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## FULL EXTRACTION OF THE SURPLUS IN BAYESIAN AND DOMINANT STRATEGY AUCTIONS

BY JACQUES CRÉMER AND RICHARD P. MCLEAN<sup>1</sup>

We consider auctions for a single indivisible object, in the case where the bidders have information about each other which is not available to the seller. We show that the seller can use this information to his own benefit, and we completely characterize the environments in which a well chosen auction gives him the same expected payoff as that obtainable were he able to sell the object with full information about each bidder's willingness to pay. We provide this characterization for auctions in which the bidders have dominant strategies, and for those where the relevant equilibrium concept is Bayesian Nash. In both set-ups, the existence of these auctions hinges on the possibility of constructing lotteries with the correct properties.

**KEYWORDS:** Auctions, optimal auctions, information structures, dominant strategy equilibrium, Bayesian Nash equilibrium.

### 1. INTRODUCTION

WE CONSIDER the situation in which an agent, the seller, possesses one indivisible unit of a good to which he attaches no value. But the good has value to a number of potential buyers, and its transfer to one of them would increase social welfare. In particular, the transfer to the buyer with the highest valuation maximizes social welfare. In this paper, we completely characterize environments in which the seller can design an auction that will enable him to capture for himself the full increase in social welfare induced by the transfer of the good to the bidder with the highest willingness to pay.

If the seller had full information about the reservation prices of potential buyers, his optimal selling strategy would be very simple. He would announce a price equal or very close to the highest reservation value. The optimal strategy for the bidder with the highest evaluation would be to accept the offer. (Note that we are treating a situation in which the seller can commit himself to a price.) As a result of the exchange, the utility of the seller increases by the full amount of the increase in social welfare, and he has been able to fully extract the surplus.

In many circumstances, however, a seller has only imperfect knowledge of the buyers' willingnesses to pay. In this case, he must find some mechanism, or auction, which will enable him to maximize his benefit from the sale of the object. The auction literature starts with this observation and shows how the seller can, by an astute choice of auction, extract the largest possible fraction of the surplus. In general, the literature has shown that this proportion is strictly less than one.

In some circumstances, the bidders will have information about each other which is not available to the seller. For instance, in auctions for petroleum drilling rights, bidders know the results of geological tests which they have

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conducted. The results of the tests of the various bidders are correlated, and therefore they have *a priori* some information about each other's willingness to pay. In Crémer and McLean (1985), we pointed out that the auctioneer could use his knowledge of the fact that the bidders have information about each other to his own advantage. In particular, we produced sufficient conditions under which the auctioneer is able to extract the full surplus.

In the present paper, we completely characterize the information structures which will guarantee that the seller can fully extract the surplus, i.e. that he can do as well as he could with full information. We do this under two alternative assumptions. First, we will study auctions which admit a dominant strategy for the bidders. Then we will turn our attention to auctions in which each bidder submits his bid in ignorance of the bids of others. In this case, the relevant equilibrium concept is Bayesian-Nash.

Not surprisingly, the requirements for full extraction of the surplus are stricter in the case of dominant strategy auctions than in the case of Bayesian auctions. To the best of our knowledge, we provide the first examples of environments where Bayesian and dominant strategy auctions yield different payoffs when the bidders are risk neutral.

In both set-ups, the possibility of extracting the full surplus hinges on the construction of lotteries with specific properties. These lotteries consist of payments by a bidder to the seller, conditional on the announcements of other bidders. We will always consider auctions in which the buyers reveal their types to the seller. In equilibrium, these announcements will reveal the bidders' true characteristics so we can consider these lotteries to be conditional on the actual types of other bidders.

A dominant strategy auction must be, more or less, a Vickrey auction to which is appended for each bidder a payment (possibly negative) which is a function only of the bids of others.<sup>2</sup> In a Vickrey auction, a bidder never pays more for an object than it is worth to him. Hence, whatever his type, he derives a nonnegative gain in utility, measurable in monetary units, from participation in this Vickrey auction. Ex ante, his type is known to the bidder. He will participate in the auction if and only if the expected gain in utility from the Vickrey auction is greater than or equal to the expected cost to him of the lottery, where the expectation is computed according to the probability distribution of the valuations of others, conditional on his own type. The auctioneer will extract the full surplus if these expected values are equal for all possible types of every bidder. Our first theorem characterizes those information structures for which such lotteries exist.

