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IMPLEMENTATION OF REDUCED FORM AUCTIONS: A GEOMETRIC APPROACH

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1. INTRODUCTION

AN AUCTION IS A MECHANISM for allocating a single indivisible object to one of several competing bidders. The *winner* is the bidder who is awarded the object. The rules of the auction specify two functions. The first is the probability with which a bidder wins, as a function of everyone's bids. The second is the payment each bidder makes to the seller, as a function of all the bids and whether or not he wins. For instance, a first-price auction awards the object to the highest bidder with probability one (providing there are no tie bids), the winner pays his bid, and the losers pay nothing.

The bidders in an auction differ significantly. These differences are captured by the bidder's *type*. A type may be the bidder's personal valuation of the object for sale, his degree of risk aversion, or perhaps his information about the object. (Maskin and Riley (1984) discuss a number of different economically meaningful examples of bidder types.) From the viewpoint of the seller and the other bidders, each bidder's type is a random variable. In this analysis we confine attention to auctions in which the types are independently and identically distributed according to a known probability distribution. The Revelation Principle asserts that every auction is strategically equivalent to an auction in which bidders bid by announcing their type and no bidder has any incentive to lie. Such an auction is called an incentive compatible direct auction. We will confine our attention to the probability functions for direct auctions, and let the incentive compatibility conditions restrict the payment functions.

Each bidder can compute the probability that he wins, conditional on his own type, by averaging over the types of the other bidders. The function relating a bidder's type to his probability of winning is the *reduced form* of the auction. The literature on "optimal" auctions usually addresses the problem of maximizing expected revenue for the seller. For this purpose, all the relevant information about the probability function of an auction is contained in its reduced form. It is the reduced form that determines each bidder's behavior and hence the seller's expected revenue. In a symmetric auction each bidder's reduced form is identical, so that expected revenue is a functional defined on reduced forms, which are functions of one variable, namely, types. This makes the seller's problem somewhat tractable. To design an auction, a seller must be able to recognize a reduced form and recover the underlying auction.

Reduced forms satisfy an intuitive feasibility condition. Given a set of types, the reduced form tells the probability that a bidder from this set of types wins. This probability must be less than or equal to the probability that there exists a bidder from the set. Matthews (1984) conjectured that if a function satisfies this feasibility condition for all measurable sets, then it is a reduced form. Proposition 3.1 states that this is indeed the case. Proposition 3.2 refines Proposition 3.1 by requiring feasibility on a smaller family of sets. Maskin and Riley (1984, Theorem 7) gave a special case of Proposition 3.2. They showed that if the types are continuously distributed on the unit interval, then any increasing step function satisfying the feasibility condition on increasing intervals is a reduced form. Matthews generalized their results to all increasing

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functions. As Moore (1984) points out, their results are inadequate for the general study of optimal auctions. Propositions 3.1 and 3.2 apply to general probability spaces and arbitrary measurable functions.

These propositions are proved by giving a simple geometric description of the set of all (implementable) reduced forms. A drawback of this approach is that it is not constructive. That is, we know which functions are reduced forms, but not which auctions generate them. When the set of types is finite, however, this geometric description suggests a convenient computational technique (linear programming) for recovering the auction underlying a reduced form. This technique can be used to construct approximate implementations when the set of types is infinite.

The next section presents the notation necessary to make the results precise. The following section states the propositions characterizing reduced forms. It is followed by an example to illustrate the results. The fifth section provides the proofs and the final section provides the linear program for approximating the underlying auction.

2. NOTATION AND BASIC DEFINITIONS

There is a measurable space $(T, \mathscr{T})$ of possible types of bidders, and the population of N bidders is independently and identically distributed according to the probability measure λ on T. We will denote a generic element of T by t, and a generic element of T^N by p (for profile). To avoid some uninteresting cases, we explicitly make the following assumption.

ASSUMPTION: *For each* $t \in T$, $\{t\} \in \mathscr{T}$.

An *auction* is a measurable function $q\colon T^N \to [0,1]^N$ satisfying for all profiles $p \in T^N$,

$$\sum_{i=1}^{N} q^i(p) \leqslant 1. \tag{2.1}$$

The ith component, $q^i(t_1, \ldots, t_N)$, is the probability that bidder i wins when each bidder j is of the type t_j. Inequality (2.1) simply says that the probability that someone wins is less than or equal to 1. It may well be strict, for instance if the seller sets a minimum bid.

Maskin and Riley (1984) show that in the i.i.d. case, a seller need only consider a *symmetric auction*. That is, an auction $q\colon T^N \to [0,1]^N$ satisfying, for all $i = 1, \ldots, N$, and all $p \in T^N$,

$$q^i(p) = q^1(\sigma^i(p)), \tag{2.2}$$

where $\sigma^i\colon T^N \to T^N$ interchanges the first and ith coordinates, i.e.,

$$\sigma^i(t_1, \ldots, t_N) = (t_i, t_2, \ldots, t_{i-1}, t_1, t_{i+1}, \ldots, t_N).$$

Let $\mathscr{Q}_1$ denote the set of all symmetric auctions.

