GAME THEORY 2 - END TERM EXAMINATION SOLUTION

November 11, 2010; Duration: 3 hours; Total marks: 60.

1. Suppose there are three agents (countries) who need to share a perfectly divisible good (oil) among themselves. The total amount of oil is 1. The preference of each agent for quantity of oil is single-peaked.

   (a) Suppose the peaks of the three agents are: $p_1 = 0.2$, $p_2 = 0.3$, $p_3 = 0.8$. Use a strategy-proof, anonymous, and efficient social choice function to compute the allocation of oil. **5 marks**

   Answer: We know that there is a unique strategy-proof, anonymous, and efficient social choice function, and it is the uniform rule social choice function. Here, the peaks add up to > 1, hence, we use the "bucket-filling" version of the uniform rule. This gives us: agents 1 and 2 get their peaks (0.2 and 0.3 respectively), and agent 3 gets the remaining (0.5).

   (b) Suppose the peaks of agents i and j satisfy $p_i < p_j$. Show that in any allocation of a strategy-proof, anonymous, and efficient social choice function, the share of agent j is greater than or equal to the share of agent i. **5 marks**

   Answer: If sum of peaks is 1, then $f_i(\succ) = p_i < p_j = f_j(\succ)$.

   If sum of peaks is greater than 1, then we fill the buckets. In that case, either the peaks of i and j are not reached, in which case $f_i(\succ) = f_j(\succ)$, or the peak of one of the agents is reached, in which case agent i's peak must be reached. But, in the latter case, agent j's water level must be higher than agent i's peak.

   If sum of peaks is less than 1, then we empty the buckets. Again, if the peaks of agents i and j are not reached, then $f_i(\succ) = f_j(\succ)$. If peak of one of the agents is reached, then agent j's peak is reached first. In that case, agent i's water level must be lower than agent j's.

2. Suppose the set of alternatives is $A = \{a, b, c, d\}$. You are given a linear order $<$ over A as: $b < a < c < d$.

   (a) Write down all the single peaked preference orderings with respect to $<$. **5 marks**

   Answer: The single-peaked preferences with respect to $<$ are shown in Table 1.

   (b) Suppose there are three agents: $N = \{1, 2, 3\}$ and the mechanism designer uses a strategy-proof, unanimous, and anonymous social choice function f with phantom peaks at b and a. Construct a single-peaked preference profile $\succ = (\succ_1, \succ_2, \succ_3)$ such that $\succ_1 \neq \succ_2 \neq \succ_3$, and $f(\succ) = c$. **5 marks**

   Answer: We know that the median voter social choice function is the unique strategy-proof, unanimous, and anonymous social choice function. If the phantom

b	d	c	c	c	a	a	a
a	c	d	a	a	b	c	c
c	a	a	d	b	c	b	d
d	b	b	b	d	d	d	b

Table 1: Single-peaked preference orderings

peaks are at b and a, to ensure median is at c, we need to put all the peaks of agents at or after c with at least one peak at c. The profile in Table 2 ensures this (there are other profiles also which does the work).

$\succ_1$	$\succ_2$	$\succ_3$
c	c	d
d	a	c
a	b	a
b	d	b

Table 2: Single-peaked preference profile

3. Let $A = \{a, b\}$ be the set of alternatives and N be the set of agents. The possible preference orderings of an agent is the set of all linear orders over A.

   - Describe a social choice function which is strategy-proof, anonymous, and unanimous (explain your answer). **5 marks**

     Answer: Define the following social choice function f: $f(P) = a$ if $s(a) = |\{i \in N : aP_ib\}| \geq s(b) = |\{i \in N : bP_ia\}|$ and $f(P) = b$ otherwise. This is clearly anonymous since it does not distinguish between agents. It is unanimous since if everyone has a particular alternative as top ranked, then the $s(\cdot)$ of that alternative is maximum, and it is chosen.