Hence, to extract the surplus with a dominant strategy auction, the seller must construct one lottery per buyer whose outcomes are conditional on the an-

<sup>2</sup> This statement would be exactly true if the set of possible valuations of the object by the bidders were a continuum. Then the auction is a public good problem where the public decision space is the subset  $\{p | \sum_{i=1}^n p_i \leq 1\}$  of  $\mathbf{R}_+^n$ , where  $p_i$  is interpreted as the probability that agent  $i$  obtains the object. Then the characterization results of Green and Laffont (1977) and Holmström (1979) carry over. The Vickrey auction corresponds to the pivotal mechanism.

nounced valuations of the others. We emphasize that the lottery for bidder  $i$  is independent of  $i$ 's type. The problem to be solved is easier in the case of Bayesian-Nash auctions. There, the seller can construct one lottery for each type of each buyer. Each lottery has an expected value of zero if the bidder is of the type by which the lottery is indexed. Otherwise, this expected value is negative and can be made as negative as we wish. This assumes, of course, that the probability distribution of the types of other bidders varies enough when one's own type changes. Then, the auction proceeds as follows: each bidder announces his type and the winner is made to pay an amount equal to his announced valuation of the object. If we stopped here, the bidder would, in general, lie. However, we can add to this the lottery corresponding to his type. Then, the expected value of lying will become negative, and all the surplus can be extracted. Theorem 2 characterizes information structures for which this is possible.

More precise interpretations are provided after the statement of each theorem in Section 2. An example is discussed in detail in Appendix A, and the reader may wish to examine it along with the discussion of Section 2. Our main results are presented in terms of discrete probability spaces. In Appendix B, we discuss the extension to the case of distributions with infinite support. Finally, Section 3 contains some concluding remarks.

The results answer a question originally posed by Myerson (1981). Partial answers have been provided by Makin and Riley (1980) and ourselves (1985). Conditions similar to those of our Theorem 2 have independently been used by Riordan and Sappington (1985). The work of d'Aspremont, Crémer, and Gérard-Varet (1987) suggests that these conditions might eventually play an important role in the theory of mechanism design.

## 2. THE MODEL

Throughout this paper, the bidders are indexed by the set  $N = \{1, 2, \dots, n\}$ . The "characteristic" of bidder  $i$  takes values in a set  $M_i = \{1, \dots, m_i\}$ . We will call  $M$  the set  $M_1 \times \dots \times M_n$  and  $M_{-i}$  the set  $M_1 \times M_2 \times \dots \times M_{i-1} \times M_{i+1} \times \dots \times M_n$ . To each characteristic  $s_i \in M_i$ , there corresponds a willingness to pay  $w^i(s_i)$  for agent  $i$ . The function  $w^i: M_i \rightarrow \mathbb{R}_+$  will be called an individual valuation function for agent  $i$ . In the sequel,  $M_i$  will be fixed but  $w^i$  will vary. Let  $w = (w^1, \dots, w^n)$ . The function  $w: M \rightarrow \mathbb{R}_+^n$  will be called a valuation function. All bidders are assumed to be risk neutral. If agent  $i$  with characteristic  $s_i$  makes an expected payment of  $x_i$  when the probability that he obtains the object is  $p_i$ , his expected payoff is  $p_i w^i(s_i) - x_i$ . As we have shown in Crémer and McLean (1985), the theory can be expanded to a more complex setting where, in particular, an agent's willingness to pay is a function not only of his own characteristic but also of the (for him, unobserved) characteristics of others. In this framework, Theorems 1 and 2 would still hold in a fundamentally unaltered form. Taking this into account would strengthen our statements of sufficiency (our conditions allow full extraction of the surplus in a wider class of auctions), but weaken our statements of necessity.

Before the auction begins, the seller has a probability distribution  $\pi$  over the elements of  $M$ , which are the states of nature for our problem. Agent  $i$  knows  $s_i$ , and we assume that his subjective probability distribution over  $M_i$  given  $s_i$  is consistent with  $\pi$ , i.e. that it is  $\pi(s_{-i}|s_i)$ , the same distribution that the seller would have were he able to observe  $s_i$ . Without loss of generality, we can assume that the marginal probability  $\pi(s_i) = \sum_{s_{-i} \in M_{-i}} \pi(s_{-i}, s_i)$  is positive. (There is some abuse of notation in using the same symbol  $\pi$  for probability distributions over different spaces, but this will lead to no confusion.)

We call a pair  $(M, \pi)$  an *information structure*. A combination  $(M, \pi, w)$  of an information structure and a valuation function defines an *auction problem*. Our task is to find necessary and sufficient conditions on information structures  $(M, \pi)$  to ensure that, for any associated problem  $(M, \pi, w)$ , the seller can find an auction that will extract the full surplus.

We invoke the revelation principle and limit ourselves to auctions that induce the bidders to truthfully reveal their characteristics. Thus, an auction is conducted in the following way: the auctioneer asks each bidder  $i$  to submit an element  $s_i$  of  $M_i$ . If  $s = (s_1, \dots, s_n) \in M$  is the vector of announced characteristics, each bidder pays an amount  $x_i(s)$  to participate in a lottery in which he wins the object with probability  $p_i(s)$ . Formally we have the following definition:

**DEFINITION:** An auction is a collection  $\{p_i, x_i\}_{i \in N}$  where  $x_i: M \rightarrow \mathbb{R}$  and  $p_i: M \rightarrow \mathbb{R}$  such that  $p_i(s) \geq 0$  for all  $i$  and  $s$  and  $\sum_{i \in N} p_i(s) \leq 1$  for all  $s$ .