The class of hierarchical auctions is particularly useful. Let $A_1, A_2, \ldots, A_K$ be pairwise disjoint nonempty subsets of T, and define the *hierarchical auction* generated by $A_1, \ldots, A_K$, denoted $q_{A_1 \cdots A_K}$, by

$$q^i_{A_1 \cdots A_K}(t_1, \ldots, t_N) = \begin{cases} \dfrac{1}{\#\{n\colon t_n \in A_j\}} \\ \qquad \text{if } t_i \in A_j \text{ and } \forall n\ t_n \notin A_1 \cup \cdots \cup A_{j-1}, \\ 0 \qquad \text{otherwise}. \end{cases}$$

That is, there is a hierarchy of types with types in A_1 at the top. If there is a bidder whose type lies in A_1, all bidders with types in A_1 tie, that is, they all have an equal

	0	$q^1 = 1$ $q^2 = 0$	0	$q^1 = 1$ $q^2 = 0$	0
A_2	$q^1 = 0$ $q^2 = 1$	$q^1 = 1$ $q^2 = 0$	$q^1 = 0$ $q^2 = 1$	$q^1 = \frac{1}{2}$ $q^2 = \frac{1}{2}$	$q^1 = 0$ $q^2 = 1$
	0	$q^1 = 1$ $q^2 = 0$	0	$q^1 = 1$ $q^2 = 0$	0
A_1	$q^1 = 0$ $q^2 = 1$	$q^1 = \frac{1}{2}$ $q^2 = \frac{1}{2}$	$q^1 = 0$ $q^2 = 1$	$q^1 = 0$ $q^2 = 1$	$q^1 = 0$ $q^2 = 1$
	0	$q^1 = 1$ $q^2 = 0$	0	$q^1 = 1$ $q^2 = 0$	0
		A_1		A_2	

FIGURE 1.—The hierarchical auction $q_{A_1 A_2}$ when $N = 2$ and $T = [0, 1]$.

chance of winning. If no bidder's type lies in A_1, then bidders in A_2 tie, etc. Clearly, hierarchical auctions are symmetric.

Figure 1 illustrates the hierarchical auction $q_{A_1 A_2}$ for the case $N = 2$ and $T = [0, 1]$. Note that each q^i vanishes outside of $\sigma^i((A_1 \cup A_2) \times T)$, and that $q^1 + q^2$ is equal to 1 on

$$\Gamma = \bigcup_i \sigma^i((A_1 \cup A_2) \times T) = ((A_1 \cup A_2) \times T) \cup (T \times (A_1 \cup A_2)),$$

and $q^1 + q^2$ vanishes outside Γ.

Given an auction, q, each bidder i can compute the probability, $Q^i(t_i)$, that he wins when his type is t_i, by

$$\text{(2.3)} \qquad Q^i(t_i) = \int_{T^{N-1}} q^i(t_1, \dots, t_N) \, d\lambda^{N-1}(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n).$$

Clearly Q^i: $T \to [0, 1]$. Tonelli's theorem implies that it is measurable.

When the auction q is symmetric, $Q^i(t)$ is independent of i for all $t \in T$. In this case, dropping the superscript i from Q, we say that Q is the *reduced form* of q and that q *implements* Q. Thus, call Q *implementable* if there is some symmetric auction q which satisfies (2.3) for each $t_i \in T$. Call the reduced form of a hierarchical auction a *hierarchical reduced form*, and let $Q_{A_1 \cdots A_K}$ denote the reduced form of $q_{A_1 \cdots A_k}$. Let $\mathscr{Q}$ denote the set of all reduced forms of symmetric auctions.

3. STATEMENT OF RESULTS

The following propositions characterizing $\mathscr{Q}$ generalize the partial results of Maskin and Riley (1984, Theorem 7) and Matthews (1984).

PROPOSITION 3.1: *Let Q: $T \to [0,1]$ be measurable. Then Q is implementable by a symmetric auction if and only if for each measurable set of types $A \in \mathscr{T}$, the following inequality is satisfied*:

$$\int_A Q(t)\,d\lambda(t) \leqslant \frac{1-\lambda(A^c)^N}{N}. \tag{3.1}$$

Furthermore, if T is a topological space and λ is a regular Borel probability on T, then $\mathscr{T}$ may be replaced by either the open subsets or the closed subsets of T.

Proposition 3.2 follows from Proposition 3.1 roughly because if condition (3.1) is violated, it must be violated on a set where Q is large.

PROPOSITION 3.2: *Let Q: $T \to [0,1]$ and for each $\alpha \in [0,1]$, set*

$$E_\alpha = \{t: Q(t) \geqslant \alpha\}.$$

Then Q is implementable if and only if for each $\alpha \in [0,1]$

$$\int_{E_\alpha} Q\,d\lambda \leqslant \frac{1-\lambda(E_\alpha^c)^N}{N}.$$

This proposition reduces the problem of deciding the implementability of a function to checking a one parameter family of inequalities.

4. AN EXAMPLE

To illustrate the propositions, consider the case of two bidders with two equally likely types, i.e., $N=2$, $T=\{1,2\}$, $\lambda(\{1\})=\lambda(\{2\})=1/2$.

In this case, a symmetric auction is defined by the four numbers $q(1,1)$, $q(1,2)$, $q(2,1)$, and $q(2,2)$, where $q(i,j)$ is the probability that bidder 1 wins when his type is i and bidder 2's type is j. The symmetry conditions imply that $q(1,1) \leqslant \frac{1}{2}$, $q(2,2) \leqslant \frac{1}{2}$, and $q(1,2)+q(2,1) \leqslant 1$. That is, the set $\mathscr{Q}_1$ of symmetric auctions can be viewed as the set

$$\left[0,\tfrac{1}{2}\right] \times \left[0,\tfrac{1}{2}\right] \times \Delta \subset R^4,$$

with generic element $q=(q(1,1), q(2,2), q(1,2), q(2,1))$, where $\Delta = \{(x_1,x_2)\colon x_1+x_2 \leqslant 1, x_i \geqslant 0\}$.

