

GAME THEORY 2  
ASSIGNMENT 1

1. Consider the sale of  $n$  objects which are homogeneous i.e., the value of each object is the same for a bidder (agent), though different bidders may have different values. Every bidder is interested in getting at most one object. In this case, the type of a bidder  $i$  is just one number  $t_i \in \mathbb{R}_+^1$ , where  $t_i$  denotes the value for a object.

Suppose there are  $m$  agents with  $m > n$ . Prove the following.

- The efficient allocation is to allocate an object each to bidders with  $n$  highest values.
  - The VCG payment of every bidder is the same, and is equal to  $(n + 1)^{\text{st}}$  highest value.
2. Google sells advertisement slots to advertisers via auctions. The auctions are run for **every** search phrase. Fix a particular search phrase - say “hotels in New Delhi”. Once this phrase is searched on Google, bidders (computer programmed agents of different companies) participate in this auction. An advertisement that can appear along side a search page is called a **slot**. For every search phrase, there is a fixed number of slots available and fixed number of bidders interested. Suppose there are  $m$  slots and  $n$  bidders for the phrase “hotels in New Delhi”. Assume  $n > m$ . The type of each bidder is a single number -  $\theta_i$  for bidder  $i$ . Type of an agent represents the value that agent derives when his advertisement is clicked. Every slot has a probability of getting clicked. This is called the **clickthrough rate (CTR)**. The CTR of slot  $i$  is  $\alpha_i$ . The CTR vector  $\alpha = (\alpha_1, \dots, \alpha_m)$  is known to everyone. The slots are naturally ordered top to bottom, and without loss of generality, let  $\alpha_1 > \alpha_2 > \dots > \alpha_m$ .

An alternative in this model represents an assignment of agents to slots (with some agents not receiving any slot). Let  $A$  be the set of all alternatives. An alternative  $a \in A$  can be described by a  $n$  dimensional vector integers in  $\{0, 1, \dots, m\}$ , where  $a_i$  indicates the slot to which agent  $i$  is assigned, and  $a_i = 0$  means agent  $i$  is not assigned to any slot. The value function of agent  $i$  is his expected value  $v_i(a) = \theta_i \alpha_{a_i}$ , where  $\alpha_0 = 0$ . Answer the following questions.

- (a) Suppose  $n = 4$  and  $m = 3$ . Let  $\theta_1 = 10, \theta_2 = 8, \theta_3 = 6, \theta_4 = 5$ . Let  $\alpha_1 = 0.8, \alpha_2 = 0.6, \alpha_3 = 0.5$ .
  - i. Describe the efficient allocation.
  - ii. Describe the VCG (pivotal) mechanism.
- (b) For the general problem,

- i. Describe the efficient allocation.
  - ii. Describe the VCG mechanism.
- (c) Google uses something called a **Generalized Second Price (GSP)** auction: (a) agents with top  $m$   $\theta_i$  values are given the slots with highest agent getting the top slot (i.e., slot with highest CTR), second highest agent getting the next top slot, and so on, (b) if an agent wins slot  $k$  with CTR  $\alpha_k$ , he pays  $\theta_{m+1}\alpha_k$  (where  $\theta_{m+1}$  is the highest losing type).
- i. Use a simple example to illustrate that GSP is not equivalent to VCG.
  - ii. Show (again using a simple example) how agents do not have a dominant strategy to bid their values.
3. Consider the usual combinatorial auction problem, but with a simpler valuation function called the “single-minded valuation”. The set of objects is  $M = \{1, \dots, m\}$ . Every agent  $i$  **desires** a bundle of objects  $S_i \subseteq M$ . The private type of agent  $i$  is a single number  $\theta_i$ . An alternative in this model can be represented by the binary variables  $x_i(S) \in \{0, 1\}$  satisfying some feasibility constraints, where  $x_i(S) = 1$  means bundle  $S$  goes to agent  $i$  and  $x_i(S) = 0$  means bundle  $S$  does not go to agent  $i$ . Given an alternative  $x$ , denote by  $\mu_i(x)$  the indicator function whether agent  $i$  gets his desire in  $x$  or not, i.e.,  $\mu_i(x) = 1$  if  $x_i(S) = 1$  and  $S_i \subseteq S$  and  $\mu_i(x) = 0$  otherwise. The value function of agent  $i$  is given by  $\theta_i\mu_i(x)$ . Essentially, agent  $i$  realizes his value  $\theta_i$  if he gets his desired set of objects.
- (a) Suppose there are four agents  $\{1, 2, 3, 4\}$  and three objects  $\{a, b, c\}$ . The desires of agents are:  $S_1 = \{a\}$ ,  $S_2 = \{a, b\}$ ,  $S_3 = \{b, c\}$ , and  $S_4 = \{a, c\}$ . Suppose  $\theta_1 = 7, \theta_2 = 8, \theta_3 = 2, \theta_4 = 10$ .
- i. Find the efficient allocation.
  - ii. Find the payment in the VCG mechanism.
4. Show that the Pivotal mechanism is the only Groves mechanism in the combinatorial auction setting which implements the efficient allocation rule and where an agent pays zero if his values for all bundles are zero.
- Further, show that the Pivotal mechanism is the only Groves mechanism in the combinatorial auction setting which implements the efficient allocation rule, which is individually rational, and where no agent is paid (i.e., payments are non-negative).
5. Suppose there is a single agent in the economy, and his type space is  $T$ . Let  $f : T \rightarrow A$  be an allocation rule. Consider another type space  $T' \subset T$  ( $T \neq T'$ ) and an allocation rule  $f' : T' \rightarrow A$ . Show that if  $f$  is DSIC, then  $f'$  is also DSIC.