If  $(M, \pi, w)$  is an auction problem and  $\{p_i, x_i\}_{i \in N}$  an auction, the utility of agent  $i$  when he announces  $t_i \in M_i$  and the other agents have announced  $s_{-i} \in M_{-i}$  is  $p_i(s_{-i}, t_i)w^i(s_i) - x_i(s_{-i}, t_i)$ . The auction  $\{p_i, x_i\}_{i \in N}$  is *individually rational* for the problem  $(M, \pi, w)$  if for each  $i \in N$ :

$$\sum_{s_{-i} \in M_{-i}} \pi(s_{-i}|s_i) [p_i(s_{-i}, s_i)w^i(s_i) - x_i(s_{-i}, s_i)] \geq 0, \quad \forall s_i \in M_i.$$

The left-hand side of the inequality is the expected payoff to agent  $i$  from participating in the auction when he is of type  $s_i$ , given that all bidders truthfully reveal their characteristics. (Our results would hold, under the appropriate definitions, if the right-hand side were replaced by  $c^i(s_i)$ , i.e. if there were a cost to participating in the auction dependent perhaps on the type of the agent.)

We will consider two types of auctions. An auction which satisfies the individual rationality constraint is a *Bayesian auction* for the problem  $(M, \pi, w)$  if it satisfies the Bayesian incentive compatibility constraints:

$$\begin{aligned} & \sum_{s_{-i} \in M_{-i}} \pi(s_{-i}|s_i) [p_i(s_{-i}, s_i)w^i(s_i) - x_i(s_{-i}, s_i)] \\ & \geq \sum_{s_{-i} \in M_{-i}} \pi(s_{-i}|s_i) [p_i(s_{-i}, t_i)w^i(s_i) - x_i(s_{-i}, t_i)] \end{aligned}$$

for all  $i \in N$ , and all  $s_i, t_i \in M_i$ .

An auction which satisfies the individual rationality constraint is a *dominant strategy auction* for the problem  $(M, \pi, w)$  if it satisfies the dominant strategy incentive compatibility constraints:

$$p_i(s_{-i}, s_i)w^i(s_i) - x_i(s_{-i}, s_i) \geq p_i(s_{-i}, t_i)w^i(s_i) - x_i(s_{-i}, t_i)$$

for all  $i \in N$ , all  $s_i, t_i \in M_i$ , and all  $s_{-i} \in M_{-i}$ .

For any auction  $\{p_i, x_i\}_{i \in N}$ , the seller's payoff (i.e. expected revenue) is  $\sum_{s \in M} \pi(s)[\sum_{i \in N} x_i(s)]$ . An auction *extracts the full surplus for the problem*  $(M, \pi, w)$ , if the payoff to the seller is equal to  $\sum_{s \in M} \{\pi(s)[\max_{i \in N} w^i(s)]\}$ . We will say that an information structure  $(M, \pi)$  guarantees full extraction of the surplus by a Bayesian (respectively dominant strategy) auction if, for any problem  $(M, \pi, w)$ , there exists a Bayesian (respectively dominant strategy) auction that extracts the full surplus.

The game associated with an optimal auction may have several equilibria, at least one of which corresponds to full extraction of the surplus. These equilibria are focal, because they ask bidders to reveal their true valuations. (For the finite type case, techniques similar to those of Maskin and Riley (1980) may allow the construction of auctions with only one equilibrium, the "good one." Whether this extends to the case of infinitely many types is an open question.)

The following lemma (whose proof is straightforward) is essential for our analysis.

**LEMMA 1:** *If  $\{p_i, x_i\}_{i \in N}$  is an auction which satisfies the individual rationality constraints for the problem  $(M, \pi, w)$ , it extracts the full surplus if and only if: (a) the individual rationality constraints hold as equalities for all  $i$  and  $s_i$  and (b) whenever  $\pi(s) > 0$ ,  $\sum_{i \in N} p_i(s)$  is equal to 1 if  $\max_{i \in N} [w^i(s_i)] > 0$  and  $p_j(s)$  is equal to 0 if  $w^j(s_j) < \max_{i \in N} [w^i(s_i)]$ .*

We can now turn to the first of our characterization theorems.

