Theory of Mechanism Design - Assignment 4

1. A seller is selling an object to an agent whose value (type) for the object lies in the interval $I \equiv [0,1]$. The seller uses an allocation rule $f : I \to [0,1]$ and a payment rule $p : I \to \mathbb{R}$. Denote the mechanism (f,p) as M.

   Fix an $\epsilon \in (0,1]$. The mechanism M satisfies only a subset of incentive constraints: for every $t \in I$ and for every $s \in I$ such that $|s-t| \leq \epsilon$,

   $$tf(t) - p(t) \geq tf(s) - p(s).$$

   Show that M is dominant strategy incentive compatible. **(10 marks)**

   Answer. We illustrate the idea with $\epsilon = 0.5$. We only need to show that if we pick a type $s \in [0,0.5)$ and a type $t \in (0.5,1]$, then the incentive constraints between these types hold. Let $x \equiv 0.5$. Since the incentive constraints between x and every type holds, we can write:

   $$\begin{aligned} sf(s) - p(s) &\geq sf(x) - p(x) \\ xf(x) - p(x) &\geq xf(s) - p(s). \end{aligned}$$

   Adding these and using $x > s$, we get $f(x) \geq f(s)$. A similar argument establishes $f(t) \geq f(x)$. Hence, we have $f(t) \geq f(s)$.

   Next, we note that

   $$\begin{aligned} sf(s) - p(s) &\geq sf(x) - p(x) \\ xf(x) - p(x) &\geq xf(t) - p(t). \end{aligned}$$

   Adding these we get

   $$\begin{aligned} sf(s) - p(s) &\geq (s-x)f(x) + xf(t) - p(t) \\ &= (s-x)f(x) + (x-s)f(t) + sf(t) - p(t) \\ &= (s-x)(f(x) - f(t)) + sf(t) - p(t) \\ &\geq sf(t) - p(t), \end{aligned}$$

   where we used $f(x) \leq f(t)$ and $s < x$ for the inequality. This shows that the incentive constraints from s to t hold. A similar argument establishes that the incentive constraints from t to s holds.

2. Consider the usual combinatorial auction problem, but with a simpler valuation function called the "single-minded valuation". The set of objects is $M = \{1, \ldots, m\}$. Every agent i **desires** a bundle of objects $S_i \subseteq M$. The private type of agent i is a single number θ_i. An alternative in this model can be represented by the binary variables $x_i(S) \in \{0, 1\}$ satisfying some feasibility constraints, where $x_i(S) = 1$ means bundle S goes to agent i and $x_i(S) = 0$ means bundle S does not go to agent i. Given an alternative x, denote by $\mu_i(x)$ the indicator function whether agent i gets his desire in x or not, i.e., $\mu_i(x) = 1$ if $x_i(S) = 1$ and $S_i \subseteq S$ and $\mu_i(x) = 0$ otherwise. The value function of agent i is given by $\theta_i \mu_i(x)$. Essentially, agent i realizes his value θ_i if he gets his desired set of objects.

   (a) Suppose there are four agents $\{1, 2, 3, 4\}$ and three objects $\{a, b, c\}$. The desires of agents are: $S_1 = \{a\}$, $S_2 = \{a, b\}$, $S_3 = \{b, c\}$, and $S_4 = \{a, c\}$. Suppose $\theta_1 = 7, \theta_2 = 8, \theta_3 = 2, \theta_4 = 10$.

      i. Find the efficient allocation.
      Answer: The efficient allocation will give objects a and c to agent 4, and no objects to other agents.

      ii. Find the payment in the VCG mechanism.
      Answer: The externality of agent 4 on other agent needs to be computed. When agent 4 leaves, other agents can get a total value of 9 (agent 1 gets a and agent 3 gets $\{b, c\}$). So, the externality of agent 4 is 9, which is his VCG payment.

3. There are four agents $N = \{1, 2, 3, 4\}$. There is a single indivisible object for sale. Each agent $i \in N$ gets a value $v_i \in \mathbb{R}_+$ if he is allocated the object or agent $(i + 1)$ is allocated the object, where we maintain the convention that if $i = 4$, then $(i + 1) \equiv 1$. Agent i gets zero value if any agent $j \notin \{i, i + 1\}$ gets the object.

