Game Theory 2
Assignment 1

1. Consider the sale of n objects which are homogeneous i.e., the value of each object is the same for a bidder (agent), though different bidders may have different values. Every bidder is interested in getting at most one object. In this case, the type of a bidder i is just one number $t_i \in \mathbb{R}^1_+$, where t_i denotes the value for a object.

   Suppose there are m agents with $m > n$. Prove the following.

   - The efficient allocation is to allocate an object each to bidders with n highest values.

     Answer: This follows since we want to maximize the sum of values. So, we pick the n highest values.

   - The VCG payment of every bidder is the same, and is equal to $(n+1)^{\text{st}}$ highest value.

     Answer: Of course, a bidder who does not win an object pays nothing. For the bidder who wins an object, we want to compute his externality on other bidders. To do so, fix a bidder who has the i^{th} highest value where $i \leq n$ (so he wins an object). In the efficient allocation, the sum of values of the other bidders is sum of n highest values minus the i^{th} highest value. In the absence of i^{th} highest bidder, the efficient allocation will give his object to $(n+1)^{st}$ highest bidder. Hence, the total value of bidders in this case is sum of $(n+1)$ highest values minus the i^{th} highest value. Hence, the externality of this bidder (the difference) is equal to $(n+1)^{st}$ highest value.

2. Google sells advertisement slots to advertisers via auctions. The auctions are run for **every** search phrase. Fix a particular search phrase - say "hotels in New Delhi". Once this phrase is searched on Google, bidders (computer programmed agents of different companies) participate in this auction. An advertisement that can appear along side a search page is called a **slot**. For every search phrase, there is a fixed number of slots available and fixed number of bidders interested. Suppose there are m slots and n bidders for the phrase "hotels in New Delhi". Assume $n > m$. The type of each bidder is a single number - θ_i for bidder i. Type of an agent represents the value that agent derives when his advertisement is clicked. Every slot has a probability of getting clicked. This is called the **clickthrough rate (CTR)**. The CTR of slot i is α_i. The CTR vector $\alpha = (\alpha_1, \ldots, \alpha_m)$ is known to everyone. The slots are naturally ordered top to bottom, and without loss of generality, let $\alpha_1 > \alpha_2 > \ldots > \alpha_m$.

   An alternative in this model represents an assignment of agents to slots (with some agents not receiving any slot). Let A be the set of all alternatives. An alternative

$a \in A$ can be described by a n dimensional vector integers in $\{0, 1, \ldots, m\}$, where a_i indicates the slot to which agent i is assigned, and $a_i = 0$ means agent i is not assigned to any slot. The value function of agent i is his expected value $v_i(a) = \theta_i \alpha_{a_i}$, where $\alpha_0 = 0$. Answer the following questions.

(a) Suppose $n = 4$ and $m = 3$. Let $\theta_1 = 10, \theta_2 = 8, \theta_3 = 6, \theta_4 = 5$. Let $\alpha_1 = 0.8, \alpha_2 = 0.6, \alpha_3 = 0.5$.

  i. Describe the efficient allocation.
  Answer: In efficiency, the slots should go to agents with top 3 θ-values, who are agents $1, 2$, and 3.

  ii. Describe the VCG (pivotal) mechanism.
  Answer: The total value obtained in the efficient allocation is $10(0.8) + 8(0.6) + 6(0.5) = 15.8$. So, agents other than agent 1 get a total value of $8(0.6)+6(0.5) = 7.8$. If agent 1 was not there, then the total value obtained in the efficient allocation is $8(0.8)+6(0.6)+5(0.5) = 12.5$. Hence, his externality is $12.5 - 7.8 = 4.7$, and his VCG payment is thus 4.7.
  Similarly, VCG payments of agents 2 and 3 are 3.1 and 2.5 respectively.