The set $\mathscr{Q}$ of reduced forms is the image of $\mathscr{Q}_1$ under the linear mapping $q \mapsto Q$ defined by the implementability conditions

$$Q(1) = \tfrac{1}{2}q(1,1) + \tfrac{1}{2}q(1,2)$$

and

$$Q(2) = \tfrac{1}{2}q(2,1) + \tfrac{1}{2}q(2,2).$$

The set $\mathscr{Q}$ is plotted in Figure 2. Observe that $\mathscr{Q}$ is bounded by lines (hyperplanes, if you will) normal to $(1,0)$, $(0,1)$, and $(1,1)$. Further, if Q is nondecreasing, i.e., $Q(2) \geqslant Q(1)$, then Q lies in the region of $\mathscr{Q}$ bounded by lines normal to $(0,1)$ and $(1,1)$. The vectors $(1,0)$, $(0,1)$, and $(1,1)$ may be interpreted as the indicator functions of the subsets $\{1\}$, $\{2\}$,

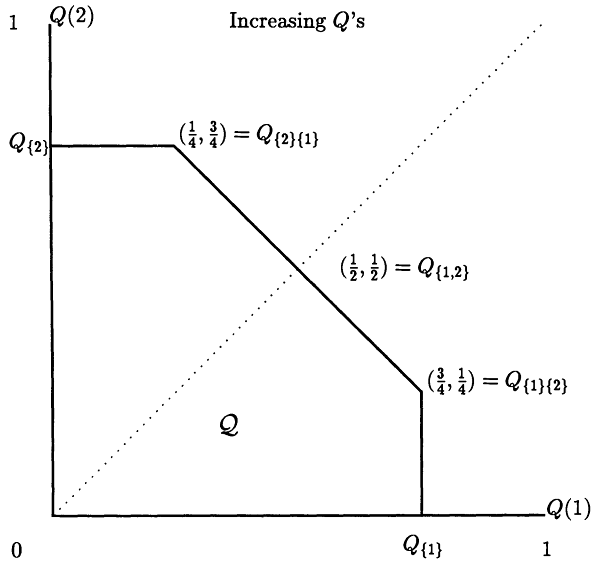

FIGURE 2.—$\mathcal{Q}$ for the case of two equiprobable types.

and $\{1,2\}$ of T. The increasing sets are $\{2\}$ and $\{1,2\}$. The extreme points of Q are the hierarchical reduced forms corresponding to hierarchies of singletons. This last property is no accident—Lemma 6.1 asserts that this is always true for the finite type case. (Figure 2 is a little misleading if the types are not equally likely. In that case, the axes must be scaled by the probability of each type in order for the slopes to be correct.)

This example suggests that by embedding $\mathcal{Q}$ in an appropriate vector space, for which indicator functions define linear functionals, then Propositions 3.1 and 3.2 become statements about the hyperplanes bounding $\mathcal{Q}$. This is indeed the case when $\mathcal{Q}$ is embedded in $L_\infty(\lambda)$.

5. PROOFS

We break the proof down into easily digestible lemmas. The first lemma states that the probability that a bidder from set A wins, $N\int_A Q\,d\lambda$, does not exceed the probability that there exists a bidder from A, $1-\lambda(A_c)^N$.

Let $\langle f, g\rangle$ denote $\int_T f(t)g(t)\,d\lambda(t)$, χ_A denote the indicator function of A, and let

$$B(A)=\frac{1-\lambda(A^c)^N}{N} \quad \text{for} \quad A\in\mathcal{T}.$$

LEMMA 5.1: *For all $A\in\mathcal{T}$, all $Q\in\mathcal{Q}$,*

$$\langle \chi_A, Q\rangle \leqslant B(A).$$

PROOF: Let q be a symmetric auction implementing Q. Calculate $\int_A Q\,d\lambda$, by integrating any of the q^i's over the cylinder $\sigma^i(A\times T^{N-1})$. (See Figure 1 for the case $N=2$.) An upper bound is given by integrating q^i over the cross shaped union of these N cylinders. This bound is tight if q^i vanishes outside the cylinder $\sigma^i(A\times T^{N-1})$.

Formally,

$$\langle \chi_A, Q\rangle = \int_A Q(t)\,d\lambda(t) = \int_{A\times T^{N-1}} q^1(p)\,d\lambda^N(p).$$

By symmetry condition (2.2),

$$\begin{aligned}\int_{A\times T^{N-1}} q^1\,d\lambda^N &= \int_{\sigma^i(A\times T^{N-1})} q^i\,d\lambda^N\\ &\leqslant \int_{\bigcup_i \sigma^i(A\times T^{N-1})} q^i\,d\lambda^N,\end{aligned}$$

with equality if q^i vanishes outside $\sigma^i(A\times T^{N-1})$.

Summing over i yields

$$N\langle \chi_A, Q\rangle = \sum_{i=1}^{N}\int_{\sigma^i(A\times T^{N-1})} q^i(p)\,d\lambda^N(p) \tag{5.1}$$

$$\leqslant \sum_{i=1}^{N}\int_{\bigcup_i\sigma^i(A\times T^{N-1})} q^i(p)\,d\lambda^N(p) \tag{5.2}$$

$$= \int_{\bigcup_i\sigma^i(A\times T^{N-1})} \sum_i q^i(p)\,d\lambda^N(p) \tag{5.3}$$

$$\leqslant \int_{\bigcup_i\sigma^i(A\times T^{N-1})} 1\,d\lambda^N(p) \tag{5.4}$$

$$= 1-\lambda(A^c)^N \tag{5.5}$$

Inequality (5.4) follows from (2.1) and holds with equality if $\Sigma q^i = 1$ on the "cross" $\bigcup \sigma^i(A\times T^{N-1})$. Equation (5.5) is an elementary probability calculation. Thus $\langle \chi_A, Q\rangle \leqslant (1-\lambda(A^c)^N)/N = B(A)$. *Q.E.D.*

Each $\langle \chi_A, \cdot\rangle$ defines a function on $\mathscr{Q}$, and according to Lemma 5.1, this function is bounded above by $B(A)$. We now show that, in fact, this bound is achieved by a hierarchical reduced form.

LEMMA 5.2: *Let Q^* be the reduced form of the hierarchical auction $q_{A_1\cdots A_K}$. For each $j=1,\ldots,K$, set $F^j = A_1\cup\cdots\cup A_j$; then for each j,*

$$\langle \chi_{F^j}, Q^*\rangle = B(F^j).$$

Furthermore Q^* is a simple function which is constant on each A_j and is zero on $(F^K)^c$.

