Game Theory 2 - Final Examination Solution

On: November 16, 2009; Duration: 3 hours

1. Show that the efficient allocation rule satisfies cycle monotonicity (without using Groves mechanism arguments). (**6 marks**)

   Answer: Fix agent i and the type profile of other agents at t_{-i}. Consider a cycle in the type graph of agent i at t_{-i}: $(t^1, t^2, \ldots, t^k, t^{k+1})$, where $t^{k+1} = t^1$. Now, let $f(t^h, t_{-i}) = a_h$ for all $1 \leq h \leq k$. Then, we have

$$
\begin{aligned}
& l_{t_{-i}}(t^1, t^2) + l_{t_{-i}}(t^2, t^3) + \ldots + l_{t_{-i}}(t^k, t^{k+1}) \\
&= [v_i(f(t^2, t_{-i}), t^2) - v_i(f(t^1, t_{-i}), t^2)] + [v_i(f(t^3, t_{-i}), t^3) - v_i(f(t^2, t_{-i}), t^3)] + \ldots \\
&+ [v_i(f(t^{k+1}, t_{-i}), t^{k+1}) - v_i(f(t^k, t_{-i}), t^{k+1})] \\
&= [v_i(a_2, t^2) - v_i(a_1, t^2)] + [v_i(a_3, t^3) - v_i(a_2, t^3)] + \ldots + [v_i(a_{k+1}, t^{k+1}) - v_i(a_k, t^{k+1})] \\
&= [v_i(a_2, t^2) + \sum_{h \neq i} v_h(a_2, t_h) - v_i(a_1, t^2) - \sum_{h \neq i} v_h(a_1, t_h)] \\
&+ [v_i(a_3, t^3) + \sum_{h \neq i} v_h(a_3, t_h) - v_i(a_2, t^3)] - \sum_{h \neq i} v_h(a_2, t_h)] + \ldots \\
&+ [v_i(a_{k+1}, t^{k+1}) + \sum_{h \neq i} v_h(a_{k+1}, t_h) - \sum_{h \neq i} v_h(a_k, t_h) - v_i(a_k, t^{k+1})] \\
&\geq 0,
\end{aligned}
$$

   where the inequality comes due to efficiency.

2. Consider a seller faced with a single buyer. The type set of the buyer is $T = \{0, 1, 2\}$. The set of allocations is $A = \{a, b, c\}$. The valuation function of the buyer is: $v(a, s) = s$ for all $s \in T$; $v(b, s) = \frac{s}{2}$ for all $s \in T$ and $v(c, s) = 1$ for all $s \in T$. Consider the following allocation rule f: $f(0) = c$, $f(1) = b$, and $f(2) = a$.

   - Draw the **type graph** of this allocation rule. (**4 marks**)

     Answer: The nodes in the type graph are $T = \{0, 1, 2\}$. The lengths of six edges are as follows:

$$\begin{aligned}
l(0,1) &= v(f(1),1) - v(f(0),1) = v(b,1) - v(c,1) = \frac{1}{2} - 1 = -\frac{1}{2} \\
l(1,0) &= v(f(0),0) - v(f(1),0) = v(c,0) - v(b,0) = 1 - 0 = 1 \\
l(0,2) &= v(f(2),2) - v(f(0),2) = v(a,2) - v(c,2) = 2 - 1 = 1 \\
l(2,0) &= v(f(0),0) - v(f(2),0) = v(c,0) - v(a,0) = 1 - 0 = 1 \\
l(1,2) &= v(f(2),2) - v(f(1),2) = v(a,2) - v(b,2) = 2 - 1 = 1 \\
l(2,1) &= v(f(1),1) - v(f(2),1) = v(b,1) - v(a,1) = \frac{1}{2} - 1 = -\frac{1}{2}.
\end{aligned}$$

- Use the type graph to conclude whether this allocation rule is DSIC. (**2 marks**)

  Answer: It is clear that none of the cycles in the type graph has negative length. Hence, this allocation rule DSIC.

