Theory of Mechanism Design - Assignment 1

1. Consider a two agent model with three alternatives $\{a, b, c\}$. Table 1 shows two preference profiles of preferences. Suppose $f(P_1, P_2) = a$. Show that if f is strategy-proof then $f(P_1', P_2') = b$. You are allowed to use the result that for any preference profile $(\bar{P}_1, \bar{P}_2)$, $f(\bar{P}_1, \bar{P}_2) \in \{\bar{P}_1(1), \bar{P}_2(1)\}$ (but do not use any other result from the lectures).

P_1	P_2	P_1'	P_2'
a	c	b	c
b	b	a	a
c	a	c	b

Table 1: Two Preference Profiles

Answer: We know that $f(P') \in \{b, c\}$. Assume for contradiction that $f(P') = c$. Consider another preference profile $P'' = (P_1', P_2)$. So, $f(P'') \in \{b, c\}$. Since $f(P') = c$, $f(P'') = c$ - else agent 2 will manipulate at P'' via P'. Since $f(P) = a$. Agent 1 will manipulate at P'' via P. This is a contradiction.

2. Let X be a set of projects. A social choice function chooses a non-empty subset of projects. Agent i has a linear ordering P_i over the set of projects X. Agent i evaluates subsets of projects by extending P_i in the following manner: for any pair of subsets of projects $S, T \subseteq X$, S is preferred to T if the highest ranked project in S (according to P_i) is better than the highest ranked project in T - if these two projects are the same, then S and T are indifferent.

   Suppose $|X| \geq 2$. Will the Gibbard-Satterthwaite result apply here? Discuss your answer.

   Answer: The set of alternatives is the set of all subsets of objects: $\{S : S \subseteq X\}$. If there are at least 2 projects then, the set of alternatives is at least 3. Now, consider two alternatives S and T such that $S \subsetneq T$. By definition of the preference ordering, any agent is either indifferent between S and T or likes T to S. Hence, the preference ordering where S is ranked higher than T can never arise. This is a restriction of the domain and we cannot apply the Gibbard-Satterthwaite result here.

3. Consider the unanimous SCF f defined as follows. If $P_1(1) = \ldots = P_n(1) = a$, then $f(P_1, \ldots, P_n) = a$. Else, $f(P_1, \ldots, P_n) = b$ for some alternative $b \in A$. In other words, f satisfies unanimity wherever possible and picks a "status-quo" alternative b otherwise. Argue how f can be manipulated if there are at least three alternatives?

Answer: This can be manipulated. Consider $N = \{1, 2\}$. Suppose agent 1 has top a. On the other hand agent 2 has top $c \notin \{a, b\}$ followed by a and then b. The outcome here is the status quo b. But agent 2 can change his preference to lift a to the top to get the outcome a which he likes over b.

4. Let A be a finite set of alternatives and $f : \mathcal{P}^n \to A$ be a social choice function that is unanimous and strategy-proof. Suppose $|A| \geq 3$.

   Now, consider another social choice function $g : \mathcal{P}^2 \to A$ defined as follows. The scf g only considers profiles of two agents, denote these two agents as 1 and 2. For any $(P_1, P_2) \in \mathcal{P}^2$, let
   $$g(P_1, P_2) = f(P_1, P_2, P_1, P_1, \ldots, P_1),$$
   i.e., the outcome of g at (P_1, P_2) coincides with the outcome of f at the profile where agents 1 and 2 have types P_1 and P_2 respectively, and all other agents have type P_1.

   Show that g is a dictatorship scf.

   Answer: Since f is unanimous, it is clear that g will be unanimous. Since f is unanimous and strategy-proof and $|A| \geq 3$, by the GS Theorem, f is a dictatorship. We consider three possible cases.

   Case 1. Agent 1 is the dictator of f. This means agent 1 gets his top alternative in f and also in g. Then agent 1 cannot manipulate g. Since agent 1 is the dictator, for any P_1 and P_2, P_2', we have $g(P_1, P_2) = g(P_1, P_2') = f(P_1, P_2, \underbrace{P_1, \ldots, P_1}_{(n-2) \text{ times}}) = f(P_1, P_2', \underbrace{P_1, \ldots, P_1}_{(n-2) \text{ times}})$. So, agent 2 also cannot manipulate g.