**THEOREM 1:** *An information structure  $(M, \pi)$  guarantees full extraction of the surplus by a dominant strategy auction if and only if for all  $i \in N$ , there do not exist  $\{\rho_i(s_i)\}_{s_i \in M_i}$ , not all equal to zero, such that:*

$$\sum_{s_i \in M_i} \rho_i(s_i) \pi(s_{-i}|s_i) = 0 \quad \text{for all } s_{-i} \in M_{-i}.$$

This is the condition which we introduced in Crémer and McLean (1985). It states that for any  $i$ , the matrix  $\Gamma_i$  whose rows are indexed by the elements of  $M_i$ , whose columns are indexed by the elements of  $M_{-i}$ , and whose generic element is  $\pi(s_{-i}|s_i)$ , is of rank  $m_i$ . The form of the condition used in Theorem 1 is intended to make comparison with Theorem 2 easier.

*Interpretation of Theorem 1*

This condition on the information structure is a spanning condition. It enables the auctioneer to construct a lottery of the type discussed in the introduction: for any  $i$ , its expected value to bidder  $i$ , whatever his type  $s_i$ , is equal to  $(-1)$  times the expected surplus,  $h_i(s_i)$ , from participation in a Vickrey auction arbitrarily chosen by the seller. A lottery for bidder  $i$  is a function  $L_i$  from  $M_{-i}$  into  $\mathbb{R}$ , which assigns a payment by  $i$  to each  $(n-1)$ -tuple of announcements by the other bidders. Hence, we need to build a lottery  $L_i$  whose expected value for each type  $s_i$  of  $i$ ,  $V_i(L_i|s_i) = \sum_{s_{-i}} \pi(s_{-i}|s_i) L_i(s_{-i})$ , is equal to  $-h_i(s_i)$ . But if the condition of Theorem 1 does not hold, we have  $\sum_i \rho_i(s_i) V_i(L_i|s_i) = 0$ , and hence this lottery can be constructed only for Vickrey auctions such that  $\sum_i \rho_i(s_i) h_i(s_i)$  is equal to zero, and therefore full extraction of the surplus is not guaranteed.

PROOF OF THE IF PART OF THEOREM 1: Let  $\{p_i^*, x_i^*\}_{i \in N}$  be a Vickrey, or second price, auction (there could be several depending on the manner in which the object is allocated when several bidders have the same willingness to pay). Let  $h_i(s_i)$  be equal to  $\sum_{s_{-i} \in M_{-i}} \pi(s_{-i}|s_i) [p_i^*(s) w^i(s_i) - x_i^*(s)]$ . Thus, the nonnegative number  $h_i(s_i)$  is the true expected benefit to agent  $i$  from participating in the auction when he is of type  $s_i$ .

Because the matrix  $\Gamma_i$  is of rank  $m_i$ , there exists a family  $\{g_i(s_{-i})\}_{s_{-i} \in M_{-i}}$  such that  $\sum_{s_{-i} \in M_{-i}} \pi(s_{-i}|s_i) g_i(s_{-i}) = h_i(s_i)$ . Let  $x'_i(s) = x_i^*(s) + g_i(s_{-i})$ . It is easy to check that  $\{p_i^*, x'_i\}_{i \in N}$  is a dominant strategy auction and, by its definition, that  $\sum_{s_{-i}} \pi(s_{-i}|s_i) [p_i^*(s) w^i(s_i) - x'_i(s)] = 0$  for all  $i$  and all  $s_i$ . It follows from Lemma 1 that  $\{p_i^*, x'_i\}_{i \in N}$  extracts the full surplus.

PROOF OF THE ONLY IF PART OF THEOREM 1: Assume that the information structure  $(M, \pi)$  guarantees full extraction of the surplus by a dominant strategy auction. Choose any  $i \in N$ , and let  $w^j(s_j) = 0$  for all  $j \neq i$  and all  $s_j \in M_j$  and  $w^i(s_i) > 0$  for all  $s_i \in M_i$ . Let  $\{p_i, x_i\}_{i \in N}$  be a dominant strategy auction that extracts the full surplus. According to Lemma 1, it must satisfy:

- (1)  $p_i(s) = 1$  for all  $s \in M$  such that  $\pi(s) > 0$ ,
- (2)  $w^i(s_i) = \sum_{s_{-i} \in M_{-i}} \pi(s_{-i}|s_i) x_i(s_{-i}, s_i)$  for all  $s_i \in M_i$ .

By incentive compatibility, there must exist a family  $\{h_i(s_{-i})\}_{s_{-i} \in M_{-i}}$  such that:

- (3)  $\pi(s_{-i}|s_i) x_i(s_{-i}, s_i) = \pi(s_{-i}|s_i) h_i(s_{-i})$ .

Combining equations (2) and (3) we obtain:  $\sum_{s_{-i} \in M_{-i}} \pi(s_{-i}|s_i) h_i(s_{-i}) = w^i(s_i)$ . This must be true for all families of positive  $w^i(s_i)$ . Hence  $\mathbb{R}_{++}^{m_i}$  is a subset of the image of  $\Gamma_i$ . Because  $\mathbb{R}_{++}^{m_i}$  contains a basis for  $\mathbb{R}^{m_i}$ , we conclude that  $\Gamma_i$  must be of rank  $m_i$  and the result is proved.

