

## GAME THEORY 2 - ASSIGNMENT 3

All the questions are concerning single object sale in the private independent value set up. Set of agents is  $N = \{1, \dots, n\}$ . Type set of agent  $i \in N$  is  $X_i = [0, h_i]$ . Agent  $i$  uses probability distribution (cdf)  $F_i$  (density function  $f_i$ ) to draw his type/value from  $X_i$ . All distributions are independent. In a mechanism  $(a, p)$ , the expected utility of agent  $i$  when he has type  $x_i$  and reports  $z_i$  is  $\alpha_i(z_i)x_i - \pi_i(z_i)$ , where  $\alpha$  and  $\pi$  denote the expected allocation rule and the expected payment rule respectively.

1. Consider a setting with one buyer whose type set is  $[0, 1]$ . Suppose the allocation rule is  $f(t) = t^\kappa$  where  $\kappa$  is a positive integer.
  - Show that this allocation rule is DSIC.
  - Find a payment rule which makes this allocation rule DSIC and the resulting mechanism satisfies IR and non-negative payments (i.e. payment of the buyer is always non-negative).
2. You need to prove that an allocation rule  $a$  is BIC if and only if it is NDE without using the cycle monotonicity arguments. Here are the steps.
  - Suppose  $a$  is BIC. Then there exists a  $p$  which satisfies the BIC constraints. Take an agent  $i \in N$  and  $x_i, z_i \in X_i$  with  $x_i > z_i$ . Write down the two constraints involving  $x_i$  and  $z_i$ . Add them to show that  $a$  is NDE.
  - Suppose  $a$  is NDE. Show that the following payment rule makes  $a$  BIC:

$$p_i(x_i, x_{-i}) = a_i(x_i, x_{-i})x_i - \int_0^{x_i} a_i(t_i, x_{-i})dt_i.$$

3. Consider a setting with two buyers whose values are distributed uniformly in the intervals  $X_1 = [0, 10]$  (buyer 1) and  $X_2 = [0, 20]$  (buyer 2). What are the reserve prices for buyer 1 and buyer 2 in the optimal mechanism, i.e., for what valuations do the buyers have zero virtual valuation? Describe the allocation and payments of buyers in an optimal mechanism when valuations are as given in Table 1. Repeat the question when  $X_1 = [0, 20]$  and  $X_2 = [0, 20]$ .
4. Consider  $N = \{1, 2\}$  (two bidders) who draw their values from the same interval  $[0, b]$  with distributions  $F_1$  (bidder 1) and  $F_2$  (bidder 2). Let the hazard rates of distributions of bidder 1 and bidder 2 be  $\lambda_1 : [0, b] \rightarrow \mathbb{R}_+$  and  $\lambda_2 : [0, b] \rightarrow \mathbb{R}_+$  respectively. Assume that the hazard rates are non-decreasing.
  - Show that  $F_1(x) = 1 - e^{-\int_0^x \lambda_1(t)dt}$  and  $F_2(x) = 1 - e^{-\int_0^x \lambda_2(t)dt}$ .

Valuations	Allocation (who gets object)	Payment of Buyer 1	Payment of Buyer 2
$(x_1 = 4, x_2 = 8)$			
$(x_1 = 2, x_2 = 12)$			
$(x_1 = 6, x_2 = 6)$			
$(x_1 = 6, x_2 = 10.5)$			
$(x_1 = 8, x_2 = 15)$			

Table 1: Description of Optimal Mechanism

- Show that if  $\lambda_1(x) \leq \lambda_2(x)$  for all  $x \in [0, b]$ , then bidder 1's value stochastically dominates bidder 2's value, i.e., bidder 2 is likely to have less value.
- Show that if  $\lambda_1(x) \leq \lambda_2(x)$  for all  $x \in [0, b]$ , then  $w_1(x) \leq w_2(x)$  for all  $x \in [0, b]$ .
- Use these to show that if  $\lambda_1(x) \leq \lambda_2(x)$  then the optimal mechanism is discriminating in the sense that it favors the bidder who is more likely to have less value.