6. Suppose there is a single agent in the economy. Let  $A = \{a, b\}$  and  $T = \{s, t\}$ . Consider the following valuation function of the agent  $v(a, s) = 1$  and  $v(a, t) = 0$  but  $v(b, s) = v(b, t) = 0$ . Consider an allocation rule  $f$  as follows:  $f(s) = a$  and  $f(t) = b$ .

- Write down the DSIC constraints for the allocation rule  $f$ .
- Show that  $f$  is DSIC.
- What payment function makes  $f$  DSIC?

7. Consider the sale of a single object to a set of  $n$  agents. Consider the following allocation rule: at every type profile  $\mathbf{t} = (t_1, \dots, t_n)$ , the allocation rule picks top  $k \leq n$  types, i.e., agent  $i$  is picked in type profile  $\mathbf{t}$  if  $t_i$  is in the top  $k$  types in  $(t_1, \dots, t_n)$ . Let  $K(\mathbf{t})$  be the set of agents picked in type profile  $\mathbf{t}$ . Then the object is awarded with equal probability to any agent in  $K(\mathbf{t})$  at type profile  $\mathbf{t}$ . Suppose the valuation function is in product form.

- Is this allocation rule DSIC? Explain with reason.
- Consider the allocation rule where the object is awarded to an agent with the second highest type, i.e., in a type profile  $\mathbf{t}$ , it chooses the set of agents who have the second highest type and awards the object with equal probability to each of them. Is this allocation rule DSIC? Explain with reason.

8. Consider an economy with a single agent and let the type set of this agent be  $T \subseteq \mathbb{R}^m$  (where  $m$  is a non-negative integer). Let  $A$  be the set of alternatives and  $v : A \times T \rightarrow \mathbb{R}$ . Consider an allocation rule  $f$ .

- Write down the constraints for  $f$  to be DSIC.
- Show that if for any  $s, t \in T$ , if  $f(s) = f(t)$ , then  $p(s) = p(t)$  for any payment function  $p$  which makes  $f$  DSIC.
- Use the previous result to show that a payment function can be written as a mapping  $p : A \rightarrow \mathbb{R}$  instead of  $p : T \rightarrow \mathbb{R}$  without loss of generality.
- Use these results to show that the DISC constraints can be written in the following form:

$$p(b) - p(a) \leq l(a, b) \quad \forall a, b \in A,$$

where for every  $a, b \in A$ ,  $l(a, b) = \inf_{t \in T: f(t)=b} [v(b, t) - v(a, t)]$ .

- Use the previous result to show that  $f$  is DSIC if and only if a (complete directed) graph whose set of nodes is  $A$  and whose length function is  $l$  has no cycle of negative length. Such a graph is called an **allocation graph**.
9. Consider an example of sale of two goods  $a$  and  $b$ . There are two agents  $N = \{1, 2\}$ . Each agent wants at most one good. Consider two possible types of agent 1:  $t_1(a) = 4, t_1(b) = 10$  and  $t'_1(a) = 0.5, t'_1(b) = 2$ . Also, consider a type of agent 2:  $t_2(a) = 1$  and  $t_2(b) = 2$ .
- Consider the type profile  $(t_1, t_2)$ . Verify that the max-min allocation rule allocates good  $a$  to agent 1 and good  $b$  to agent 2 at this type profile.
  - Consider the type profile  $(t'_1, t_2)$ . Verify that the max-min allocation rule allocates good  $a$  to agent 2 and good  $b$  to agent 1.
  - Suppose  $T_1$  contains  $t_1$  and  $t'_1$ . Show that the max-min allocation rule does not satisfy cycle-monotonicity, and hence, not DSIC.