**THEOREM 2:** *An information structure  $(M, \pi)$  guarantees full extraction of the surplus by a Bayesian auction if and only if for all  $i \in N$ , there does not exist*

$s_i \in M_i$  and a family  $\{\rho_i(t_i)\}_{t_i \in M_i \setminus s_i}$  such that<sup>3</sup>

$$(a) \quad \rho_i(t_i) \geq 0 \quad \text{for all } t_i \in M_i \setminus s_i$$

and

$$(b) \quad \pi(s_{-i}|s_i) = \sum_{t_i \neq s_i} \rho_i(t_i) \pi(s_{-i}|t_i), \quad \text{for all } s_{-i} \in M_{-i}.$$

### Interpretation of Theorem 2

The condition of Theorem 2 states that the probability distributions on  $M_{-i}$  associated with each type in  $M_i$  are not too positively correlated. (This is strictly a property of conditional distributions: it does not imply that when the utility of one agent increases, the utility of others also increases.) Hence, there is enough leeway to build the lotteries, discussed in the introduction, whose expected values are zero conditional on one's true type, and negative conditional on other types. Indeed, suppose that (a) and (b) in Theorem 2 do not hold. Then for some  $s_i$ , we have  $\pi(s_{-i}|s_i) = \sum_{t_i \neq s_i} \rho(t_i) \pi(s_{-i}|t_i)$  with  $\rho(t_i) \geq 0$  for all  $t_i \in M_i \setminus s_i$ . Then, for any lottery whose expected value is zero conditional on  $s_i$ , it cannot be true that for all  $t_i \neq s_i$  the lottery has a negative expected value when conditioned on  $t_i$ .

**PROOF OF THE IF PART OF THEOREM 2:** Assume that for any  $i$  and for any  $s_i \in M_i$ , there is no family  $\{\rho_i(t_i)\}_{t_i \in M_i \setminus s_i}$  which satisfies conditions (a) and (b) of the theorem. By Farkas' Lemma (see, for instance, Mangasarian (1969)), there exists, for each  $i$ , a family  $\{g_i(s)\}_{s \in M}$  such that  $\sum_{s_{-i}} \pi(s_{-i}|s_i) g_i(s_{-i}, s_i)$  is positive and  $\sum_{s_{-i}} \pi(s_{-i}|t_i) g_i(s_{-i}, s_i)$  is nonpositive for all  $t_i \neq s_i$ . Let  $\varepsilon$  be equal to  $\sum_{s_{-i}} \pi(s_{-i}|s_i) g_i(s_{-i}, s_i)$  and let  $h_i(s)$  be  $g_i(s) - \varepsilon$ . We obtain:

$$\sum_{s_{-i}} \pi(s_{-i}|s_i) h_i(s_{-i}, s_i) = 0$$

and

$$\sum_{s_{-i}} \pi(s_{-i}|t_i) h_i(s_{-i}, s_i) < 0 \quad \text{for all } t_i \neq s_i.$$

Let  $\{p_i\}_{i \in N}$  satisfy the conditions of Lemma 1 and, for all  $i \in N$  and for all  $s \in M$ , let  $x_i(s) = \sum_{t_{-i} \in M_{-i}} \pi(t_{-i}|s_i) p_i(t_{-i}, s_i) w^i(s_i) - \gamma_i(s_i) h_i(s)$  for some collection  $\{\gamma_i(s_i)\}_{s_i \in M_i}$ . The auction  $\{p_i, x_i\}_{i \in N}$  satisfies the condition of Lemma 1, and hence extracts the full surplus. Furthermore, it is easy to check that the definition of  $x_i$  implies that  $\sum_{s_{-i}} \pi(s_{-i}|s_i) [p_i(s_{-i}, t_i) w^i(s_i) - x_i(s_{-i}, t_i)]$  is equal to zero if  $t_i$  is equal to  $s_i$  and otherwise can be made arbitrarily small by choosing  $\gamma_i(s_i)$  sufficiently large. The incentive compatibility constraint is then met and the result is proved.

<sup>3</sup> This is very close in spirit to condition B in d'Aspremont and Gérard-Varet (1982). See d'Aspremont et al. (1987) for more details.



PROOF OF THE ONLY IF PART OF THEOREM 2: We assume that for some  $i \in N$ , there exist an  $s_i \in M_i$  and a family  $\{\rho_i(t_i)\}_{t_i \in M_i \setminus s_i}$  such that (a)  $\rho_i(t_i) \geq 0$  for all  $t_i \in M_i \setminus s_i$  and (b)  $\pi(s_{-i}|s_i) = \sum_{t_i \neq s_i} \rho_i(t_i) \pi(s_{-i}|t_i)$ ,  $\forall s_{-i} \in M_{-i}$ .

