text{if } x_1 + x_2 = 0 \end{cases}$$

   (a) Suppose the cost of agent 1 is publicly known and assume that $c_1 > 0$. On the other hand, suppose cost of agent 2 is his private information, but it is commonly known that c_2 is drawn from $[0.5, 1.5]$ using uniform distribution.

      i. Describe all Bayes Nash equilibria of this game. **(4 marks)**

      ANSWER. A strategy for agent 1 is $s_1 \in \{0, 1\}$. A strategy for agent 2 is a map $s_2 : [0.5, 1.5] \to \{0, 1\}$. If $s_1 = 1$, then all types of agent 2 must choose $s_2(c_2) = 0$ as a best response (since $c_2 > 0$). Hence, if $(s_1 = 1, s_2)$ is a Bayes Nash equilibrium, then, the expected payoff of agent 1 is $1 - c_1$. Expected payoff of agent 1 by choosing strategy 0 is 0 - this is because all types of agent 2 chooses zero in s_2. As a result choosing, $s_1 = 1$ is a best response if $c_1 \leq 1$. So, one Bayes Nash equilibrium is when

$$c_1 \leq 1 : s_1 = 1, s_2(c_2) = 0 \ \ \forall \ c_2.$$

      If $s_1 = 0$, then agent 2 gets positive payoff if $1 - c_2 \geq 0$ or $c_2 \leq 1$. So, $s_2(c_2) = 1$ if $c_2 < 1$ and $s_2(c_2) = 0$ if $c_2 > 1$ is a best response. If $c_2 = 1$, then agent 2 can choose either action. So, any Bayes Nash equilibrium with $s_1 = 0$ must have this strategy for agent 2. Hence, expected payoff of agent

1 from $s_1 = 0$ is $\frac{1}{2} \times 1 = \frac{1}{2}$, where $\frac{1}{2}$ is the probability with which agent 2 is likely to choose 1. Agent 1's expected payoff from choosing 1 is $1 - c_1$. So, if $s_1 = 0$ is a Bayes Nash equilibrium, then $1 - c_1 \leq \frac{1}{2}$ or $c_1 \geq \frac{1}{2}$. So, another Bayes Nash equilibrium is when

$$c_1 \geq \frac{1}{2} : s_1 = 0, s_2(c_2) = 1 \;\forall\; c_2 \in [0.5, 1), s_2(c_2) = 0 \;\forall\; c_2 \in (1, 1.5].$$

ii. Are there values of c_1 for which *all* Bayes Nash equilibria involve Player 2 shirking at all types? **(2 marks)**

Answer. See previous part: if $c_1 < 0.5$, all equilibria must involve $s_1 = 1$ and $s_2(c_2) = 0$ for all c_2.

(b) Suppose the costs of both the agents are their respective private information. Further, each agent's cost is drawn from $[0.5, 1.5]$ using a uniform distribution. Call a strategy of an agent i a **cutoff** strategy if there is a number $c_i^* \in [0.5, 1.5]$ such that for all types with cost less than c_i^*, i chooses one action and for all types with cost greater than c_i^*, he chooses the other action.

i. Show that every Bayes Nash equilibrium has cutoff strategies for both the agents. **(4 marks)**

Answer. Fix a strategy of agent i: $s_i : [0.5, 1.5] \to \{0, 1\}$. Let $\pi(s_i)$ denote the probability that agent i uses $s_i(c_i) = 1$. Formally, because c_i is uniformly distributed,

$$\pi(s_i) := L(\{c_i : s_i(c_i) = 1\}),$$

where L assigns the probability measure using the uniform distribution on $[0.5, 1.5]$.