PROOF: The proof proceeds by straightforward calculation. Set

$$\Gamma_j = \bigcup_{i=1}^{N} \sigma^i(F^j\times T^{N-1}),$$

i.e., the set of profiles in T^N in which at least one bidder has a type in $A_1\cup\cdots\cup A_j$. Simple calculations reveal that $\Sigma_{i=1}^N q^i_{A_1\cdots A_K}$ vanishes outside of Γ_K and is identically 1 on Γ_K. This can be seen recursively: Γ_1 is the set of profiles for which some bidder has a type in A_1. If there is exactly one bidder, his $q^i = 1$; if there are two, then they each have $q^i = 1/2$, etc., so $\Sigma q^i = 1$ on Γ_1. Now suppose no bidder has type in A_1, but some bidder has type in A_2; then his $q^i = 1$ if he is alone in A_2, etc. Thus $\Sigma q^i = 1$ on $\Gamma_2 \backslash \Gamma_1$, etc. If no bidder's type lies in any A_j, then no one wins. Thus Σq^i vanishes outside of Γ_K. See Figure 1 for an example. It follows that (5.2) and (5.4) hold with equality for F^j (see the proof of Lemma 5.1) and so

$$\langle \chi_{F^j}, Q^* \rangle = B(F^j).$$

Clearly Q^* is constant on each A_j and vanishes outside their union. In particular, a hierarchical reduced form is a simple function. *Q.E.D.*

Lemma 5.1 shows that each $Q \in \mathscr{Q}$ satisfies the inequalities (3.1). The next lemma shows that if a simple function separates Q from $\mathscr{Q}$, then Q violates (3.1) on some set of types.

LEMMA 5.3: *Let $\overline{Q}$: $T \to [0,1]$ be measurable and suppose the simple function $\Sigma_{j=1}^{L} \alpha_j \chi_{A_j}$ separates $\overline{Q}$ from $\mathscr{Q}$. That is, for all $Q \in \mathscr{Q}$,*

$$\left\langle \sum_{j=1}^{L} \alpha_j \chi_{A_j}, \overline{Q} \right\rangle > \left\langle \sum_{j=1}^{L} \alpha_j \chi_{A_j}, Q \right\rangle. \tag{5.6}$$

Then for some measurable $A \subset T$, $\langle \chi_A, \overline{Q} \rangle > B(A)$.

PROOF: Without loss of generality, take the A_j's to be pairwise disjoint and numbered so that $\alpha_1 > \alpha_2 > \cdots > \alpha_K > 0 \geqslant \alpha_{K+1} > \cdots > \alpha_L$. (Inequality (5.6) implies that at least one $\alpha_j > 0$, since $\overline{Q} \geqslant 0$ and $0 \in \mathscr{Q}$.) Let Q^* be the reduced form of the hierarchical auction generated by $A_1, \dots, A_K$. If for any $k = 1, \dots, K-1$ we have $\langle \chi_{A_1 \cup \cdots \cup A_k}, \overline{Q} - Q^* \rangle > 0$, then we are done, since $\langle \chi_{A_1 \cup \cdots \cup A_k}, Q^* \rangle = B(A_1 \cup \cdots \cup A_k)$ by Lemma 5.2. So suppose

$$\langle \chi_{A_1 \cup \cdots \cup A_k}, \overline{Q} - Q^* \rangle \leqslant 0 \tag{5.7}$$

for all $k = 1, \dots, K-1$. To ease notation, let $Q_j^* = \langle \chi_{A_j}, Q^* \rangle$ and $\overline{Q}_j = \langle \chi_{A_j}, \overline{Q} \rangle$. Then (5.6) implies

$$\sum_{j=1}^{L} \alpha_j (\overline{Q}_j - Q_j^*) > 0.$$

Taking the $j = 1$ term to the right and dividing by $\alpha_1 > 0$ yields

$$\sum_{j=2}^{L} \frac{\alpha_j}{\alpha_1} (\overline{Q}_j - Q_j^*) > (Q_1^* - \overline{Q}_1) \geqslant 0, \tag{5.8}$$

where the second inequality follows from (5.7) for $k = 1$. If $K > 1$, then $\alpha_2 > 0$, so $\alpha_1/\alpha_2 > 1$, and multiplying the left-hand side of (5.8) by α_1/α_2 strengthens the inequality. Then, taking the $j = 2$ term to the right yields

$$\sum_{j=3}^{L} \frac{\alpha_j}{\alpha_2} (\overline{Q}_j - Q_j^*) > (Q_1^* - \overline{Q}_1) + (Q_2^* - \overline{Q}_2) \geqslant 0,$$

where the second inequality follows from (5.7) for $k = 2$. Continue in this fashion until arriving at

$$0 \geqslant \sum_{j=K+1}^{L} \frac{\alpha_j}{\alpha_K} (\overline{Q}_j - Q_j^*) > (Q_1^* - \overline{Q}_1) + \cdots + (Q_K^* - \overline{Q}_K),$$

where the first inequality follows from $\alpha_K > 0$, $\alpha_j \leqslant 0$ for $j > K$, and $Q_j^* = 0$ for $j > K$. Thus $\langle \chi_{A_1 \cup \cdots \cup A_K}, \bar{Q} \rangle > \langle \chi_{A_1 \cup \cdots \cup A_K}, Q^* \rangle = B(A_1 \cup \cdots \cup A_K)$, completing the proof of the lemma. *Q.E.D.*

To complete the proof of Proposition 3.1, we must show that if $\bar{Q} \notin \mathscr{Q}$, then it is separated from $\mathscr{Q}$ by a simple function. This will be accomplished using a separating hyperplane argument, after establishing some topological preliminaries. The set $\mathscr{Q}$ of implementable functions is clearly convex. We want to embed it as a compact set in an appropriate linear space whose dual contains the simple functions as a dense set. Matthews (1984) embeds $\mathscr{Q}$ in the space of measures on T, endowed with the topology of weak convergence of measures. This is the wrong space for the separating hyperplane argument. Instead, treat $\mathscr{Q}$ as a subset of $L_\infty(\lambda)$, the set of λ-essentially bounded measurable functions on T. Embed the set of symmetric auctions $\mathscr{Q}_1$ in $L_\infty(\lambda^N)^N$. For brevity denote $L_p(\lambda^N)^N$ by L_p^N and $L_p(\lambda)$ by L_p.