We now choose  $w$  with the following properties:  $w^j(s_j) < w^i(s_i) < w^i(t_i)$  for all  $j \neq i$ , all  $s_j \in M_j$  and all  $t_i \in M_i \setminus s_i$ . We will assume that a Bayesian auction  $\{p_i, x_i\}_{i \in N}$  exists which fully extracts the surplus for the problem  $\{M, \pi, w\}$  and show that this leads to a contradiction. We would have, by Lemma 1:

$$(4) \quad p_i(s) = 1 \quad \text{for all } s \in M \text{ such that } \pi(s) > 0,$$

$$(5) \quad \sum_{s_{-i}} \pi(s_{-i}|t_i) [w^i(t_i) - x_i(s_{-i}, t_i)] = 0 \quad \text{for all } t_i \in M_i.$$

By Bayesian incentive compatibility we also obtain:

$$(6) \quad \sum_{s_{-i}} \pi(s_{-i}|t_i) [w^i(t_i) - x_i(s_{-i}, s_i)] \leq 0 \quad \text{for all } t_i \neq s_i.$$

Because  $w^i(s_i)$  is less than  $w^i(t_i)$  for all  $t_i$ , and  $\pi(s_{-i}|t_i)$  is positive for at least one  $s_{-i} \in M_{-i}$ , we have:

$$(7) \quad \sum_{s_{-i}} \pi(s_{-i}|t_i) [w^i(s_i) - x_i(s_{-i}, s_i)] < 0 \quad \text{for all } t_i \neq s_i.$$

Multiplying both sides of (7) by  $\rho_i(t_i)$  and summing over all  $t_i$  in  $M_i \setminus s_i$  we obtain:

$$(8) \quad \sum_{s_{-i}} \pi(s_{-i}|s_i) [w^i(s_i) - x_i(s_{-i}, s_i)] < 0$$

which contradicts Lemma 1, and the result is proved.

### 3. CONCLUDING REMARKS

The conditions of Theorems 1 and 2 will be met by “nearly all” information structures. In “nearly all” auctions, the seller should be able to extract the full surplus, which implies that asymmetry of information between buyers and sellers should be of no practical importance. Economic intuition and informal evidence (we know of no way to test such a proposition) suggest that this result is counterfactual, and several explanations can be suggested.

First, the assumption that a common knowledge probability distribution  $\pi$  exists is very strong. Though economic theorists have found this assumption convenient because it makes strategic problems with incomplete information analytically tractable, little discussion has been devoted to its ramifications for “real life” problems of mechanism design. Presumably, the seller in an auction would have to invest in costly research to determine  $\pi$  prior to computing the optimal auction. This costly information gathering, not explicitly modeled in auction problems, may result in less profitable but vastly simpler auctions being used in practice. Furthermore the seller would have to be able to share the information that he has gathered with the buyers in a credible way, in order to ensure that  $\pi$  is indeed common and this raises a host of difficult issues.

A second difficulty is linked to the fact that the penalties associated with lying (the  $g_i(s_{-i})$  in the proof of Theorem 1) may have to be quite large. Introducing risk aversion in our analysis would modify the results in directions that will be a topic for future work, but note that the buyers in many auctions are firms for which the assumption of risk neutrality is appropriate. The same issue would arise if we introduced limited liability.

Finally, we should stress that in our opinion the independence assumption should be used only with great caution when deriving optimal auctions, at least in the case of finitely many types. It does enable the derivation of results that on the surface look more “realistic” (there is no full extraction of the surplus). However, the derivation of these results rely on a very “unrealistic” assumption. Furthermore, the results of this paper show that a small deviation from this assumption can induce fundamentally different results.

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## APPENDIX A

In this appendix we briefly present an example to show how the theory developed in Section 2 applies. When there are two bidders and  $m_1$  and  $m_2$  are both equal to three, the conditions of Theorems 1 and 2 are equivalent. Hence, we consider an environment with two bidders such that  $m_1 = m_2 = 4$ : the characteristics  $s_1$  and  $s_2$  can each take four possible values denoted respectively  $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  and  $(\beta_1, \beta_2, \beta_3, \beta_4)$ . The probabilities of the different states of the world can most easily be represented in matrix form:

	bidder 2			
bidder 1	$\beta_1$	$\beta_2$	$\beta_3$	$\beta_4$
$\alpha_1$	.01	.04	.04	.02
$\alpha_2$	.08	.01	.08	.04
$\alpha_3$	.16	.02	.04	.16
$\alpha_4$	.03	.12	.03	.12

For instance,  $\pi(\alpha_1, \beta_3)$ , the probability that  $v_1$  is equal to  $\alpha_1$  and  $v_2$  to  $\beta_3$ , is equal to .04. It is easy to check that this information structure satisfies the conditions of Theorem 2, but not those of Theorem 1 because 12 times the first row plus 3 times the third is equal to 6 times the second plus 4 times the fourth.