Consider the payoff of the other agent $j \neq i$ by following a strategy $s_j : [0.5, 1.5] \to \{0, 1\}$. If the cost of agent j is c_j, then using $s_j(c_j) = 0$ gives a payoff equal to $\pi(s_i)$. On the other hand using $s_j(c_j) = 1$ gives a payoff equal to $1 - c_j$ (independent of s_i). Hence, $s_j(c_j) = 1$ is a best response to s_i if $1 - c_j \geq \pi(s_i)$ and $s_j(c_j) = 0$ is a best response to s_i if $1 - c_j \leq \pi(s_i)$. Hence, if we set $c_j^* = 1 - \pi(s_i)$, then the best response to s_i is a cutoff strategy with respect to c_j^*.

ii. Compute all Bayes Nash equilibria of this game. **(4 marks)**

Answer. From the earlier part, every Bayes Nash equilibria must involve cutoff strategies. Let (s_1, s_2) be a Bayes Nash equilibria with cutoffs (c_1^*, c_2^*). Then, $\pi(s_i) = c_i^* - \frac{1}{2}$ for each $i \in \{1, 2\}$. The best response condition requires that agent i should be indifferent between choosing 1 or 0 at $c_i = c_i^*$:

$$1 - c_i^* = \pi(s_j) = c_j^* - \frac{1}{2}.$$

That is: $c_1^* + c_2^* = \frac{3}{2}$.

Hence, every Bayes Nash equilibria are with cutoff strategies (c_1^*, c_2^*) with

$$c_1^*, c_2^* \in [0.5, 1.5], c_1^* + c_2^* = 1.5.$$

Finally, we verify that each of them is indeed a Bayes Nash equilibrium. To see this, take agent $i \in \{1, 2\}$. His expected payoff at $c_i < c_i^*$ by choosing 1: $1 - c_i \geq 1 - c_i^* = c_j^* - 0.5$. But $c_j^* - 0.5$ is the payoff of choosing 0. Hence, choosing 1 is best response. Similarly, if $c_i > c_i^*$, identical argument shows that choosing 0 is a best response. At $c_i = c_i^*$, both are best responses. This completes the proof.

2. Consider only pure strategies in the following Bertrand game. Two firms are setting prices in $[0, 1]$. If firms set prices (p_1, p_2), then demand for each firm $i \in \{1, 2\}$ is

$$D_i(p_1, p_2) := 1 - 2p_i + p_j,$$

where $j \neq i$ is the other firm. The utility of firm $i \in \{1, 2\}$ at prices (p_1, p_2) is

$$u_i(p_1, p_2) = p_i D_i(p_1, p_2).$$

(a) Compute all the never best response strategies of each firm. **(3 marks)**

Answer. Fix strategy (price) of firm j and note that utility of firm $i \neq j$ is

$$u_i(p_i, p_j) = p_i(1 - 2p_i + p_j).$$

This is a strictly concave function in p_i. So, the first order condition gives

$$p_i = \frac{1}{4}(1 + p_j).$$

Since value of $p_j \in [0, 1]$, the best response prices lie in $[\frac{1}{4}, \frac{1}{2}]$. So, never best response strategies are $[0, \frac{1}{4})$ and $(\frac{1}{2}, 1]$.

(b) Carry out iterative elimination of never best response strategies. For instance, both the firms delete their never best response strategies in the original game. Then, they find out the never best response strategies in the new game and delete those. This process is carried out forever (possibly infinite times). Do you converge to a strategy profile after infinite rounds of elimination? If you converge, can you verify if the strategy profile that survived this iterative elimination is a Nash equilibrium? **(3 marks)**

ANSWER. First round of elimination gives $[\frac{1}{4}, \frac{1}{2}]$ that survive. Second round of elimination gives $[\frac{5}{16}, \frac{3}{8}]$, third round of elimination gives $[\frac{21}{64}, \frac{11}{32}]$, fourth round of elimination gives $[\frac{85}{256}, \frac{43}{128}]$.

So, the lower bound limit is the following:

$$\frac{1}{4}, \frac{5}{16}, \frac{21}{64}, \frac{85}{256}, \ldots, \frac{4^0 + \ldots + 4^{n-1}}{4^n}, \ldots$$

So, a generic element of this sequence is $\frac{4^0+\ldots+4^{n-1}}{4^n}$, which converges to $\frac{1}{4} \times \frac{4}{3} = \frac{1}{3}$. as n tends to ∞.

