Game Theory 2 - End Term Examination

November, 2011; Duration: 3 hours; Total marks: [50]

Note: Notations and terminologies are as discussed in class.

1. Let A be a set of alternatives and $\mathcal{P}$ be the set of all reflexive, complete, and transitive preference orderings over A. A **domain** is any subset $\mathcal{D} \subseteq \mathcal{P}$ of preference orderings. We will say a domain is **connected** if for every pair of alternatives $a, b \in A$, there is a preference ordering $P_i \in \mathcal{D}$, where $P_i(1) = a$ and $P_i(2) = b$ (i.e., a is the top and b is second ranked).

   - Verify (with arguments) whether the following domains are connected: (a) single-peaked (b) house allocation. (**4 marks**).

     Answer: (a) In single-peaked domain, any alternative can be a peak. if the alternative to the extreme left is the peak, then the alternative to the extreme right cannot be second ranked. So, this domain is not connected.

     (b) On the other hand, the house allocation domain is not connected. Consider two alternatives (assignments) a and b where an agent i gets the same house. Then agent i is always indifferent between a and b. As a result, if a is top ranked then b cannot be second ranked.

   - Let $N = \{1, \ldots, n\}$ be the set of agents. A social choice function $f : \mathcal{D}^n \to A$ is **monotone** if for any two profiles $P, P' \in \mathcal{D}$ with $f(P) = a$ and for all $b \neq a$, we have for all $i \in N$, aP'_ib if aP_ib, then $f(P') = a$. Suppose $\mathcal{D}$ is connected. Show that if f is monotone then it is strategy-proof. (**8 marks**)

     Answer: Suppose f is monotone and consider two profiles $P \equiv (P_i, P_{-i})$ and $P' \equiv (P'_i, P_{-i})$. Suppose $f(P) = a$ and $f(P') = b$. Assume for contradiction that bP_ia. Consider a preference ordering P''_i such that $P''_i(1) = b$ and $P''_i(2) = a$. By assumption for any alternative $c \in A \setminus \{a\}$, if aP_ic then aP''_ic and if bP_ic then bP''_ic. By monotonicity, from profile P, we have $f(P'') = a$, and from profile P'', $f(P'') = b$. This is a contradiction.

2. Consider the house allocation model. Let f be a social choice function defined as follows. At any preference profile $\succ \equiv (\succ_1, \ldots, \succ_n)$, agent 1 is given his best object according to his preference ordering $\succ_1$, then agent 2 is given the second best object according to agent 1's preference ordering $\succ_1$, then agent 3 is given the third best object according to agent 1's preference ordering $\succ_1$, and so on.

   - Is f strategy-proof? Is f efficient? (argue or give counter examples). (**6 marks**).

     Answer: f is strategy-proof since agents other than agent 1 cannot change the outcome and agent 1 always gets his most preferred object. But f is not efficient.

Consider three agents $\{1, 2, 3\}$ and three objects $\{a, b, c\}$ with preferences as: $a \succ_1 b \succ_1 c$, $c \succ_2 a \succ_2 b$, and $b \succ_3 c \succ_3 a$. According to f, the allocation is: agent 1 gets a, agent 2 gets b, and agent 3 gets c. But the allocation: agent 1 gets a, agent 2 gets c, and agent 3 gets b improves utilities of agents 2 and 3, and keeps utility of agent 1 the same.

3. Consider the house allocation model with three agents $N = \{1, 2, 3\}$ and three objects $M = \{a, b, c\}$. Let f be a social choice function defined as follows. At any preference profile $\succ \equiv (\succ_1, \ldots, \succ_n)$, if $\succ_2 (1) = a$, then agent 1 gets the best element in $\{b, c\}$ according to his preference ordering $\succ_1$, agent 2 gets a, and agent 3 gets the remaining object (i.e., a serial dictatorship with the highest priority to agent 2, followed by agent 1, and finally to agent 3). In all other cases, agent 1 gets the best object in M, agent 2 gets the best remaining object according to $\succ_2$, and agent 3 gets the remaining object (i.e., a serial dictatorship with the highest priority to agent 1, followed by agent 2, and finally to agent 3).