     To verify strategy-proofness, suppose $f(P) = a$. Then, agent i will manipulate if $P_i(1) = b$. In that case, by stating a as his top-ranked alternative, f still chooses a. Similarly, if $f(P) = b$, agent i will manipulate if $P_i(1) = a$, and by stating his peak to be b, f still selects b. So, no manipulation is possible.

   - Call a social choice function f **positively responsive** if for two profiles P and P' with $f(P) = a$ and P' such that $\{i \in N : aP_ib\} \subseteq \{i \in N : aP'_ib\}$, implies $f(P') = a$. Show that if f is strategy-proof, then it is positively responsive. **5 marks**

     Answer: You can directly apply monotonicity here. Note here that P and P' are two profiles where conditions for monotonicity hold. Hence, $f(P') = a$ if f is monotone. We know that if f is strategy-proof then it is monotone.

You can also construct intermediate profiles where only one agent's type changes from P_i to P_i' and show that in each of these profiles a must be chosen - this will follow from strategy-proofness.

4. Let A be a finite set of alternatives and $\mathcal{L}(A)$ be the set of all probability distributions over A. Consider a random social choice function $f : \mathcal{P}^n \rightarrow \mathcal{L}(A)$, where $\mathcal{P}$ is the set of all linear orders over A. Fix an agent i and the preference profile of other agents at P_{-i}. Consider two preference orderings of agent i (see Table 3): P_i and P_i'. Assume that

$$\{a \in A : aP_ix\} = \{a \in A : aP_i'y\}.$$

P_i	P_i'
$\cdot$	$\cdot$
$\cdot$	$\cdot$
x	y
y	x
$\cdot$	$\cdot$
$\cdot$	$\cdot$

Table 3: Preference orderings

If f is strategy-proof, then show the following:

(a) $f_x(P_i, P_{-i}) + f_y(P_i, P_{-i}) = f_x(P_i', P_{-i}) + f_y(P_i', P_{-i})$. **5 marks**

Answer: Denote $X = \{a \in A : aP_ix\} = \{a \in A : aP_i'y\}$. We need to apply the first-order stochastic dominance (FSD) condition multiple times. First, apply it at y in P_i:

$$f_x(P_i, P_{-i}) + f_y(P_i, P_{-i}) + \sum_{a \in X} f_a(P_i, P_{-i}) \geq f_x(P_i', P_{-i}) + f_y(P_i', P_{-i}) + \sum_{a \in X} f_a(P_i', P_{-i}).$$

Next, apply it at x in P_i':

$$f_x(P_i', P_{-i}) + f_y(P_i', P_{-i}) + \sum_{a \in X} f_a(P_i', P_{-i}) \geq f_x(P_i, P_{-i}) + f_y(P_i, P_{-i}) + \sum_{a \in X} f_a(P_i, P_{-i}).$$

This gives us

$$f_x(P_i, P_{-i}) + f_y(P_i, P_{-i}) + \sum_{a \in X} f_a(P_i, P_{-i}) = f_x(P_i', P_{-i}) + f_y(P_i', P_{-i}) + \sum_{a \in X} f_a(P_i', P_{-i}). \tag{1}$$

If X is empty, then we are done. Suppose X is not empty, then define b to be the lowest ranked alternative in X under P_i and c to be the lowest ranked alternative in X under P'_i. Note that $X = \{b\} \cup \{a \in X : aP_i b\} = \{c\} \cup \{a \in X : aP'_i c\}$. We now apply FSD condition on b at P_i and on c at P'_i.

$$\begin{aligned}\sum_{a \in X} f_a(P_i, P_{-i}) &\geq \sum_{a \in X} f_a(P'_i, P_{-i}) \\ \sum_{a \in X} f_a(P'_i, P_{-i}) &\geq \sum_{a \in X} f_a(P_i, P_{-i}).\end{aligned}$$