The upper bound of the limit is the following:

$$\frac{1}{2}, \frac{3}{8}, \frac{11}{32}, \frac{43}{128}, \ldots, \frac{2^0 + 2^1 + \ldots + 2^{2n-3}}{2^{2n-1}}, \ldots.$$

So, a generic term is $\frac{2^0+2^1+\ldots+2^{2n-3}}{2^{2n-1}}$ which converges to $\frac{1}{4} \times \frac{4}{3} = \frac{1}{3}$ as n tends to ∞. So, we converge to $(\frac{1}{3}, \frac{1}{3})$ after infinite rounds of elimination of never best responses. It is easily checked that $\frac{1}{3}$ is a best response if $p_j = \frac{1}{3}$. Hence, this is also a Nash equilibrium.

3. A food chain F has shops in two different cities $\{1, 2\}$. In each city $i \in \{1, 2\}$, there is a local firm ℓ_i which is trying to open a shop. But they make these decisions sequentially.

(a) First, ℓ_1 decides whether to open or not open.

(b) If ℓ_1 decides to open the shop, the food chain F either *slashes* prices or *does not slash* prices in city 1. Else, F takes no action.

(c) Then, firm ℓ_2 observes the choices of ℓ_1 and F, and decides to open or not open.

(d) If ℓ_2 decides to open the shop, the food chain F either *slashes* prices or *does not slash* prices in city 2. Else, F takes no action.

If a firm ℓ_i decides not to open the store, then ℓ_i gets 1 payoff and F gets 5 payoff in city i. If firm ℓ_i opens the store and F slashes prices, both ℓ_i and F receive a payoff of 0. If firm ℓ_i opens the store and F does not slash the price, both ℓ_i and F receive a payoff of 2. Payoff of F is the sum of its payoffs in all cities.

(a) Write the extensive form of this game (via a game tree figure). **(2 marks)**

(b) What are the (pure) strategies of each player in this game? **(2 marks)**

(c) Compute the subgame perfect equilibria of this game. **(3 marks)**

(d) Write down the reduced strategic form of this game. Is there a pure strategy Nash equilibrium which produces a different outcome than the subgame perfect equilibrium? **(3 marks)**

ANSWER. The extensive form game and the subgame perfect equilibrium (using backward induction) is shown in Figure 1.

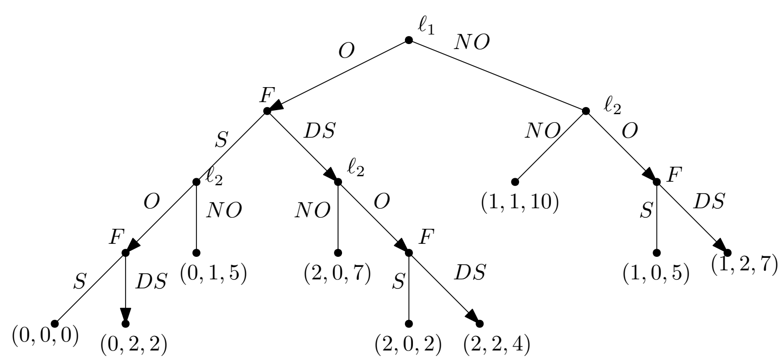

Figure 1: Subgame perfect equilibrium

Store ℓ_1 has one decision vertex. So, its strategy is either to open (O) or not open (NO). Store ℓ_2 has 3 decision vertices but at each vertex it can choose from $\{O, NO\}$. So, its strategy is a vector from:

$$\{O, NO\} \times \{O, NO\} \times \{O, NO\}.$$

The firm has four decision vertices with two actions: Slash (S) or not slash (NS) prices. So F has sixteen pure strategies chosen from:

$$\{S, NS\} \times \{S, NS\} \times \{S, NS\} \times \{S, NS\}.$$

The reduced strategic form game is defined by these strategies and corresponding payoffs (given in Figure 1). A Nash equilibrium of this game is the following:

$$(NO, (NO, NO, NO), (S, S, S, S)).$$

This is Nash equilibrium because if local stores are choosing NO, then irrespective of what action F takes, he gets payoff 10 - so S is a best response. Local store ℓ_1 by choosing O gets you to $O \to S \to NO$ path giving him a payoff of zero. Hence, NO is a best response for ℓ_1. Local store ℓ_2 can change his payoff only if it chooses a different action at decision vertex after ℓ_1 has chosen NO. In that case, ℓ_2 choosing O gives us the path $NO \to O \to S$, which gives ℓ_2 a payoff of 0. Hence, (NO, NO, NO) is a best response for ℓ_2.