Theory of Mechanism Design - Midterm Examination

September, 2015; Duration: 3 hours; Total marks: 45.

Explain your answers clearly, but avoid unnecessary elaboration.

You can use all the results proved in class.

1. Let A be a finite set of alternatives. Suppose $\succ$ is a strict linear order over A. Call a pair $a, b \in A$ **consistent** in a strict linear order P if aPb if and only if $a \succ b$, i.e., the ranking of a and b are the same in P and $\succ$. Else, we call $a, b \in A$ inconsistent in P.

   Construct the following domain $\mathcal{D}$ using $\succ$ as follows. First, consider P^1, which is the *inverse* of the strict linear order $\succ$ (in other words, every pair of alternatives is inconsistent in P^1). We add P^1 to $\mathcal{D}$. Having added, $(P^1, \ldots, P^{k-1})$, we choose P^k as follows. We look for $a, b \in A$ such that (i) a and b are inconsistent in P^{k-1}, (ii) a and b are consecutively ranked in P^{k-1}, and (iii) for any $\{x, y\} \neq \{a, b\}$ with x and y inconsistent and consecutively ranked in P^{k-1}, if $aP^{k-1}b$ and $xP^{k-1}y$, then $aP^{k-1}x$. In other words, a and b are the best ranked consecutively ranked and inconsistent pair of alternatives in P^{k-1}. If no such pair a, b exists, then our domain is $\mathcal{D} = \{P^1, \ldots, P^{k-1}\}$.

   Else, we swap a and b in P^{k-1} to form P^k, i.e., the rank of a in P^k is the rank of b in P^{k-1}, the rank of b in P^k is the rank of a in P^{k-1}, and the rank of all other alternatives remain the same. We then add $P^k \in \mathcal{D}$ and repeat the procedure.

   As an example, suppose $A = \{a, b, c, d\}$ and $a \succ b \succ c \succ d$. Then, we start from $dP^1cP^1bP^1a$. Now, the top inconsistent and consecutively ranked pair is d, c. So, P^2 is $cP^2dP^2bP^2a$. Then, the top inconsistent and consecutively ranked pair in P^2 is d, b. So, P^3 is $cP^3bP^3dP^3a$. Then, we will get P^4 as $bP^4cP^4dP^4a$, and so on.

   - Find $\mathcal{D}$ when $A = \{a, b, c, d, e\}$ and $a \succ b \succ c \succ d \succ e$. (**2 marks**)
   - Is every $P \in \mathcal{D}$ single-peaked with respect to $\succ$ (with any set of alternatives)? Either prove this or provide a counter example. (**6 marks**)
   - Suppose $|N| = 2$. Give an example of a social choice function that is strategy-proof, unanimous, and anonymous in this domain. (**2 marks**)

2. Consider the two-sided matching problem with the same number of men and women.

   - Suppose man m is ranked first by all the women. Show that m is matched to the same woman in every stable matching. (**3 marks**)
   - Suppose man m is ranked last by all the women. Show that m is matched to the same woman in every stable matching. (**3 marks**)

- Let μ and μ' be two stable matchings. Define $\mu \wedge^m \mu'$ as follows: for every $m \in M$,

$$(\mu \wedge^m \mu')(m) := \min_{\succ_m}(\mu(m), \mu'(m)).$$

Show that $(\mu \wedge^m \mu')$ is a stable matching. (**6 marks**)

3. Consider the one-sided matching problem (house allocation problem). Assume that the number of agents equal number of houses.

   - The TTC mechanism and the serial dictatorship mechanism are both efficient and strategy-proof. Clearly define an appealing axiom that is satisfied by the TTC mechanism but violated by the serial dictatorship? (**2 marks**)
   - Suppose all the agents were assigned houses in the first stage of the TTC. What can you say about the preferences of the agents? (**2 marks**)

4. Let A be a finite set of alternatives and $\mathcal{L}(A)$ be the set of all probability distributions over A. Consider a random social choice function $f : \mathcal{P}^n \to \mathcal{L}(A)$, where $\mathcal{P}$ is the set of all linear orders over A. Fix an agent i and the preference profile of other agents at P_{-i}. Consider two preference orderings of agent i: P_i and P_i'. Suppose $x, y \in A$ are such that x and y are consecutively ranked in both P_i and P_i' with xP_iy and yP_ix. Suppose for any $a \notin \{x, y\}$, the rank of a in P_i and P_i' is the same. Hence, P_i' is obtained from P_i by swapping only x and y.

   If f is strategy-proof, then show the following:

   (a) $f_x(P_i, P_{-i}) + f_y(P_i, P_{-i}) = f_x(P_i', P_{-i}) + f_y(P_i', P_{-i})$. (**4 marks**)

   (b) for any $a \notin \{x, y\}$, $f_a(P_i, P_{-i}) = f_a(P_i', P_{-i})$. (**4 marks**)

   (c) $f_x(P_i, P_{-i}) \geq f_x(P_i', P_{-i})$. (**4 marks**)

5. Consider the private good allocation of one unit of a divisible commodity to n agents, with each agent having single peaked preference over their consumption space $[0, 1]$.

   - Consider a preference profile where peak of agent i is larger than the peak of agent j. Does the uniform rule allocate agent i at least as much as it allocates to agent j in this profile? Argue your answer. (**4 marks**)
   - Consider a preference profile where all the agents have the same peak except agent i, who has higher peak than all the other agents. Under what conditions (on the peaks) does agent i get his peak amount in the uniform rule? (**3 marks**)