Game Theory 2 - Assignment 3 Solutions

All the questions are concerning single object sale in the private independent value set up. Set of agents is $N = \{1, \ldots, n\}$. Type set of agent $i \in N$ is $X_i = [0, h_i]$. Agent i uses probability distribution (cdf) F_i (density function f_i) to draw his type/value from X_i. All distributions are independent. In a mechanism (a, p), the expected utility of agent i when he has type x_i and reports z_i is $\alpha_i(z_i)x_i - \pi_i(z_i)$, where α and π denote the expected allocation rule and the expected payment rule respectively.

1. Consider a setting with one buyer whose type set is $[0, 1]$. Suppose the allocation rule is $f(t) = t^\kappa$ where κ is a positive integer.

   - Show that this allocation rule is DSIC.

     Answer: It is clear that f is non-decreasing. Hence, it is DSIC.

   - Find a payment rule which makes this allocation rule DSIC and the resulting mechanism satisfies IR and non-negative payments (i.e. payment of the buyer is always non-negative).

     Answer: Let $U : [0, 1] \to \mathbb{R}$ be the utility function of the buyer (under truth-telling). In the single agent case, BIC and DSIC concepts are equivalent. We know that for all $x \in [0, 1]$,

$$\begin{aligned} U(x) &= U(0) + \int_0^x f(t)dt \\ &= U(0) + \int_0^x t^\kappa dt \\ &= U(0) + \frac{x^{\kappa+1}}{\kappa+1}. \end{aligned}$$

     Now, IR requires that $U(x) \geq 0$ for all $x \in [0, 1]$. This is equivalent to requiring $U(0) \geq 0$. Let $p : [0, 1] \to \mathbb{R}$ be the payment function. Non-negative payments require that $p(x) = xf(x) - U(x) \geq 0$ for all $x \in [0, 1]$. Equivalently, for all $x \in [0, 1]$, $x^{\kappa+1} \geq U(x) = U(0) + \frac{x^{\kappa+1}}{\kappa+1}$. This is equivalent to requiring that $U(0) \leq 0$. Hence, IR and non-negative payments are equivalent to requiring that $U(0) = 0$. Hence, we conclude that any mechanism which satisfies IR and non-negative payments must have for all $x \in [0, 1]$,

$$U(x) = \frac{x^{\kappa+1}}{\kappa+1}.$$

     Hence, the payment for all $x \in [0, 1]$ is given by $p(x) = xf(x) - U(x) = x^{\kappa+1} - U(x)$, which comes out to be

$$p(x) = \frac{kx^{\kappa+1}}{\kappa+1}.$$

2. You need to prove that an allocation rule a is BIC if and only if it is NDE without using the cycle monotonicity arguments. Here are the steps.

   - Suppose a is BIC. Then there exists a p which satisfies the BIC constraints. Take an agent $i \in N$ and $x_i, z_i \in X_i$ with $x_i > z_i$. Write down the two constraints involving x_i and z_i. Add them to show that a is NDE.

     Answer: This is straightforward.

   - Suppose a is NDE. Show that the following payment rule makes a BIC:

     $$p_i(x_i, x_{-i}) = a_i(x_i, x_{-i})x_i - \int_0^{x_i} a_i(t_i, x_{-i})dt_i.$$

     Answer: The expected payment function of this payment scheme is given by:

     $$\pi_i(x_i) = x_i\alpha_i(x_i) - \int_0^{x_i} \alpha_i(t_i)dt_i \qquad \forall\ x_i \in X_i.$$

     Now, take any $x_i, z_i \in X_i$. We need to show that

     $$\pi_i(z_i) - \pi_i(x_i) \leq z_i[\alpha_i(z_i) - \alpha_i(x_i)].$$

     We do the proof for $z_i > x_i$. The other case is similar. Substituting for π_i, we get

     $$\pi_i(z_i) - \pi_i(x_i) = [z_i\alpha_i(z_i) - x_i\alpha_i(x_i)] - [\int_0^{z_i} \alpha_i(t_i)dt_i - \int_0^{x_i} \alpha_i(t_i)dt_i].$$

     The second set of terms $\int_0^{z_i} \alpha_i(t_i)dt_i - \int_0^{x_i} \alpha_i(t_i)dt_i$ is greater than or equal to $(z_i - x_i)\alpha_i(x_i)$. Hence, we get

     $$\begin{aligned}\pi_i(z_i) - \pi_i(x_i) &\leq z_i\alpha_i(z_i) - x_i\alpha_i(x_i) - (z_i - x_i)\alpha_i(x_i) \\ &= z_i\alpha_i(z_i) - z_i\alpha_i(x_i) = z_i[\alpha_i(z_i) - \alpha_i(x_i)]\end{aligned}$$

