

THEORY OF MECHANISM DESIGN - ASSIGNMENT 1

1. Consider a two agent model with three alternatives $\{a, b, c\}$. Table 1 shows two preference profiles of preferences. Suppose $f(P_1, P_2) = a$. Show that if f is strategy-proof then $f(P'_1, P'_2) = b$. You are allowed to use the result that for any preference profile (\bar{P}_1, \bar{P}_2) , $f(\bar{P}_1, \bar{P}_2) \in \{\bar{P}_1(1), \bar{P}_2(1)\}$ (but do not use any other result from the lectures).

| | | | |
|-------|-------|--------|--------|
| P_1 | P_2 | P'_1 | P'_2 |
| a | c | b | c |
| b | b | a | a |
| c | a | c | b |

Table 1: Two Preference Profiles

2. Let X be a set of projects. A social choice function chooses a non-empty subset of projects. Agent i has a linear ordering P_i over the set of projects X . Agent i evaluates subsets of projects by extending P_i in the following manner: for any pair of subsets of projects $S, T \subseteq X$, S is preferred to T if the highest ranked project in S (according to P_i) is better than the highest ranked project in T - if these two projects are the same, then S and T are indifferent.

Suppose $|X| \geq 2$. Will the Gibbard-Satterthwaite result apply here? Discuss your answer.

3. Consider the unanimous SCF f defined as follows. If $P_1(1) = \dots = P_n(1) = a$, then $f(P_1, \dots, P_n) = a$. Else, $f(P_1, \dots, P_n) = b$ for some alternative $b \in A$. In other words, f satisfies unanimity wherever possible and picks a “status-quo” alternative b otherwise. Argue how f can be manipulated if there are at least three alternatives?
4. Let A be a finite set of alternatives and $f : \mathcal{P}^n \rightarrow A$ be a social choice function that is unanimous and strategy-proof. Suppose $|A| \geq 3$.

Now, consider another social choice function $g : \mathcal{P}^2 \rightarrow A$ defined as follows. The scf g only considers profiles of two agents, denote these two agents as 1 and 2. For any $(P_1, P_2) \in \mathcal{P}^2$, let

$$g(P_1, P_2) = f(P_1, P_2, P_1, P_1, \dots, P_1),$$

i.e., the outcome of g at (P_1, P_2) coincides with the outcome of f at the profile where agents 1 and 2 have types P_1 and P_2 respectively, and all other agents have type P_1 .

Show that g is a dictatorship scf.

5. Let the number of alternatives be m . Show that the number of single-peaked preference orderings with respect to $<$ (an exogenously given ordering of alternatives) is 2^{m-1} .
6. Consider the single-peaked domain model. A social choice function f is manipulable by a group of agents $K \subseteq N$ if for some preference profile (P_K, P_{-K}) there exists some preference profile P'_K of agents in K such that $f(P'_K, P_{-K}) \succ_i f(P_K, P_{-K})$ for all $i \in K$. A social choice function f is **group strategy-proof** if cannot be manipulated by any group of agents. Is the median voter SCF group strategy-proof?
7. Let $A = [0, 1]$ and assume that agents have single peaked preferences over $A = [0, 1]$. Consider the following social choice function.

DEFINITION 1 *A social choice function f is a **generalized median voter social choice function** if there exists weights y_S for every $S \subseteq N$ satisfying*

- (a) $y_\emptyset = 0, y_N = 1$ and
- (b) $y_S \leq y_T$ for all $S \subseteq T$

such that for all preference profile P , $f(P) = \max_{S \subseteq N} z(S)$, where $z(S) = \min\{y_S, P_i(1) : i \in S\}$.

Show that a generalized median voter SCF is strategy-proof.

8. Let A be a finite set of alternatives and \succ be a linear order over A . Suppose $a_L, a_R \in A$ be two alternatives such that $a \succ a_L$ for all $a \in A \setminus \{a_L\}$ and $a_R \succ a$ for all $a \in A \setminus \{a_R\}$ - in other words, a_L is the “left-most” alternative and a_R is the “right-most” alternative with respect to \succ .

Let \mathcal{S} be the set of all possible single-peaked strict orderings over A with respect to \succ . An SCF $f : \mathcal{S}^n \rightarrow A$ maps the set of preference profiles of n agents to A .

Let $P_i(1)$ denote the peak of agent i in P_i . Suppose f satisfies the following property (call it property II). There is an alternative $a^* \in A$ such that for any preference profile $(P_1, \dots, P_n) \in \mathcal{S}^n$, where $P_i(1) \in \{a_L, a_R\}$ for all $i \in N$ with at least one agent’s peak at a_L and at least one agent’s peak at a_R , $f(P_1, \dots, P_n) = a^*$.

- (a) Suppose f is strategy-proof, efficient, anonymous, and satisfies property II. Then, give a precise (simplified) description of f (using a^*), i.e., for every preference profile P , what is $f(P)$?
 - (b) Can f be strategy-proof, anonymous, and satisfy property II, but not efficient (give a formal argument or an example)?
9. In the private divisible good allocation model, discuss a social choice function that is strategy-proof and efficient but not anonymous.