A vector  $L = (y_1, y_2, y_3, y_4)$  in  $\mathbb{R}^4$  is interpreted as a lottery for, let us say, bidder 1 if we think of player 1 paying  $y_i$  when bidder 2 announces  $\beta_i$ . Consider the following four lotteries for player 1:  $L(\alpha_1) = (-1000, 100, 100, 100)$ ,  $L(\alpha_2) = (100, -2000, 100, 100)$ ,  $L(\alpha_3) = (100, -800, -400, 100)$ , and  $L(\alpha_4) = (-2000, 500, -2000, 500)$ . If bidder 2 is truthful, then for  $i = 1, \dots, 4$ , the expected value of  $L(\alpha_i)$  to bidder 1, conditioned on the event that  $s_i = \hat{\alpha}_i$ , is zero if  $\alpha_i$  is equal to  $\hat{\alpha}_i$  and negative otherwise. Assume now that bidder 1 thinks that bidder 2 will tell the truth, and offer him the following choices. If he announces  $\alpha_i$ , he will get the object if  $w_1(\alpha_i)$  is greater than  $w_2(\beta)$ , where  $\beta$  is the bid of player 2. He will have to pay  $w_1(\alpha_i)$  times the probability that  $w_2(\beta)$  is smaller than  $w_1(\alpha_i)$  and be forced to participate in the lottery  $L(\alpha_i)$  multiplied by some constant. If the constant is large enough the expected surplus of the bidder will be zero if he announces the truth; and negative

otherwise. The same construction can be made for bidder 2 with the following lotteries:  $L(\beta_1) = (-3000, 500, 500, -3000)$ ,  $L(\beta_2) = (100, 2000, 200, 100)$ ,  $L(\beta_3) = (-1000, 1000, -1000, 0)$ ,  $L(\beta_4) = (-3000, -2000, 500, 500)$ . Hence, there exists a recipe for building a Bayesian-Nash auction which extracts the full surplus.

We now show that full extraction of the surplus by a dominant strategy auction cannot be guaranteed with this information structure. First, assume that the valuation functions satisfy the following condition:

$$0 < w^1(\alpha_1) < w^1(\alpha_2) < w^2(\beta_i) < w^1(\alpha_3) < 3w^1(\alpha_3) \leq w^1(\alpha_4).$$

Let  $\{p_i, x_i\}$  be a dominant strategy auction that extracts the full surplus. By Lemma 1, for all  $\beta_i$ ,  $p_1(\alpha_3, \beta_i)$  and  $p_1(\alpha_4, \beta_i)$  are equal to 1, while  $p_1(\alpha_1, \beta_i)$  and  $p_1(\alpha_2, \beta_i)$  are equal to 0. Substitution in the incentive compatibility constraints shows that the following equations must hold for all  $\beta_i$ :  $x_1(\alpha_1, \beta_i) = x_1(\alpha_2, \beta_i)$ ,  $x_1(\alpha_3, \beta_i) = x_1(\alpha_4, \beta_i)$ ,  $w^1(\alpha_2) - x_1(\alpha_3, \beta_i) \leq -x_1(\alpha_2, \beta_i)$ , and  $w^1(\alpha_3) - x_1(\alpha_3, \beta_i) \geq -x_1(\alpha_1, \beta_i)$ . Using these equations, the values of  $p_i$  derived above, and the fact that, by Lemma 1, the individual rationality constraints hold with equality we obtain:  $20w^1(\alpha_4) \leq 4w^1(\alpha_3)$ , which contradicts the hypothesis that  $3w^1(\alpha_3)$  is less than  $w^1(\alpha_4)$ .

## APPENDIX B

In this appendix, we outline an approach to the full extraction problem when bidders' types can be drawn from sets that are not necessarily finite.

To keep matters technically simple, we only deal with dominant strategy auctions. (See the remarks at the end.) The player set is again denoted  $N = \{1, \dots, n\}$ . For each  $i \in N$ , the set of characteristics of bidder  $i$  is  $M_i = [0, 1]$ . The symbols  $M, M_{-i}, s_i \in M_i, s_{-i} \in M_{-i}$  retain the interpretations given in the text. The probabilistic structure on  $M$  is specified by a distribution function  $F: M \rightarrow \mathbb{R}$ . Let  $F_i, F_{-i}$  and  $F_i(\cdot | s_i)$  denote, respectively, the marginal distribution of  $F$  on  $M_i$ , the marginal distribution of  $F$  on  $M_{-i}$  and the conditional distribution of  $F$  on  $M_{-i}$  given  $s_i \in M_i$ . A property  $Q$  is said to hold  $F$ -a.e. if  $\int_C dF = 0$  where  $C$  is the subset of  $M$  for which property  $Q$  does not hold. The term " $F_i$ -a.e." has an analogous definition. Let  $L_1(F)$  denote the set  $\{f: M \rightarrow \mathbb{R} | \int_M |f| dF < \infty\}$ .  $L_1(F_i)$  has an analogous definition. A valuation function  $w^i$  for  $i$  is a nonnegative element of  $L_1(F_i)$ . An information structure  $(M, F)$  and an auction problem  $(M, F, w)$  are defined as in the text. An auction is a collection  $\{p_i, x_i\}_{i \in N}$  where  $p_i: M \rightarrow \mathbb{R}_+$  and  $x_i: M \rightarrow \mathbb{R}$  are elements of  $L_1(F)$  and  $\sum_i p_i(s)$  is smaller than or equal to 1 for all  $s \in M$ . The auction  $\{p_i, x_i\}$  is individually rational for  $(M, F, w)$  if for each  $i \in N$  the following inequality holds:

$$\int_{M_{-i}} [p_i(s_{-i}, s_i) w^i(s_i) - x_i(s_{-i}, s_i)] dF_i(s_{-i} | s_i) \geq 0$$

for  $F_i$ -almost all  $s_i \in M_i$ .

An auction  $\{p_i, x_i\}$  extracts the full surplus for  $(M, F, w)$  if  $\int_M [\sum_{i \in N} x_i(s)] dF = \int_M \max_{i \in N} [w^i(s_i)] dF$ . The following analog of Lemma 1 holds:

**LEMMA 1A:** *Let  $\{p_i, x_i\}_{i \in N}$  be an individually rational auction for  $(M, F, w)$ . This auction extracts the full surplus if and only if (a) the individual rationality constraints hold with equality for all  $i$  and for  $F_i$ -almost all  $s_i \in M_i$  and (b) for  $F$ -almost all  $s$ ,  $\sum_{i \in N} p_i(s) = 1$  if  $\max_j \{w^j(s_j)\} > 0$  and  $p_i(s) = 0$  if  $w^i(s_i) < \max_j \{w^j(s_j)\}$ .*

An individually rational auction is a dominant strategy auction if for all  $i \in N$  and for  $F_i$ -almost all  $s_i \in M_i$ :

$$p_i(s_{-i}, s_i) w^i(s_i) - x_i(s_{-i}, s_i) \geq p_i(s_{-i}, t_i) w^i(s_i) - x_i(s_{-i}, t_i),$$

for all  $s_{-i} \in M_{-i}$  and for all  $t_i \in M_i$ .

**THEOREM 1A:** *An information structure  $(M, F)$  guarantees full extraction of the surplus by a dominant strategy auction if and only if for each  $i \in N$  and for every  $v_i \in L_1(F_i)$  that is  $F_i$ -almost*

everywhere nonnegative, there exists a function  $g_i : M_{-i} \rightarrow \mathbb{R}$  such that

$$v_i(s_i) = \int_{M_{-i}} g_i(s_{-i}) dF_i(s_{-i}|s_i), \quad \text{for all } s_i \in M_i.$$

Several remarks are in order. First, the finite type case treated in the paper is a “special case” of Theorem 1A. In particular, suppose  $F$  has finite support  $M_0$  and let  $\tilde{M}_i = \{s_i \in M_i | (s_{-i}, s_i) \in M_0 \text{ for some } s_{-i}\}$ . Each  $s \in M_0$  is an atom of the probability measure induced on  $M$  by  $F$ , i.e., there is a probability measure  $\pi_0$  on  $M_0$  that assigns positive probability to each  $s \in M_0$ . This  $\pi_0$  can be extended to a probability measure  $\tilde{\pi}$  on  $\tilde{M}_1 \times \cdots \times \tilde{M}_n = \tilde{M}$  as follows:  $\tilde{\pi}(s) = \pi_0(s)$  if  $s \in M_0$  and  $\tilde{\pi}(s) = 0$  if  $s \in \tilde{M} \setminus M_0$ . Note that  $\tilde{\pi}_i(s_i)$ , the marginal density of  $\tilde{\pi}$  on  $\tilde{M}_i$ , is greater than zero for all  $s_i \in \tilde{M}_i$ . Finally, the integral of Theorem 1A reduces to  $\sum_{s_{-i} \in \tilde{M}_{-i}} g_i(s_{-i}) \pi(s_{-i}|s_i)$ . Thus, Theorem 1A holds if and only if Theorem 1 holds when the latter is applied to the finite auction problem  $(\tilde{M}, \tilde{\pi})$ .

One is naturally led to investigate whether or not an infinite dimensional version of Theorem 2 holds as well. The situation is not quite as straightforward as the dominant strategy case since Farkas Lemma in the infinite dimensional setting requires extra topological assumptions that are automatically satisfied in the finite dimensional framework (see, for example, Theorem III.4 in Hurwicz (1958)). This technical, though important, problem is a topic for further research.

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