3. What is the revelation principle (for dominant strategy incentive compatible mechanisms)? (**2 marks**)

   Answer: If a mechanism (M, g) implements a social choice function (f, p) in dominant strategies then the direct mechanism (f, p) is strategy-proof.

4. What is the class of Groves mechanisms? What is the Vickrey-Clarke-Groves (VCG) mechanism (pivotal mechanism)? (**4 marks**)

   Answer: Let f^e be the efficient allocation rule. The allocation rule in the Groves class of mechanisms is f^e. The payment of agent i when the type profile is (t_i, t_{-i}) is given by

   $$p_i(t_i, t_{-i}) = h_i(t_{-i}) - \sum_{j \neq i} v_j(f^e(t_i, t_{-i}), t_j),$$

   where $h_i : T_{-i} \to \mathbb{R}$.

   The VCG (pivotal) mechanism is characterized by the following h_i function for every i and every $t_{-i} \in T_{-i}$:

   $$h_i(t_{-i}) = \max_{a \in A} \sum_{j \neq i} v_j(a, t_j).$$

5. Compute the allocation and payment in the Vickrey-Clarke-Groves (VCG) mechanism (pivotal mechanism) for valuations in Tables 1 and 2. In both tables, valuations are for a combinatorial auction with two objects $\{a, b\}$. In Table 1, there are two bidders and in Table 2, there are three bidders. The value of a bidder is zero if he does not get

any object. (**4+4 marks**)

Answer: In both the tables, bidder 1 wins object a and bidder 2 wins object b. In Table 1, externality of bidder 1 is zero and that of bidder 2 is 3 (13 - 10). Hence, both of them pay zero. In Table 2, externality of bidder 1 is $11 - 10 = 1$ and that of bidder 2 is $14 - 10 = 4$. So, bidder 1 pays 1 and bidder 2 pays 4.

Notice that the two bidders in Table 1 continue to exist in Table 2. Does the extra bidder help them in Table 2? (**2 marks**)

Answer: Clearly not since the allocation is the same but they pay more in Table 2 than in Table 1.

	$\{a\}$	$\{b\}$	$\{a,b\}$
$v_1(\cdot)$	10	13	13
$v_2(\cdot)$	5	10	10

Table 1: Combinatorial auction with two bidders

	$\{a\}$	$\{b\}$	$\{a,b\}$
$v_1(\cdot)$	10	13	13
$v_2(\cdot)$	5	10	10
$v_3(\cdot)$	1	1	1

Table 2: Combinatorial auction with three bidders

6. Table 3 shows the preference ordering of three men and three women in a marriage market model. Use the Deferred Acceptance Algorithm to compute a men-optimal and a women-optimal stable matching. (**6 marks**)

$\succ_{m_1}$	$\succ_{m_2}$	$\succ_{m_3}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$
w_2	w_1	w_1	m_1	m_3	m_1
w_3	w_2	w_3	m_3	m_1	m_2
w_1	w_3	w_2	m_2	m_2	m_3

Table 3: Preference orderings of men and women

Answer: The men-proposing version goes as follows:

- $m_1 \to w_2$, $m_2 \to w_1$, and $m_3 \to w_1$.
- (w_1, m_3) and (w_2, m_1).
- $m_2 \to w_2$.

- (w_2, m_1).
- $m_2 \to w_3$.
- (w_3, m_2).

So, men-optimal stable matching by DAA is $\mu_M(m_1) = w_2, \mu_M(m_2) = w_3, \mu_M(m_3) = w_1$.

The women-proposing version goes as follows:

- $w_1 \to m_1$, $w_2 \to m_3$, and $w_3 \to m_1$.
- (m_3, w_2), (m_1, w_3).
- $w_1 \to m_3$.
- (m_3, w_1).
- $w_2 \to m_1$.
- (m_1, w_2).
- $w_3 \to m_2$.
- (m_2, w_3).

So, women-optimal stable matching by DAA is $\mu_W(m_1) = w_2, \mu_W(m_2) = w_3, \mu_W(m_3) = w_1$. Note that $\mu_M = \mu_W$.

