Game Theory 2 - Assignment 4 Solution

The first three questions are for voting models. The final three questions are in house allocation models and house allocation models with existing tenants.

1. Show that the Gibbard-Satterthwaite theorem does not hold when the number of alternatives ($|A|$) is equal to 2.

   Solution: If $|A| = \{a, b\}$, then define $f(P) = a$ if $|\{i \in N : P_i(1) = a\}| \geq |\{i \in N : P_i(1) = b\}|$ and $f(P_1, P_2) = b$ otherwise.

   Now, fix a preference profile P and agent i. Suppose $f(P) = a$. Note that an agent can only manipulate by reporting a different top ranked alternative. Agent i manipulates if $P_i(1) = b$. In that case, by reporting P'_i with $P'_i(1) = a$, the outcome is unchanged. Similarly, if $f(P) = b$, agent i manipulates if $P_i(1) = a$. In that case, if he reports $P'_i(1) = b$, then the outcome is unchanged. Hence, f is strategy-proof.

2. Let $\mathcal{P}$ be the set of all linear orders over the set of alternatives A (with $|A| \geq 3$). Let $\mathcal{D}$ be a domain of orderings such that $\mathcal{P} \subseteq \mathcal{D}$. Using the Gibbard-Satterthwaite theorem, show that every onto and strategy-proof social choice function $f : \mathcal{D}^n \to A$ is a dictatorship.

   Solution: Assume for contradiction that $f : \mathcal{D}^n \to A$ is a strategy-proof and onto social choice function which is not a dictatorship. Denote by $\hat{f}$ the restriction of f to $\mathcal{P}^n$, i.e, for all $P \in \mathcal{P}^n$, $\hat{f}(P) = f(P)$. We note the following.

   - $\hat{f}$ is strategy-proof. If this is not true, then f is also not strategy-proof.
   - $\hat{f}$ satisfies unanimity. Consider a preference profile $P \in \mathcal{P}^n$ such that $P_1(1) = P_2(1) = \ldots = P_n(1) = a$. Since f is onto, there is a profile $R = (R_1, \ldots, R_n) \in \mathcal{D}^n$ such that $f(R) = a$. Let $R' = (P_1, R_2, \ldots, R_n)$. Note that $R' \in \mathcal{D}^n$. We argue that $f(R') = a$, else agent 1 can manipulate via R at R'. Similarly, letting $R'' = (P_1, P_2, R_3, \ldots, R_n)$, we can show that $f(R'') = a$ (else agent 2 will manipulate at R'' via R'), and so on. We will thus get $f(P) = b = \hat{f}(P)$.
   - $\hat{f}$ is not a dictatorship. Since f is not a dictatorship for every agent i, there is a preference profile $R = (R_1, \ldots, R_n) \in \mathcal{D}^n$ such that $f(R) \notin R_i(1)$ - note that since preference orderings are not anti-symmetric $R_i(1)$ is a set of alternatives. Let $f(R) = a$. Consider a preference profile $R' = (R_i, P_{-i})$, where P_j is a linear order with $P_j(1) = a$. We argue that $f(R') = a$. We can go from R to R' by changing the preference ordering of one agent at a time. In each stage, the outcome must be a due to strategy-proofness - else the agent whose preference is changing will manipulate. Now, consider the preference profile $P = (P_i, P_{-i})$

where $P_i(1) \in R_i(1)$. Let $P_i(1) = b$. By definition $b \neq a$. We argue that $f(P) \neq b$. Assume for contradiction $f(P) = b$. Then, agent i can manipulate at R' via P. This is a contradiction since f is strategy-proof. Hence, we establish that for every agent i, there is a preference profile $P \in \mathcal{P}$ such that $f(P) = \hat{f}(P) \neq b$. So, $\hat{f}$ is not a dictatorship.

But Gibbard-Satterwaite theorem says that if $\hat{f}$ is strategy-proof and unanimous, then it has to be a dictatorship. This is a contradiction.

3. Consider a two agent model with three alternatives $\{a, b, c\}$. Table 1 shows two preference profiles of linear orders. Suppose $f(P_1, P_2) = a$. Show that if f is strategy-proof then $f(P'_1, P'_2) = b$. You are allowed to use the result that for any preference profile $(\bar{P}_1, \bar{P}_2)$, $f(\bar{P}_1, \bar{P}_2) \in \{\bar{P}_1(1), \bar{P}_2(1)\}$ (but do not use any other result from the lectures).

P_1	P_2	P'_1	P'_2
a	c	b	c
b	b	a	a
c	a	c	b

Table 1: Two Preference Profiles

Solution: We know that $f(P') \in \{b, c\}$. Assume for contradiction that $f(P') = c$. Consider another preference profile $P'' = (P'_1, P_2)$. So, $f(P'') \in \{b, c\}$. Since $f(P') = c$, $f(P'') = c$ - else agent 2 will manipulate at P'' via P'. Since $f(P) = a$. Agent 1 will manipulate at P'' via P. This is a contradiction.

4. Show that in the TTC mechanism no agent can get a house which is lower in ranking than his initial endowment house.

   Solution: In the TTC mechanism, an agent points to a house not owned by him if he prefers that house to his endowed house. Whenever he points to his endowed house, he gets it. So, if an agent gets a house which he is not endowed, he must prefer it to the endowed house.

5. In a house allocation model, consider the preferences of four agents over four houses as shown in Table 2.

   - Run the TTC mechanism for some fixed endowment.
   - Run the fixed priority mechanism for some priority.

   Solution: Run the TTC and fixed priority mechanisms.

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$
a_3	a_3	a_1	a_2
a_1	a_2	a_4	a_1
a_2	a_1	a_3	a_3
a_4	a_4	a_2	a_4

Table 2: An example for housing model

6. We consider the house allocation model with existing tenants. Consider the following form of manipulation by a coalition of agents in the TTC mechanism. A coalition of agents S exchange their houses before the start of the mechanism (i.e., they end up with an endowment which is different from their actual endowment). Now, the TTC mechanism is executed. Do you think each agent in S will now get a house which is either the same house he gets if he had not done the manipulation or a house which is higher ranked than the house he gets if he does not do manipulation?

   Solution: Such a manipulation is possible. We give an example with four agents with four houses. Let $N = \{1, 2, 3, 4\}$ and the set of houses be $\{a_1, a_2, a_3, a_4\}$. The initial endowments of houses are given by a^*: $a^*(i) = a_i$ for all $i \in N$. The preferences of agents are shown in Table 3 - some of the preferences of agents are not shown completely, implying that it can be anything in the parts not shown.

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$
a_2	a_1	a_4	a_4
a_3		a_2	
		a_3	

Table 3: An example for housing model

   If we run the TTC mechanism on this problem, the outcome will be a: $a(1) = a_2, a(2) = a_1, a(3) = a_3, a(4) = a_4$.

   Now, suppose agents 2 and 3 swap their endowments. So, the initial endowments of agents look as a': $a'(1) = a_1, a'(2) = a_3, a'(3) = a_2, a'(4) = a_4$. If we run the TTC mechanism on this problem, the outcome will be $\hat{a}$: $\hat{a}(1) = a_3, \hat{a}(2) = a_1, \hat{a}(3) = a_2, \hat{a}(4) = a_4$. Note that $\hat{a}(2) = a(2)$ and $\hat{a}(3) \succ_3 a(3)$. Hence, agents 2 and 3 successfully manipulated their initial endowments.