(b) For the general problem,

  i. Describe the efficient allocation.
  Answer: Agents with top m θ values get the top m slots with ith ($i \leq m$) highest θ value agent getting the ith slot.

  ii. Describe the VCG mechanism.
  Answer: Without loss of generality, assume that $\theta_1 \geq \theta_2 \geq \ldots \theta_n$. In efficiency, agents 1 to m get a slot. In particular, agent j ($j \leq m$) gets slot j with clickthrough rate α_j. Any agent j pays zero if he is not allocated a slot, i.e., $j > m$. For any agent $j \leq m$, we need to compute his externality. Note that the total value of agents other than agent j in an efficient allocation is

$$\sum_{i=1}^{j-1} \theta_i \alpha_i + \sum_{i=j+1}^{m} \theta_i \alpha_i.$$

  If agent j is removed, then the total value of agents other than agent j in an efficient allocation is

$$\sum_{i=1}^{j-1} \theta_i \alpha_i + \sum_{i=j+1}^{m+1} \theta_i \alpha_{i-1}.$$

  So, the externality of agent j is

$$\theta_{m+1} \alpha_m + \sum_{i=j+1}^{m} \theta_i (\alpha_{i-1} - \alpha_i),$$

where we assume that the summation term for $j = m$ is zero.

(c) Google uses something called a **Generalized Second Price (GSP)** auction: (a) agents with top m θ_i values are given the slots with highest agent getting the top slot (i.e., slot with highest CTR), second highest agent getting the next top slot, and so on, (b) if an agent wins slot k with CTR α_k, he pays $\theta_{m+1}\alpha_k$ (where θ_{m+1} is the highest losing type).

i. Use a simple example to illustrate that GSP is not equivalent to VCG.
Answer: In the previous example, agent 1 will pay $5(0.8) = 4$ in the GSP. This is clearly different from what he should pay in the VCG mechanism.

ii. Show (again using a simple example) how agents do not have a dominant strategy to bid their values.
Answer: In the example above, fix the bids of agents other than agent 2 at (agent 1: 10, agent 3: 6, agent 4: 5). Now, let agent 2 not bid truthfully, and bid $10 + \epsilon$ ($\epsilon > 0$) to become the highest bidder. So, he gets the top slot with clickthrough rate 0.8. So, his value is now $8 \times 0.8 = 6.4$ (remember, his true type is $\theta_2 = 8$). He pays $5 \times 0.8 = 4$. So, his net utility is 2.4. If he is truthful he pays $5 \times 0.6 = 3$, and gets a value of $8 \times 0.6 = 4.8$. So, his net utility of being truthful is 1.8. So, deviation is profitable, and truth-telling is not a dominant strategy.

3. Consider the usual combinatorial auction problem, but with a simpler valuation function called the "single-minded valuation". The set of objects is $M = \{1, \ldots, m\}$. Every agent i **desires** a bundle of objects $S_i \subseteq M$. The private type of agent i is a single number θ_i. An alternative in this model can be represented by the binary variables $x_i(S) \in \{0, 1\}$ satisfying some feasibility constraints, where $x_i(S) = 1$ means bundle S goes to agent i and $x_i(S) = 0$ means bundle S does not go to agent i. Given an alternative x, denote by $\mu_i(x)$ the indicator function whether agent i gets his desire in x or not, i.e., $\mu_i(x) = 1$ if $x_i(S) = 1$ and $S_i \subseteq S$ and $\mu_i(x) = 0$ otherwise. The value function of agent i is given by $\theta_i \mu_i(x)$. Essentially, agent i realizes his value θ_i if he gets his desired set of objects.

(a) Suppose there are four agents $\{1, 2, 3, 4\}$ and three objects $\{a, b, c\}$. The desires of agents are: $S_1 = \{a\}$, $S_2 = \{a, b\}$, $S_3 = \{b, c\}$, and $S_4 = \{a, c\}$. Suppose $\theta_1 = 7, \theta_2 = 8, \theta_3 = 2, \theta_4 = 10$.

i. Find the efficient allocation.
Answer: The efficient allocation will give objects a and c to agent 4, and no objects to other agents.

ii. Find the payment in the VCG mechanism.

Answer: The externality of agent 4 on other agent needs to be computed. When agent 4 leaves, other agents can get a total value of 9 (agent 1 gets a and agent 3 gets $\{b, c\}$). So, the externality of agent 4 is 9, which is his VCG payment.

4. Show that the Pivotal mechanism is the only Groves mechanism in the combinatorial auction setting which implements the efficient allocation rule and where an agent pays zero if his values for all bundles are zero.

