Game Theory - Assignment 1

Due date: **August 24, 2022**.

1. Consider the following game with two players $\{b, s\}$: Player b is the buyer of *time* from Player s who has 1 unit of time to sell. Strategies of the players are as follows:

   - b chooses a price $p_b \in \{3, 4\}$.
   - s chooses a price $p_s \in \{3, 4\}$.

   Depending on the prices (p_b, p_s) chosen by the players, an amount of time $q(p_b, p_s)$ is given by Player s to Player b:

   - $q(p_b, p_s) = 0$ if $p_b < p_s$.
   - $q(p_b, p_s) = 1$ if $p_s = 3 \leq p_b$ at a per unit price $p_s = 3$.
   - $q(p_b, p_s) = \frac{2}{3}$ if $p_s = 4 = p_b$ at a per unit price $p_b = 4$.

   Utilities of players are as follows. Player s incurs a cost $c \in [0, 4]$ by giving one unit of time and Player b gets a value $v \in [3, 5]$ per unit time of Player s. The per unit price paid by b to s is the price chosen by the seller (p_s). Utilities are

   $$u_b(p_b, p_s; v) = q(p_b, p_s)(v - p_s)$$
   $$u_s(p_b, p_s; c) = q(p_b, p_s)(p_s - c),$$

   where u_b is the utility of buyer and u_s is utility of seller. Answer the following.

   - Show that for every $v \neq 4$, Player b has a weakly dominant strategy in this game. Specify the weakly dominant strategy. What happens at $v = 4$?
   - Show that for $c \geq 3$ and $c \leq 1$, Player s has a weakly dominant strategy in this game.
   - What happens if $c \in (1, 3)$ for Player s?

2. Three indivisible objects (houses) need to be assigned to three agents. Each agent needs to be assigned a unique house. Each agent has a strict preference ordering over the set of objects.

The agents play an *allocation game* to allocate objects. First, agent 1 goes and selects an object from the three objects. Second, agent 2 goes and selects an object from the remaining two objects. Finally, agent 3 gets the remaining object.

Write down the strategic form game by clearly specifying the strategies of the players.

3. An indivisible good is sold to 3 buyers. If any buyer i gets $q_i \in \{0, 1\}$ quantity of the goods makes a payment of p_i, her payoff is

$$q_i v_i - p_i.$$

Payment p_i can be positive, negative or zero (some buyers may be *paid* or compensated).

The seller asks each buyer to place a bid. If (b_1, b_2, b_3) are the bids of the buyers then the highest bidder wins (with ties broken in favor of highest indexed bidder[1]). If bidder i wins, she pays $\max_{j \neq i} b_j$. Out of this payment, the seller returns

$$\frac{1}{3} \min_{j \neq i} b_j$$

to highest and second highest bidder and

$$\frac{1}{3} \max_{j \neq i} b_j$$

to the lowest bidder.

Show that bidding their own value is a weakly dominant strategy for each bidder.

[1]For instance, if buyer 1 and 2 are joint winners, buyer 1 wins the object.