3. Consider a setting with two buyers whose values are distributed uniformly in the intervals $X_1 = [0, 10]$ (buyer 1) and $X_2 = [0, 20]$ (buyer 2). What are the reserve prices for buyer 1 and buyer 2 in the optimal mechanism, i.e., for what valuations do the buyers have zero virtual valuation? Describe the allocation and payments of buyers in an optimal mechanism when valuations are as given in Table 1. Repeat the question when $X_1 = [0, 20]$ and $X_2 = [0, 20]$.

   Answer: First the case of $X_1 = [0, 10]$ and $X_2 = [0, 20]$. The virtual valuation of buyer 1 is 5 and that of buyer 2 is 10. Table 1 is filled based on this.

   When $X_1 = [0, 20]$ and $X_2 = [0, 20]$, we are in the symmetric case. We know the optimal auction is the second-price auction with the reserve price equal to $v^{-1}(0)$. The reserve price in this case is 10. Now, we can fill the table using the second-price auction with reserve price 10. This is shown in Table 2.

Valuations	Allocation (who gets object)	Payment of Buyer 1	Payment of Buyer 2
$(x_1 = 4, x_2 = 8)$	Unsold	0	0
$(x_1 = 2, x_2 = 12)$	Buyer 2	0	10
$(x_1 = 6, x_2 = 6)$	Buyer 1	5	0
$(x_1 = 6, x_2 = 10.5)$	Buyer 1	5.5	0
$(x_1 = 8, x_2 = 15)$	Buyer 2	0	13

Table 1: Description of Optimal Mechanism for $X_1 = [0, 10]$ and $X_2 = [0, 20]$.

Valuations	Allocation (who gets object)	Payment of Buyer 1	Payment of Buyer 2
$(x_1 = 4, x_2 = 8)$	Unsold	0	0
$(x_1 = 2, x_2 = 12)$	Buyer 2	0	10
$(x_1 = 6, x_2 = 6)$	Unsold	0	0
$(x_1 = 6, x_2 = 10.5)$	Buyer 2	0	10
$(x_1 = 8, x_2 = 15)$	Buyer 2	0	10

Table 2: Description of Optimal Mechanism for $X_1 = [0, 20]$ and $X_2 = [0, 20]$.

4. Consider $N = \{1, 2\}$ (two bidders) who draw their values from the same interval $[0, b]$ with distributions F_1 (bidder 1) and F_2 (bidder 2). Let the hazard rates of distributions of bidder 1 and bidder 2 be $\lambda_1 : [0, b] \to \mathbb{R}_+$ and $\lambda_2 : [0, b] \to \mathbb{R}_+$ respectively. Assume that the hazard rates are non-decreasing.

   - Show that $F_1(x) = 1 - e^{-\int_0^x \lambda_1(t)dt}$ and $F_2(x) = 1 - e^{-\int_0^x \lambda_2(t)dt}$.

     Answer: Note that $\lambda_1(x) = \frac{dF_1(x)}{1-F_1(x)}$. Hence, $\int_0^x \lambda_1(t)dt = -\log(1 - F_1(x))$. This gives us the desired result.

   - Show that if $\lambda_1(x) \leq \lambda_2(x)$ for all $x \in [0, b]$, then bidder 1's value stochastically dominates bidder 2's value, i.e., bidder 2 is likely to have less value.

     Answer: If $\lambda_1(x) \leq \lambda_2(x)$, then we get $F_1(x) \leq F_2(x)$ by using the previous result.

   - Show that if $\lambda_1(x) \leq \lambda_2(x)$ for all $x \in [0, b]$, then $w_1(x) \leq w_2(x)$ for all $x \in [0, b]$.

     Answer: This follows straightforwardly - $w_1(x) = x - \frac{1}{\lambda_1(x)} \leq x - \frac{1}{\lambda_2(x)}$.

   - Use these to show that if $\lambda_1(x) \leq \lambda_2(x)$ then the optimal mechanism is discriminating in the sense that it favors the bidder who is more likely to have less value.

     Answer: Since $\lambda_1(x) \leq \lambda_2(x)$, we know that $F_1(x) \leq F_2(x)$. It implies that bidder 2 is more likely to be the lowest value bidder. But $w_1(x) \leq w_2(x)$ implies that bidder 2 is likely to win more often. This shows that the optimal mechanism is discriminating in the sense that it favors bidder 2.