   Answer: Groves payments are $p_i(t_i, t_{-i}) = h_i(t_{-i}) - \sum_{j \neq i} v_j(f(t_i, t_{-i}), t_j)$. But $p_i(0, t_{-i}) = h_i(t_{-i}) - \sum_{j \neq i} v_j(f(0, t_{-i}), t_j)$. Note that $\sum_{j \neq i} v_j(f(0, t_{-i}), t_j) = \max_{a \in A} \sum_{j \neq i} v_j(a, t_j)$. Then, $p_i(0, t_{-i}) = 0$ implies that $h_i(t_{-i}) = \max_{a \in A} \sum_{j \neq i} v_j(a, t_j)$. This is the payment in the Pivotal mechanism.

   Further, show that the Pivotal mechanism is the only Groves mechanism in the combinatorial auction setting which implements the efficient allocation rule, which is individually rational, and where no agent is paid (i.e., payments are non-negative).

   Answer: This is similar to above. But here, individual rationality will give $p_i(0, t_{-i}) \leq 0$ and non-negative payment gives $p_i(0, t_{-i}) = 0$. The rest of the argument is the same.

5. Suppose there is a single agent in the economy, and his type space is T. Let $f : T \to A$ be an allocation rule. Consider another type space $T' \subset T$ ($T \neq T'$) and an allocation rule $f' : T' \to A$. Show that if f is DSIC, then f' is also DSIC.

   Answer: Suppose p implements f. Note that $p : T \to \mathbb{R}$. Consider the restriction of p to T', and call it p'. It can now be verified that p' implements f'.

6. Suppose there is a single agent in the economy. Let $A = \{a, b\}$ and $T = \{s, t\}$. Consider the following valuation function of the agent $v(a, s) = 1$ and $v(a, t) = 0$ but $v(b, s) = v(b, t) = 0$. Consider an allocation rule f as follows: $f(s) = a$ and and $f(t) = b$.

   - Write down the DSIC constraints for the allocation rule f.

     Answer: DISC constraints are:

     $$v(a = f(s), s) - p(s) \geq v(b = f(t), s) - p(t)$$
     $$v(b = f(t), t) - p(t) \geq v(a = f(s), t) - p(s).$$

     Rewriting using the values of v,

     $$p(s) - p(t) \leq 1$$
     $$p(t) - p(s) \leq 0.$$

- Show that f is DSIC.

  Answer: Clearly, the only cycle in this graph has a length equal to $1 + 0 = 1$, which is positive. Hence, f is DSIC.

- What payment function makes f DSIC?

  Answer: By setting $p(s) = 1$ and $p(t) = 0$, we get a payment rule.

7. Consider the sale of a single object to a set of n agents. Consider the following allocation rule: at every type profile $\mathbf{t} = (t_1, \ldots, t_n)$, the allocation rule picks top $k \leq n$ types, i.e, agent i is picked in type profile $\mathbf{t}$ if t_i is in the top k types in $(t_1, \ldots, t_n)$. Let $K(\mathbf{t})$ be the set of agents picked in type profile $\mathbf{t}$. Then the object is awarded with equal probability to any agent in $K(\mathbf{t})$ at type profile $\mathbf{t}$. Suppose the valuation function is in product form.

   - Is this allocation rule DSIC? Explain with reason.

     Answer: We need to check if this allocation rule is non-decreasing. It is indeed non-decreasing because by increasing one's type, an agent's position in a type profile cannot decrease.

   - Consider the allocation rule where the object is awarded to an agent with the second highest type, i.e., in a type profile $\mathbf{t}$, it chooses the set of agents who have the second highest type and awards the object with equal probability to each of them. Is this allocation rule DSIC? Explain with reason.

     Answer: This is not a DSIC rule. Suppose an agent is the second highest type in a type profile. So, he must get the object in that profile with positive probability. But by increasing his type sufficiently he will become the highest type and no longer get the object. Hence, his allocation probability decreases by increasing his type. So, this allocation rule is not non-decreasing, and not DSIC.

8. Consider an economy with a single agent and let the type set of this agent be $T \subseteq \mathbb{R}^m$ (where m is a non-negative integer). Let A be the set of alternatives and $v : A \times T \to \mathbb{R}$. Consider an allocation rule f.