   Case 2. Agent 2 is the dictator of f. This means agent 2 gets his top alternative in f and also in g. So, agent 2 cannot manipulate. Agent 1 cannot change the outcome by changing his preference. So, he also cannot manipulate g.

   Case 3. Agent $i \notin \{1, 2\}$ is the dictator of f. Then, $g(P_1, P_2) = f(P_1, P_2, \underbrace{P_1, \ldots, P_1}_{(n-2) \text{ times}}) = P_1(1)$. Hence, agent 1 cannot manipulate g. Also, $g(P_1, P_2) = g(P_1, P_2')$. So, agent 2 also cannot manipulate g.

   This concludes the proof that g is strategy-proof.

5. Let the number of alternatives be m. Show that the number of single-peaked preference orderings with respect to $<$ (an exogenously given ordering of alternatives) is 2^{m-1}.

Answer. Consider the alternative at the k-th position from left with respect to $<$. If we fix the peak at this alternative, then we have to choose $(k-1)$ alternatives to the left and the remaining to the right. The $(k-1)$ alternatives can take any of the remaining $(m-1)$ positions. Once we choose $(k-1)$ positions out of $(m-1)$ positions, we know which alternative goes where due to single-peakedness. Hence, the total number of orderings where this alternative can be the peak is

$$C(k-1, m-1),$$

where $C(k-1, m-1)$ is the total number of ways to choose $(k-1)$ positions from $(m-1)$ positions. Hence, the total number of single-peaked orderings is $\sum_{k=1}^{m} C(k-1, m-1) = \sum_{k=0}^{m-1} C(k, m-1) = 2^{m-1}$.

6. Consider the single-peaked domain model. A social choice function f is manipulable by a group of agents $K \subseteq N$ if for some preference profile (P_K, P_{-K}) there exists some preference profile P'_K of agents in K such that $f(P'_K, P_{-K}) P_i f(P_K, P_{-K})$ for all $i \in K$. A social choice function f is **group strategy-proof** if cannot be manipulated by any group of agents. Is the median voter SCF group strategy-proof?

   Answer. Yes, the median voter SCF is group strategy-proof. The proof is similar to the proof that shows that the median voter SCF is strategy-proof. A group of agents can shift a median if they can shift their peak to the other side of the median, and this will shift the outcome to the other side, which this agent will not like.

7. Let $A = [0, 1]$ and assume that agents have single peaked preferences over $A = [0, 1]$. Consider the following social choice function.

   DEFINITION **1** *A social choice function f is a* **generalized median voter** *social choice function if there exists weights y_S for every $S \subseteq N$ satisfying*

   (a) $y_\emptyset = 0$, $y_N = 1$ and

   (b) $y_S \leq y_T$ for all $S \subseteq T$

   such that for all preference profile P, $f(P) = \max_{S \subseteq N} z(S)$, where $z(S) = \min\{y_S, P_i(1) : i \in S\}$.

   Show that a generalized median voter SCF is strategy-proof.

   Answer. Consider a generalized median voter SCF f. Consider a preference profile P with peaks of agents $(p_1, \ldots, p_n)$. Fix an agent $i \in N$. If $p_i = f(P) = z(S)$ for some $S \subseteq N$, then agent i has no incentive to manipulate. We consider the case when $p_i \neq f(P)$. Suppose $f(P) = z(S)$ for some $S \subseteq N$. Hence, $z(S) \geq z(T)$ for all $T \subseteq N$. Note that f only depends on the peaks of the agents. We consider two cases.