   - Is f strategy-proof? Is f non-bossy, i.e., can an agent change the outcome at a profile without changing the object assigned to him? (**10 marks**)

     Answer: f is strategy-proof. Agents 1 and 3 cannot change the priority. So, they have no incentive to manipulate. Agent 2 can change the priority. But he will not manipulate if he gets the top priority. When agent 2 gets the second priority, he can change the priority by saying that his top is a, and in this case he gets a. But a is not his top according to his true preference. So he gets an object which is at least his second preferred object. But that he could have got even if he did not change the priority. So, he does not gain by manipulation.

     f is also non-bossy. Note that if the serial dictatorship with a given priority is non-bossy. So, if an agent does not change his own allocation in f, it does not change the priority in f. So, by the same reasoning, it is non-bossy - other agents will continue to choose the best from same set of available objects to them.

4. Consider the marriage market model (with n men and n women) where every man m ranks women as:

$$w_1 \succ_m w_2 \succ_m \ldots \succ_m w_n$$

   and every woman w ranks men as:

$$m_1 \succ_w m_2 \succ_w \ldots \succ_w m_n.$$

   What are the outcomes of the men-proposing and the women-proposing versions of the deferred acceptance algorithm at this profile? (**6 marks**)

Answer: If men propose, then all of them propose to w_1, who rejects all except m_1. Then, all but m_1 propose to w_2, who rejects all except m_2, and so on. So, the outcome is m_i is matched to w_i for all $i \in \{1, \ldots, n\}$. The same outcome is achieved if women propose.

5. Consider the model of private good allocation of a single divisible good to n agents with each agent having single-peaked preference. Let f be a strategy-proof, efficient, and anonymous allocation rule.

   - Show that if the sum of the peaks of the agents is greater than 1, then an agent who gets a share strictly less than his peak gets more than $\frac{1}{n}$ share in f, i.e., if for some profile $\succ \equiv (\succ_1, \ldots, \succ_n)$ we have $\sum_{i \in N} p_i(\succ_i) > 1$ and $f_j(\succ) < p_j(\succ_j)$ for some agent j, then $f_j(\succ) \geq \frac{1}{n}$. (**8 marks**)

     Answer: We know that the only strategy-proof, efficient, and anonymous allocation rule is the uniform allocation rule. In the uniform allocation rule, if the sum of the peaks is greater than one, then we fill the bucket. So, an agent who gets strictly less than his peak must be the last agent is the filling process to get allocated, and hence his share must be (weakly) larger than the share of every other agent. Clearly, the agent who gets the largest share cannot get strictly less than $\frac{1}{n}$ - if this happens then the total share is less than 1, which is not possible.

6. Consider a model with three alternatives $A = \{a, b, c\}$ and three agents $N = \{1, 2, 3\}$. Let f be a randomized social choice function, i.e., $f : \mathcal{P}^3 \to \mathcal{L}(A)$, where $\mathcal{P}$ is the set of all linear orders over A and $\mathcal{L}(A)$ is the set of all lotteries over A. The output of f for the two profiles $P \equiv (P_1, P_2, P_3)$ and $P' \equiv (P'_1, P'_2, P'_3)$ shown in Table 1 are:

$$f_a(P) = \frac{5}{6}, f_b(P) = \frac{1}{6}, f_c(P) = 0$$
$$f_a(P') = 0, f_b(P') = \frac{2}{3}, f_c(P') = \frac{1}{3}.$$

Is there a strategy-proof and unanimous randomized social choice function $f' : \mathcal{P}^3 \to \mathcal{L}(A)$ such that $f'(P) = f(P)$ and $f'(P') = f(P')$? Explain your answer. (**8 marks**)

P_1	P_2	P_3	P'_1	P'_2	P'_3
a	a	b	c	b	b
b	c	c	a	a	c
c	b	a	b	c	a

Table 1: Preference Profiles

Answer: We know that a strategy-proof and unanimous RSCF must be a random dictatorship. Suppose it is a random dictatorship with weights $\beta_1, \beta_2, \beta_3$. Note that by setting $\beta_1 = \frac{1}{3}, \beta_2 = \frac{1}{2}$, and $\beta_3 = \frac{1}{6}$ we get a random dictatorship whose outcome coincides with f at profiles P and P'.