   - Suppose the values of the agents are $v_1 = 10, v_2 = 4, v_3 = 7, v_4 = 5$. Who should get the object according to the efficient allocation rule and what should be the payment according to the VCG (pivotal) mechanism?

     Answer: If agent 1 gets the object, then agents 1 and 4 realize value and nobody else. So, total value realized is 15. Similarly, if agent 2 gets the object, then the total value realized is 14. If agent 3 gets the object, then the total value realized is 11. If agent 4 wins the object, then the total value realized is 12. So, efficiency requires that the object be given to agent 1.

     Agents 2 and 3 have no externality. For agent 1, the total value of other agents is 5 (agent 4 realizes value). If agent 1 is absent, then the maximum total value of other agents is 12 (when agent 4 wins). So, his externality is 7. For agent 4, the total value of other agents is 10 (agent 1 realizes value). If agent 4 is absent, then the maximum total value of other agents is 14. So, externality of agent 4 is 4.

- Suppose we use the affine maximizer rule where we use a weight of $\lambda_i = 1$ for $i \in \{1, 2\}$ and $\lambda_i = 0.5$ for $i \in \{3, 4\}$. Assume $\kappa(a) = 0$ for all alternatives a. Who should get the object according to this allocation rule? Describe a payment consistent with a generalized Groves mechanism for this allocation rule.

  Answer: Now, if agent 1 gets the object, then the total weighted value is 10+0.5(5)=12.5. Similarly, if agent 2 gets it, then the total weighted value is 4 + 10=14. If agent 3 gets it, then the total weighted value is 3.5+5=8.5. If agent 4 gets it, then the total weighted value is 6. Hence, agent 2 must get the object.

  There are many generalized Groves payment rules. One of them is the following. Agent i at profile (t_i, t_{-i}) gets paid an amount

  $$\frac{1}{\lambda_i}\sum_{j \neq i} \lambda_j t_j(a),$$

  where $f(t_i, t_{-i}) = a$.

4. Show that the Pivotal mechanism is the only Groves mechanism in the combinatorial auction setting which implements the efficient allocation rule and where an agent pays zero if his values for all bundles are zero.

   Answer: Groves payments are $p_i(t_i, t_{-i}) = h_i(t_{-i}) - \sum_{j \neq i} v_j(f(t_i, t_{-i}), t_j)$. But $p_i(0, t_{-i}) = h_i(t_{-i}) - \sum_{j \neq i} v_j(f(0, t_{-i}), t_j)$. Note that $\sum_{j \neq i} v_j(f(0, t_{-i}), t_j) = \max_{a \in A} \sum_{j \neq i} v_j(a, t_j)$. Then, $p_i(0, t_{-i}) = 0$ implies that $h_i(t_{-i}) = \max_{a \in A} \sum_{j \neq i} v_j(a, t_j)$. This is the payment in the Pivotal mechanism.

   Further, show that the Pivotal mechanism is the only Groves mechanism in the combinatorial auction setting which implements the efficient allocation rule, which is individually rational, and where no agent is paid (i.e., payments are non-negative).

   Answer: This is similar to above. But here, individual rationality will give $p_i(0, t_{-i}) \leq 0$ and non-negative payment gives $p_i(0, t_{-i}) = 0$. The rest of the argument is the same.

5. Consider the public good provision problem with alternatives a_0 (public good is not provided) and a_1 (public good is provided). Assume that the value for a_0 is zero for all the agents and the value for a_1 is θ_i for agent i. Consider the allocation rule that chooses a_1 at a profile $(\theta_1, \ldots, \theta_n)$ if $\sum_{i \in N} \theta_i \geq C$, where C is the cost of providing the public good, and chooses a_0 otherwise.

   Illustrate the generalized pivotal mechanism as a generalized Groves mechanism (by choosing appropriate h_i functions) in this model.

Answer: If we use $h_i(\theta_{-i}) = \max(\sum_{j \neq i} \theta_j - C, 0)$, it will correspond to the generalized pivotal mechanism.