- Suppose $p_i = z(T)$ for some $T \subseteq N$. Since $p_i \neq z(S)$, we have $p_i = z(T) < z(S)$ - this follows from the fact that $p_j \geq z(S)$ for all $j \in S$. This implies that $i \notin S$. By reporting a lower peak $p'_i < p_i$, agent i cannot change the outcome. By reporting a higher peak, the outcome will become $> z(S)$, which the agent prefers less than $z(S)$ due to single-peakedness.
- Suppose $p_i \neq z(T)$ for all $T \subseteq N$ such that $i \in T$. Then, $p_i > z(T)$ for all $T \subseteq N$ such that $i \in T$. Then, by reporting a higher peak $p'_i > p_i$, agent i cannot change the outcome. By reporting a lower peak $p'_i < p_i$, if the outcome changes, then it must change to p'_i. In that case, $i \in S$ - because if $i \notin S$, by reporting lower peak, $z(S) \geq z(T)$ for all $T \subseteq N$, and hence the outcome does not change. If the outcome changes to p'_i, then $p'_i < z(S)$. Since $i \in S$, we have $z(S) < p_i$. Hence, i likes $z(S)$ to p'_i due to single-peakedness.

Answer

8. Let A be a finite set of alternatives and $\succ$ be a linear order over A. Suppose $a_L, a_R \in A$ be two alternatives such that $a \succ a_L$ for all $a \in A \setminus \{a_L\}$ and $a_R \succ a$ for all $a \in A \setminus \{a_R\}$ - in other words, a_L is the "left-most" alternative and a_R is the "right-most" alternative with respect to $\succ$.

   Let $\mathcal{S}$ be the set of all possible single-peaked strict orderings over A with respect to $\succ$. An SCF $f : \mathcal{S}^n \to A$ maps the set of preference profiles of n agents to A.

   Let $P_i(1)$ denote the peak of agent i in P_i. Suppose f satisfies the following property (call it property Π). There is an alternative $a^* \in A$ such that for any preference profile $(P_1, \ldots, P_n) \in \mathcal{S}^n$, where $P_i(1) \in \{a_L, a_R\}$ for all $i \in N$ with at least one agent's peak at a_L and at least one agent's peak at a_R, $f(P_1, \ldots, P_n) = a^*$.

   (a) Suppose f is strategy-proof, efficient, anonymous, and satisfies property Π. Then, give a precise (simplified) description of f (using a^*), i.e., for every preference profile P, what is $f(P)$?

   Answer. Since f is strategy-proof, efficient, and anonymous, it must be a median voter SCF. Further, for any preference ordering $(P_1, \ldots, P_n) \in \mathcal{S}^n$, where $P_i(1) \in \{a_L, a_R\}$ for all $i \in N$, $f(P_1, \ldots, P_n) = a^*$. This means that all the phantom peaks are located at a^*. This follows from the fact that the location of a phantom peak can be obtained as outcome of f by placing suitable number of agent peaks at a_L and the remaining agent peaks at a_R.

   Hence, the outcome of f is median of agent peaks and $(n-1)$ phantom peaks located at a^*. This simplifies f as follows. For any preference profile $P \equiv (P_1, \ldots, P_n)$, if $P_i(1) \succ a^*$ for all $i \in N$, then $f(P)$ is the "left-most" agent peak, i.e., $f(P) = P_j(1)$, where $P_j(1) = P_k(1)$ or $P_k(1) \succ P_j(1)$ for all $k \in N \setminus \{j\}$.

Similarly, if $a^* \succ P_i(1)$ for all $i \in N$, then $f(P)$ is the "right-most" agent peak, i.e., $f(P) = P_j(1)$, where $P_j(1) = P_k(1)$ or $P_j(1) \succ P_k(1)$ for all $k \in N \setminus \{j\}$. Otherwise, $f(P) = a^*$.

(b) Can f be strategy-proof, anonymous, and satisfy property Π, but not efficient (give a formal argument or an example)?

Answer. Yes. Suppose $f(P) = a^*$ for all preference profiles P. This constant f is not efficient but anonymous and strategy-proof. It trivially satisfies property Π.

9. In the private divisible good allocation model, discuss a social choice function that is strategy-proof and efficient but not anonymous.

Answer. Consider a priority over agents σ. The agent with the highest priority takes his peak amount. Then, the agent with the next highest priority takes his peak amount and so on. The last agent in the priority consumes all the remaining units. This is clearly strategy-proof and efficient but not anonymous.