Since L_∞ is the dual of L_1 under the duality $\langle f, g \rangle = \int_T f(t) g(t)\, d\lambda(t)$, topologize L_∞ with its weak*, or $\sigma(L_\infty, L_1)$, topology. Similarly give L_∞^N its $\sigma(L_\infty^N, L_1^N)$ topology.

Bear in mind that strictly speaking, an element of L_∞ is not a measurable function, but an equivalence class of measurable functions, where two functions are equivalent if they differ only on a set of λ-measure zero. This means that if we show that $Q \in \mathscr{Q} \subset L_\infty$, we can only conclude that there is a reduced form agreeing with Q λ-almost everywhere. To show that Q is actually implementable requires an additional argument.

Equation (2.3) defines a function Λ: $\mathscr{Q}_1 \to Q$ mapping each q to its reduced form. The next lemma describes the main topological results.

LEMMA 5.4:

(5.9) $\mathscr{Q}_1$ *is* $\sigma(L_\infty^N, L_1^N)$ *compact.*

(5.10) Λ: $\mathscr{Q}_1 \to \mathscr{Q}$ *is* $\sigma(L_\infty^N, L_1^N)$, $\sigma(L_\infty, L_1)$ *continuous.*

(5.11) $\mathscr{Q}$ *is* $\sigma(L_\infty, L_1)$ *compact.*

PROOF: Since $\mathscr{Q}_1$ is a subset of the unit ball of L_∞^N under the norm given by $\|q\| = \max_{\iota = 1, \ldots, N} \|q^\iota\|_\infty$, by the Banach–Alaoglu theorem, for (5.9) we need only prove that $\mathscr{Q}_1$ is $\sigma(L_\infty^N, L_1^N)$ closed. Let q_ν be a net in $\mathscr{Q}_1$ converging in the $\sigma(L_\infty^N, L_1^N)$ topology to q. First we verify that $q^\iota(p) = q^1(\sigma^\iota(p))$. Observe that σ^i: $T^N \to T^N$ is measurable, $\sigma^\iota = (\sigma^\iota)^{-1}$, and $\lambda^N \circ \sigma^\iota = \lambda^N$. Since each $q_\nu \in \mathscr{Q}_1$, for any $f \in L_1^N$, we have

$$\int_{T^N} f(p) q_\nu^\iota(p)\, d\lambda^N(p) = \int_{T^N} f(p) q_\nu^1(\sigma^\iota(p))\, d\lambda^N(p)$$

$$= \int_{T^N} f(\sigma^\iota(\pi)) q_\nu^1(\pi)\, d\lambda^N(\pi).$$

(The first equality follows from (2.2) and the second equality is just the transformation of variables $\pi = \sigma^\iota(p)$.) Taking limits on each side yields

$$\int_{T^N} f(p) q^\iota(p)\, d\lambda^N(p) = \int_{T^N} f(\sigma^\iota(\pi)) q^1(\pi)\, d\lambda^N(\pi).$$

Transforming variables on the right-hand side, we get

$$\int_{T^N} f(p) q^\iota(p)\, d\lambda^N(p) = \int_{T^N} f(p) q^1(\sigma^\iota(p))\, d\lambda^N(p).$$

Since f is arbitrary, conclude that $q^\iota = q^1 \circ \sigma^\iota$. Similar arguments show that $0 \leqslant \Sigma q^\iota \leqslant 1$, and so $q \in \mathscr{Q}_1$. This completes the proof of (5.9).

To check continuity of Λ, let $q_\nu \to q$ in the $\sigma(L_\infty^N, L_1^N)$ topology. Let $f \in L_1(\lambda)$ and define $\tilde{f} \in L_1(\lambda^N)$ by $\tilde{f}(t_1, \ldots, t_N) = f(t_1)$. Then

$$\langle \Lambda q_\nu, f \rangle = \int_{T^N} \tilde{f}(p) q_\nu^1(p)\, d\lambda^N(p).$$

Since $q_\nu \to q$, the right-hand side converges to $\int \tilde{f}_q\, d\lambda^N = \langle \Lambda q, f \rangle$. Since f is arbitrary, $\Lambda q_\nu \to \Lambda q$ in the $\sigma(L_\infty, L_1)$ topology, proving (5.10).

This shows that $\mathscr{Q}$ is compact, since continuous images of compact sets are compact. *Q.E.D.*

Proof of Proposition 3.1: Lemma 5.1 shows that the condition (3.1) is necessary for implementability. To see the converse, suppose $\overline{Q} \notin \mathscr{Q}$. Since $\mathscr{Q}$ is $\sigma(L_\infty, L_1)$ compact and convex, a separating hyperplane theorem implies that there exists a nonzero $f \in L_1$ satisfying $\langle f, \overline{Q} \rangle > \max\{\langle f, Q \rangle: Q \in \mathscr{Q}\}$. Since simple functions are norm dense in L_1, we may take f to be simple. By Lemma 5.3 then, $\langle \chi_A, \overline{Q} \rangle > B(A)$ for some $A \in \mathscr{T}$, violating the feasibility inequality (3.1).

If λ is regular, then the indicator function χ_A can be replaced by the indicator of either an open or closed set without disturbing the strictness of the inequality.

So far, since L_∞ identifies functions differing only on sets of measure zero, all we have proven is that if Q satisfies condition (3.1) for all $A \in \mathscr{T}$, then there is a symmetric auction q^* whose reduced form Q^* agrees with Q λ-almost everywhere. The following argument shows that we can modify q^* on a set of λ^N-measure zero to implement Q itself.