   - Write down the constraints for f to be DSIC.

     Answer: The DSIC constraints are

     $$p(s) - p(t) \leq v(f(s), s) - v(f(t), s) = l(t, s) \qquad \forall\ s, t \in T.$$

   - Show that if for any $s, t \in T$, if $f(s) = f(t)$, then $p(s) = p(t)$ for any payment function p which makes f DSIC.

     Answer: If $f(s) = f(t)$ then $l(t, s) = l(s, t) = 0$. This implies that $p(s) - p(t) \leq 0$ and $p(t) - p(s) \leq 0$. This gives us $p(s) = p(t)$.

- Use the previous result to show that a payment function can be written as a mapping $p : A \to \mathbb{R}$ instead of $p : T \to \mathbb{R}$ without loss of generality.

  Answer: Fix $a \in A$. By the previous result, for all $s, t \in T$ such that $f(s) = f(t) = a$, we have $p(s) = p(t)$. Hence, $p : A \to \mathbb{R}$.

- Use these results to show that the DISC constraints can be written in the following form:

$$p(b) - p(a) \leq l(a, b) \qquad \forall\ a, b \in A,$$

  where for every $a, b \in A$, $l(a, b) = \inf_{t \in T : f(t) = b}[v(b, t) - v(a, t)]$.

  Answer: Consider two types $s, t \in T$ such that $f(s) = a$ and $f(t) = b$. DSIC constraints say, using the previous result,

$$p(b) - p(a) \leq v(b, t) - v(a, t) \qquad \forall\ s, t \in T.$$

  But s does not appear anywhere in the above constraint. So, we can write, for all $t \in T$ such that $f(t) = b$, we must have

$$p(b) - p(a) \leq v(b, t) - v(a, t) \qquad \forall\ a \in A.$$

  Alternatively, we can write,

$$p(b) - p(a) \leq \inf_{t \in T : f(t) = b}[v(b, t) - v(a, t)] \qquad \forall\ a, b \in A.$$

- Use the previous result to show that f is DSIC if and only if a (complete directed) graph whose set of nodes is A and whose length function is l has no cycle of negative length. Such a graph is called an **allocation graph**.

  Answer: Clearly, the above inequalities are in potential inequality form. So, the appropriate complete directed graph can be interpreted.

9. Consider an example of sale of two goods a and b. There are two agents $N = \{1, 2\}$. Each agent wants at most one good. Consider two possible types of agent 1: $t_1(a) = 4, t_1(b) = 10$ and $t'_1(a) = 0.5, t'_1(b) = 2$. Also, consider a type of agent 2: $t_2(a) = 1$ and $t_2(b) = 2$.

   - Consider the type profile (t_1, t_2). Verify that the max-min allocation rule allocates good a to agent 1 and good b to agent 2 at this type profile.

     Answer: The allocation that maximizes the minimum utility must assign a good to every agent. In that case, there are two possible allocations to consider: one where agent 1 gets good a and agent 2 gets good b and the other where agent 1 gets good b and agent 2 gets good a. The minimum utility in the first allocation is 4 (agent 1 on good a). The minimum utility in the second allocation is 1 (agent 2 on good a). Hence, the max-min allocation rule chooses the first allocation.

- Consider the type profile (t'_1, t_2). Verify that the max-min allocation rule allocates good a to agent 2 and good b to agent 1.

  Answer: Now, in the first allocation agent 1 gets utility of 0.5 on good a, which is the minimum.

- Suppose T_1 contains t_1 and t'_1. Show that the max-min allocation rule does not satisfy cycle-monotonicity, and hence, not DSIC.

  Answer: The 2-cycle condition will be violated. To see this,

$$\begin{aligned} l_{t_2}(t_1, t'_1) + l_{t_2}(t'_1, t_1) &= v_1(f(t'_1, t_2), t'_1) - v_1(f(t_1, t_2), t'_1) \\ &+ v_1(f(t_1, t_2), t_1) - v_1(f(t'_1, t_2), t_1) \\ &= 2 - 0.5 + 4 - 10 = -4.5 < 0. \end{aligned}$$