This gives us

$$\sum_{a \in X} f_a(P'_i, P_{-i}) = \sum_{a \in X} f_a(P_i, P_{-i}). \tag{2}$$

Equations 1 and 2 gives us

$$f_x(P_i, P_{-i}) + f_y(P_i, P_{-i}) = f_x(P'_i, P_{-i}) + f_y(P'_i, P_{-i}).$$

(b) $f_x(P_i, P_{-i}) \geq f_x(P'_i, P_{-i})$. **5 marks**

Answer: We apply FSD condition on x at P_i

$$f_x(P_i, P_{-i}) + \sum_{a \in X} f_a(P_i, P_{-i}) \geq f_x(P'_i, P_{-i}) + \sum_{a \in X} f_a(P'_i, P_{-i}).$$

Using Equation 2, we conclude that $f_x(P_i, P_{-i}) \geq f_x(P'_i, P_{-i})$.

5. There are four houses $\{a_1, a_2, a_3, a_4\}$. Four agents $\{1, 2, 3, 4\}$ are tenants of these houses with agent i owning house a_i. The preference ordering of agents over houses are shown in Table 4.

- Consider the matching a^*: $a^*(1) = a_3$, $a^*(2) = a_4$, $a^*(3) = a_1$, and $a^*(4) = a_2$. Is a^* in the core? Explain your answer. **5 marks**

  Answer: The matching a^* is not in the core. Agents 1 and 2 are endowed with houses $\{a_1, a_2\}$ and by assigning a_2 to agent 1 and a_1 to agent 2, both agents get houses which they prefer than the houses they are matched to in a^*. Hence $\{1, 2\}$ will block matching a^*.

- Find a core matching of this house allocation problem. **5 marks**

  Answer: The unique core matching can be found using the top trading cycle mechanism. One can verify that the unique core matching is $\hat{a}$, where $\hat{a}(1) = a_2, \hat{a}(2) = a_1, \hat{a}(3) = a_4, \hat{a}(4) = a_3$.

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$
a_2	a_1	a_4	a_1
a_3	a_2	a_3	a_3
a_1	a_3	a_2	a_2
a_4	a_4	a_1	a_4

Table 4: Preference orderings over houses

6. Let $M = \{a_1, \ldots, a_n\}$ be the set of houses and $N = \{1, \ldots, n\}$ be the set of agents. Consider the fixed priority social choice function f^σ associated with the priority $\sigma : N \to N$, where $\sigma(i) = i$ for all $i \in N$ (i.e., agent 1 first, agent 2 second, and so on).

   We want to *simulate* the outcome of f^σ using the deferred acceptance algorithm, i.e., the outcome of the deferred acceptance algorithm is the same as the outcome of f^σ at every preference profile. We endow each house $j \in M$ with an arbitrary linear order $\succ_j$ over agents in N. What should be $(\succ_1, \ldots, \succ_j)$ such that the *house-proposing* version of the deferred acceptance algorithm simulates f^σ (explain)? **5 marks**

   Answer: Consider the following preference ordering of houses over agents: for every house $j \in M$, $\succ_j$ is the linear order induced by σ, i.e., $i \succ_j k$ if and only if $\sigma(i) < \sigma(k)$. With such preference ordering of houses over agents, all houses will propose to agent $\sigma(1)$ first in the deferred acceptance algorithm, and $\sigma(1)$ will choose his top ranked house, and reject everything else. Then, the rejected houses will propose to $\sigma(2)$, who will choose his top ranked house, and reject everything else, and so on. This clearly simulates f^σ.

7. Show that the deferred acceptance algorithm returns a stable matching. **5 marks**

   Answer: Consider the men-proposing version of the deferred acceptance algorithm. Suppose the deferred acceptance algorithm is not stable. Then for some profile $\succ$, some pair (m, w) blocks the matching μ produced by the deferred acceptance algorithm. This means, $w \succ_m \mu(m)$. Since $\mu(m) \neq w$ and $w \succ_m \mu(m)$, m must have proposed to w before proposing to $\mu(m)$. But w must have rejected m. This means $\mu^{-1}(w) \succ_w m$. This is a contradiction since the pair (m, w) blocks μ. A similar argument works for the women-proposing version of the deferred acceptance algorithm.