First note that each hierarchical auction $q_{\{t\}}$ is measurable, since singletons are measurable by assumption. Let $A^* = \{t: Q^*(t) \neq Q(t)\}$, the negligible set of problem points. If $t \in A^*$, then $\lambda(\{t\}) = 0$, and

$$\int_{T^{N-1}} q_{\{t\}}^1(t, t_2, \ldots, t_N)\, d\lambda^{N-1}(t_2, \ldots, t_N) = 1, \tag{5.12}$$

since $q_{\{t\}}^1 = 1$, except when t occurs in $\{t_2, \ldots, t_N\}$, which happens with probability $1 - \lambda(\{t\}^c)^{N-1} = 0$.

Now define a new function $\hat{q}^1: T^N \to [0,1]$ by

$$\hat{q}^1(t_1, \ldots, t_N) = \begin{cases} Q(t_1) \cdot q_{\{t_1\}}^1(t_1, \ldots, t_N) & \text{if } (t_1, \ldots, t_N) \in A^* \times (T \backslash A^*)^{N-1}, \\ 0 & \text{otherwise}. \end{cases} \tag{5.13}$$

This function $\hat{q}^1$ is measurable, and it follows from (5.12) that for every $t \in A^*$,

$$\int_{T^{N-1}} \hat{q}_{\{t\}}^1(t, t_2, \ldots, t_N)\, d\lambda^{N-1}(t_2, \ldots, t_N) = Q(t). \tag{5.14}$$

Next define $q^1: T^N \to [0,1]$ by

$$q^1(t_1, \ldots, t_N) = \begin{cases} q^{*1}(t_1, \ldots, t_N) & \text{if } \{t_1, \ldots, t_N\} \cap A^* = \varnothing, \\ \hat{q}^1(t_1, \ldots, t_N) & \text{if } \{t_1, \ldots, t_N\} \cap A^* \neq \varnothing. \end{cases} \tag{5.15}$$

Then, for all $t \in T$,

$$\int_{T^{N-1}} q^1(t, t_2, \ldots, t_N)\, d\lambda^{N-1}(t_2, \ldots, t_N) = Q(t). \tag{5.16}$$

To see this, first note that if $t \notin A^*$, then $q^1(t, t_2, \ldots, t_N)$ agrees with $q^{*1}(t, t_2, \ldots, t_N)$ unless some $t_2, \ldots, t_N$ belongs to A^*, a λ^{N-1}-measure zero event. Since q^* implements Q^* and Q^* agrees with Q outside A^*, (5.16) is satisfied for $t \notin A^*$. If $t \in A^*$, then equation (5.14) implies (5.16).

Finally, we show that setting $q^i = q^1 \circ \sigma^i$ yields a symmetric auction q: $T^N \to [0,1]^N$ implementing Q. Clearly, q is symmetric, and by (5.16) it implements Q. It remains to show that q satisfies the feasibility condition (2.1).

If $\{t_1, \dots, t_N\} \cap A^* = \varnothing$, then by (5.15) $\sum_{i=1}^N q^i(t_1, \dots, t_N) = \sum_{i=1}^N q^{*i}(t_1, \dots, t_N) \leqslant 1$, since q^* is a symmetric auction. If $\{t_1, \dots, t_N\} \cap A^* \neq \varnothing$, there are two cases. If more than one of $t_1, \dots, t_N$ belongs to A^*, then by (5.13) and (5.15), $q^1(t_1, \dots, t_N) = 0$, and hence each $q^i(t_1, \dots, t_N) = 0$. If exactly one of $t_1, \dots, t_N$ belongs to A^*, say $t_k \in A^*$, then again by (5.13) and (5.15), $q^k(t_1, \dots, t_N) = Q(t_k) \leqslant 1$ and for $i \neq k$, $q^i(t_1, \dots, t_N) = 0$. This shows that the modified auction q is feasible, and the proof is finally complete. *Q.E.D.*

We prove Proposition 3.2 for simple functions in the next lemma. Then a limiting argument proves the general case.

LEMMA 5.5: *Let Q: $T \to [0,1]$ be a simple function, $Q = \sum_{j=1}^K \alpha_j \chi_{A_j}$, where the A_j's are numbered so that $\alpha_1 > \alpha_2 > \cdots > \alpha_K \geqslant 0$, and the A_j's partition T. Set $E_k = \bigcup_{j=1}^k A_j$, $k = 1, \dots, K$.*

If for each $k = 1, \dots, K$,

$$\int_{E_k} Q\, d\lambda \leqslant B(E_k); \tag{5.17}$$

then Q is implementable.

PROOF: Proposition 3.1, shows we need only prove that for any measurable $A \subset T$, $\int_A Q\, d\lambda \leqslant B(A)$. Define f: $[0,1] \to [0,1]$ by $f(x) = (1-(1-x)^N)/N$. Then f is a monotone and concave, and $B(A) = f(\lambda(A))$. Define the continuous piecewise linear function g: $[0,1] \to [0,1]$ by the conditions $g(0) = 0$ and g has slope α_k on $(\lambda(E_{k-1}), \lambda(E_k))$, where we set $E_0 = \varnothing$. Since $\alpha_1 > \alpha_2 > \cdots > \alpha_K$, g is concave. Note that

$$g(\lambda(E_k)) = \sum_{j=1}^k \alpha_j \lambda(A_j) = \int_{E_k} Q\, d\lambda,$$

so by the concavity of f and (5.17), $g \leqslant f$ on $[0,1]$. Let $A \subset T$ be an arbitrary measurable set of types and define the continuous piecewise linear function h: $[0, \lambda(A)] \to [0,1]$ by the conditions $h(0) = 0$ and h has slope α_k on $(\lambda(E_{k-1} \cap A), \lambda(E_k \cap A))$. See Figure 3. Then $h \leqslant g$ on $[0, \lambda(A)]$. In particular, $h(\lambda(A)) \leqslant f(\lambda(A))$. But $h(\lambda(A))$ is by construction $\sum_{k=1}^K \alpha_k \lambda(A_k \cap A)$, which is just $\int_A Q\, d\lambda$.