Remaining questions are concerning single object sale in the private independent value set up. Set of agents is $N = \{1, \dots, n\}$. Type set of agent $i \in N$ is $X_i = [0, h_i]$. Agent i uses probability distribution (cdf) F_i (density function f_i) to draw his type/value from X_i. All distributions are independent. In a mechanism (a, p), the expected utility of agent i when he has type x_i and reports z_i is $\alpha_i(z_i)x_i - \pi_i(z_i)$, where α and π denote the expected allocation rule and the expected payment rule respectively. In a mechanism (a, p), the utility of an agent when he has type x_i and everyone reports (z_i, x_{-i}) is $a_i(z_i, x_{-i})x_i - p_i(z_i, x_{-i})$.

7. Consider a setting with two buyers whose values are distributed uniformly in the intervals $X_1 = [0, 15]$ (buyer 1) and $X_2 = [0, 18]$ (buyer 2).

   - Describe the optimal auction. (**4 marks**)

     Answer: The virtual valuations of two buyers are given by $v_1(x_1) = 2x_1 - 15$ and $v_2(x_2) = 2x_2 - 18$. The reserve prices of buyer 1 and 2 are: $r_1 = 7.5, r_2 = 9$. The object is given to a buyer in $\arg\max_{i \in \{1,2\}} \{v_i(x_i) : x_i \geq r_i\}$. The payment of buyer i is zero if he does not win the object and is given by

     $$q_i(x_{-i}) = \inf\{x_i : v_i(x_i) \geq 0, v_i(x_i) \geq v_j(x_j) \text{ for } j \neq i\}$$

if he wins the object.

- Describe the allocation and payment in the optimal auction when $x_= x_2 = 10$ (both buyers have value 10) and when $x_1 = x_2 = 8$. (**2 marks**)

  Answer: If $x_1 = x_2 = 10$, then $v_1(x_1) = 5$ and $v_2(x_2) = 2$. Hence, bidder 1 wins and pays an amount equal to 8.5 (since $2 \times 8.5 - 15 = v_2(x_2)$). If $x_1 = x_2 = 8$, bidder 2 has value less than his reserve price. So, bidder 1 wins and pays an amount equal to his reserve price which is 7.5.

- Illustrate using valuations for this example that the optimal auction can be inefficient due to reserve price and due to asymmetric buyers. (**2 marks**)

  Answer: Let $\epsilon > 0$ be arbitrarily close to zero. Let $x_1 = 10$ and $x_2 = 10 + \epsilon$. We see that $v_1(x_1) = 5$ and $v_2(x_2) = 2 + 2\epsilon$. Hence, bidder 1 wins, which is inefficient. This is inefficiency due to asymmetric bidders. Now, let $x_1 = 8$ and $x_2 = 8 + \epsilon$. Bidder 2 does not qualify because his value is less than the reserve price. Hence, the allocation is inefficient due to reserve price.

- Describe the optimal auction if $X_1 = [0, 15]$ and $X_2 = [0, 15]$. (**2 marks**)

  Answer: This is the symmetric case. The reserve price of both the bidders is 7.5. Hence, the optimal auction is the second-price Vickrey auction with a reserve price of 7.5.

8. Consider the following allocation rules for allocating a single indivisible object to a set of buyers. For each of them state and argue if it is dominant strategy incentive compatible.

- Object goes to the lowest bidder (lowest in value). (**2 marks**)

  Answer: As the lowest bidder increases his value, he no longer remains the lowest bidder and loses his object. This means the allocation rule is not non-decreasing. Hence, it is not DSIC.

- Object goes to the bidder with the highest value if **ALL** values are greater than r (a positive number), else it goes to a fixed bidder (say, bidder 1). (**2 marks**)

  Answer: This allocation rule is not non-decreasing for bidder 1. Consider a type profile where everyone except bidder 1 has value $> r$, whereas bidder 1 has value $< r$. So, bidder 1 is winning the object in this case. By increasing his value $> r$ but keeping it less than the highest value, bidder 1 loses the object. Hence, the allocation rule is not non-decreasing, and not DSIC.

- Object goes to the highest and the second highest bidder with equal probability. (**2 marks**)

  Answer: This allocation rule is clearly non-decreasing, and hence DSIC.