Thus $\int_A Q\, d\lambda \leqslant f(\lambda(A)) = B(A)$, proving the lemma. *Q.E.D.*

PROOF OF PROPOSITION 3.2: Proposition 3.1 implies that the inequalities (3.1) are necessary, so it suffices to prove sufficiency. Let Q: $T \to [0,1]$ satisfy $\int_{E_\alpha} Q\, d\lambda \leqslant B(E_\alpha)$ for each $\alpha \in [0,1]$. Construct a sequence of simple functions Q_n converging uniformly and monotonely to Q from below, by setting $Q_n(t) = k/2^n$ on $\{t$: $(k/2^n) \leqslant Q(t) < (k+1)/2^n\}$. Then $\int_{E_{k/2^n}} Q_n\, d\lambda \leqslant \int_{E_{k/2^n}} Q\, d\lambda \leqslant B(E_{k/2^n})$, where the first inequality follows from $Q_n \leqslant Q$ and the second by hypothesis. By Lemma 5.5, Q_n is implementable. The Lebesgue dominated convergence theorem implies that Q_n converges in the $\sigma(L_\infty, L_1)$ topology to Q. By Lemma 5.4, $\mathscr{Q}$ is $\sigma(L_\infty, L_1)$-closed, so Q is implementable. *Q.E.D.*

6. CONSTRUCTING AN IMPLEMENTATION

Propositions 3.1 and 3.2 tell us when a function $\overline{Q}$ is implementable. They do not however tell us which auction implements it. A "practical" approach is to start by approximating $\overline{Q}$ from below by a simple function, $Q = \sum_{j=1}^m \alpha_j \chi_{A_j}$. (The proof of

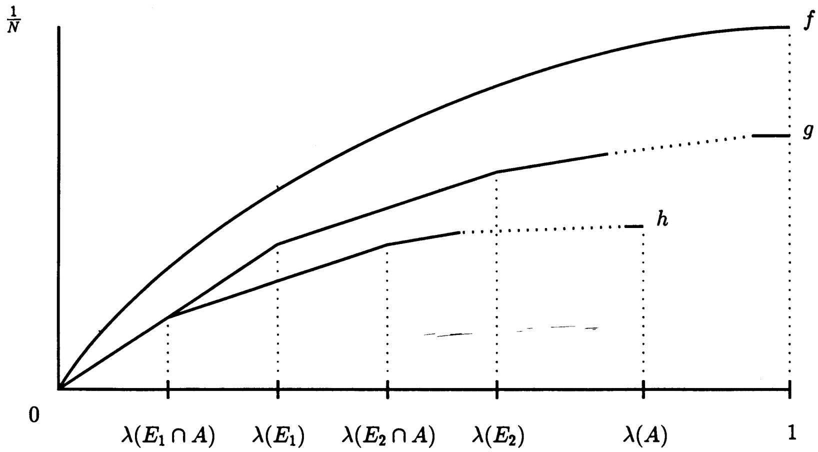

FIGURE 3.—Proof of Lemma 5.5.

Proposition 3.2 describes how to do this.) Since $Q \leqslant \overline{Q}$, Proposition 3.1 guarantees that Q is implementable.

Consider the finite type space $\hat{T} = \{1, \dots, m\}$ with probability measure $\hat{\lambda}$ defined by $\hat{\lambda}(\{j\}) = \lambda(A_j)$. Note that if T is already finite, this step is unnecessary. Let $\hat{\mathscr{Q}}$ denote the reduced forms on $\hat{T}$. Since $Q = \sum_{j=1}^{m} \alpha_j \chi_{A_j}$ is a reduced form, $(\alpha_1, \dots, \alpha_m) \in \hat{\mathscr{Q}}$. If $\hat{q}$ implements $(\alpha_1, \dots, \alpha_m)$, then $q = \sum \hat{q}_j \chi_{A_j}$ implements Q. We have thus reduced the problem of constructing an approximate implementation to the problem of constructing an implementation on a finite set of types.

Note that if T has cardinality m, but λ assigns probability zero to some types, then $L_\infty(\lambda)$ is not isomorphic to R^m, because in L_∞ we identify functions differing only on sets of measure zero. However, L_∞ is isomorphic to $R^n \subset R^m$, where n is the number of types with positive λ-measure.

We now prove a lemma about the structure of $\mathscr{Q}$ for finite sets of types. Recall that x is an *extreme point* of a convex set C if x is not a proper convex combination of two distinct points of C; that is, if there do not exist $y, z \in C$ with $y \neq z$ and $0 < \alpha < 1$ satisfying $x = \alpha y + (1-\alpha) z$. For finite T, $\mathscr{Q}$ is the convex hull of its extreme points. (In general, the Krein–Milman theorem and Lemma 5.4 tell us that $\mathscr{Q}$ is the $\sigma(L_\infty, L_1)$-closure of the convex hull of its extreme points.)

Call a hierarchical auction $q_{A_1 \cdots A_K}$ *singular* if each A_j is a singleton.

LEMMA 6.1: *Suppose the set of types T is finite. Then Q is an extreme point of $\mathscr{Q} \subset L_\infty$ if and only if Q is either the reduced form of a singular hierarchical auction or $Q = 0$.*

PROOF: Suppose first that Q is the reduced form of the singular hierarchical auction $q_{\{t_1\} \cdots \{t_K\}}$. Set $F^j = \{t_1, \dots, t_j\}$ for each $j = 1, \dots, K$. By Lemma 5.2, $\sum_{k=1}^{j} Q(t_k)\lambda(\{t_k\}) = B(F^j)$ for $j = 1, \dots, K$, and $Q(t) = 0$ for $t \notin F^K$. Suppose that $Q = \alpha Q^1 + (1-\alpha) Q^2$ where $Q^1, Q^2 \in \mathscr{Q}$, and $0 < \alpha < 1$. By Proposition 3.1, since Q^1 and Q^2 are reduced forms, $\sum_{k=1}^{j} Q^i(t_k)\lambda(t_k) \leqslant B(F^j)$ for $i = 1, 2$ and $j = 1, \dots, K$. This implies for $j = 1$, that

$$Q^1(t_1)\lambda(\{t_1\}) = Q^2(t_1)\lambda(\{t_1\}) = Q(t_1)\lambda(\{t_1\}) = B(\{t_1\}).$$

Also, considering $j = 2$,

$$\begin{aligned} Q^1(t_1)\lambda(\{t_1\}) + Q^1(t_2)\lambda(\{t_2\}) &= Q^2(t_1)\lambda(\{t_1\}) + Q^2(t_2)\lambda(\{t_2\}) \\ &= B(\{t_1, t_2\}), \end{aligned}$$

so it follows that

$$Q^1(t_2)\lambda(\{t_2\}) = Q^2(t_2)\lambda(\{t_2\}) = Q(t_2)\lambda(\{t_2\}) = B(\{t_1,t_2\}) - B(\{t_1\}).$$

Continuing in this fashion yields

$$Q^1(t)\lambda(t) = Q^2(t)\lambda(t) = Q(t)\lambda(t)$$

for all $t \in F^K$. But $Q^i \geqslant 0$, so for $t \notin F^K$,

$$Q^1(t)\lambda(t) = Q^2(t)\lambda(t) = Q(t) = 0.$$

Thus $Q^1 = Q^2 = Q$ λ-almost everywhere, i.e., $Q^1 = Q^2 = Q$ in L_∞, so Q is an extreme point of $\mathscr{Q}$.

For the converse, let $\hat{\mathscr{Q}}$ denote the convex hull of the set of singular hierarchical reduced forms and zero. Then $\hat{\mathscr{Q}}$ is compact and convex. Suppose $Q \geqslant 0$ and $Q \notin \hat{\mathscr{Q}}$. Then Q can be strictly separated from $\hat{\mathscr{Q}}$ by some vector $f \in R^m$. That is, $f \cdot Q > f \cdot Q'$ for all Q' in $\hat{\mathscr{Q}}$. Note that f defines a simple function on T and that the same argument used to prove Lemma 5.3 shows that $Q \notin \hat{\mathscr{Q}}$. This completes the proof of the lemma. *Q.E.D.*

If T has cardinality K, then there are $M = \sum_{m=1}^{K} \binom{K}{m}(m!)$ singular hierarchies. Enumerate their reduced forms, $Q^1, \ldots, Q^M$. By Lemma 6.1, since $Q \in \mathscr{Q}$, it can be written as a convex combination of 0 and $Q^1, \ldots, Q^M$. Given the representation $Q = \sum \beta_l Q^l$, if q^l is the hierarchical auction implementing Q^l, then $q = \sum \beta_l q^l$ implements Q. The problem is reduced to writing Q as a convex combination of hierarchical auctions and 0, all viewed as points in R^K. While I do not know a closed form solution to this problem,[2] the following linear program is equivalent:

$$\underset{\beta, d}{\text{minimize}} \sum_{j=1}^{K} d_j$$

[2] Chen (1986) offered a closed form construction to implement a simple reduced form. This construction is flawed, as the following example shows. Let $T = [0,1]$, λ be Lebesgue measure (the uniform distribution), and $N = 2$. Let $E_1 = [0, 1/3]$ and $E_2 = (1/3, 1]$.

Define q to be the auction generated by the hierarchy $E_1 E_2$. That is,

$$q^1(t_1,t_2) = \begin{cases} 1 & \text{if } t_1 \in E_1 \text{ and } t_2 \in E_2, \\ 1/2 & \text{if } t_1, t_2 \in E_1 \text{ or } t_1, t_2 \in E_2, \\ 0 & \text{if } t_1 \in E_2 \text{ and } t_2 \in E_1. \end{cases}$$

Its reduced form Q is given by

$$Q(t) = \begin{cases} 5/6 & \text{if } t \in E_1, \\ 1/3 & \text{if } t \in E_2. \end{cases}$$

By construction, Q is implementable and so satisfies (3.1). Using Chen's notation, we set $E_j^* = E_1$ and $G = E_2$. Then his construction yields

$$q^1(t_1,t_2) = \begin{cases} 1/2 & \text{if } t_1 \in E_1 \text{ and } t_2 \in E_2, \\ 3/2 & \text{if } t_1, t_2 \in E_1, \\ 1/5 & \text{if } t_1, t_2 \in E_2, \\ 3/5 & \text{if } t_1 \in E_2 \text{ and } t_2 \in E_1. \end{cases}$$

This is clearly not feasible. The error in the proof occurs in Chen's equation (6) where the computation $\int_{A_j} Q\, d\lambda = \alpha_j$ is made. The correct computation is $\int_{A_j} Q\, d\lambda = \alpha_j \lambda(A_j)$.

subject to

$$\sum_{l=1}^{M} \beta_l Q^l + d = Q,$$

$$\sum_{l=1}^{M} \beta_l \leqslant 1,$$

$$d \geqq 0,$$

$$\beta \geqq 0.$$

A feasible initial point is $\beta = 0 \in R^M$, $d = Q \in R^K$. Each optimum satisfies $d = 0$, and $Q = \sum_{i=1}^{M} \beta_l Q^l$. Furthermore, many algorithms, including the simplex algorithm, compute a basic solution, that is, a solution in which all but at most $(K+1)$ β_l's are zero.[3] The practicality of this method is limited, as M grows very rapidly with K.

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[3] I am indebted to Joel Franklin for this point.