Game Theory 2 - Final Examination

On: November 10, 2008; Duration: 3 hours

1. Describe the VCG mechanism for the combinatorial auction examples in Table 1 and Table 2. Both examples have two objects and three bidders. (**3+3** marks)

   Answer: For Example 1, efficient allocations are buyer 1 getting object 2 and buyer 2 or 3 getting object 1. In either case, buyer 1 pays $(8+9)-(8+0)=9$ and whoever wins object 1 pays 8 (buyer who does not win anything pays nothing).

   For example 2, efficient allocation is to allocate both the objects to buyer 3, and he pays $(8+6)-0=14$.

	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
$v_1(\cdot)$	0	6	10	10
$v_2(\cdot)$	0	8	5	8
$v_3(\cdot)$	0	8	9	9

Table 1: Example 1

	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
$v_1(\cdot)$	0	0	6	6
$v_2(\cdot)$	0	8	8	8
$v_3(\cdot)$	0	0	0	16

Table 2: Example 2

2. Consider a seller faced with a single buyer. The type set of the buyer is $T=\{0,1,2\}$. The set of allocations is $A=\{a,b,c\}$. The valuation function of the buyer is: $v(a,s)=0$ for all $s\in T$; $v(b,s)=\frac{s}{2}$ for all $s\in T$ and $v(c,s)=s$ for all $s\in T$. Consider the following allocation rule f: $f(0)=a$, $f(1)=b$, and $f(2)=c$.

   - Draw the **type graph** of this allocation rule with appropriate edge lengths. (**5 marks**)
   - Use the type graph to conclude whether this allocation rule is DSIC. (**2 marks**)
   - If f is DSIC, then find a payment rule which makes f DSIC. (**3 marks**)

   Answer: For any $s,t\in T$, $l(s,t)=v(f(t),t)-v(f(s),t)$. So, we get the following:

$$
\begin{aligned}
l(0,1) &= v(b,1) - v(a,1) = \frac{1}{2} \\
l(1,0) &= v(a,0) - v(b,0) = 0 \\
l(0,2) &= v(c,2) - v(a,2) = 2 \\
l(2,0) &= v(a,0) - v(c,0) = 0 \\
l(1,2) &= v(c,2) - v(b,2) = 1 \\
l(2,1) &= v(b,1) - v(c,1) = -\frac{1}{2}.
\end{aligned}
$$

The type graph is shown in Figure 1. Easy to verify that no cycle has negative length in this graph (the only edge with negative weight is (2,1), and cycles involving it have positive length). So, f is DSIC. One of the payments which makes f DSIC can be computed by fixing payment of type 0 to zero, and making the payments of other types equal to the shortest path from type 0 to other types: $p(0) = 0$, and $p(1) = dist_f(0,1) = \frac{1}{2}$ and $p(2) = dist_f(0,2) = \frac{3}{2}$. Of course, there are other payment rules as well.

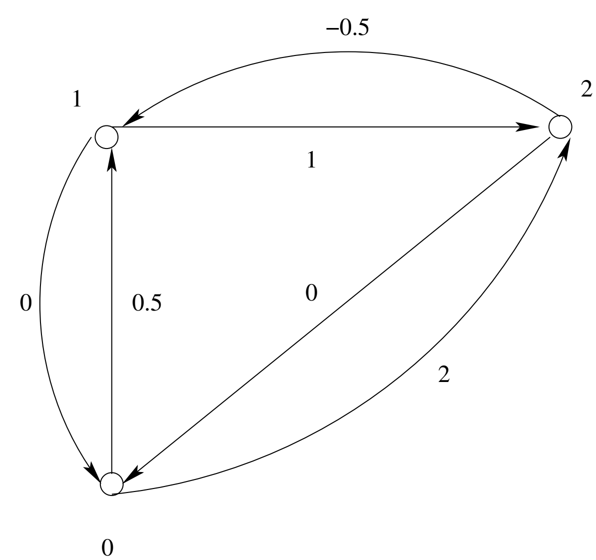

Figure 1: Type graph

Remaining questions are concerning single object sale in the private independent value set up. Set of agents is $N = \{1, \ldots, n\}$. Type set of agent $i \in N$ is $X_i = [0, h_i]$. Agent i uses probability distribution (cdf) F_i (density function f_i) to draw his type/value from X_i. All distributions are independent. In a mechanism (a, p), the expected utility of agent i when he has type x_i and reports z_i is $\alpha_i(z_i)x_i - \pi_i(z_i)$, where α and π denote the expected allocation rule and the expected payment rule respectively. In a mechanism (a,p), the utility of an agent when he has type x_i and everyone reports (z_i, x_{-i}) is $a_i(z_i,x_{-i})x_i - p_i(z_i,x_{-i})$.

3. Suppose there are two bidders, i.e., $N = \{1,2\}$. Suppose a is a DSIC allocation rule.

   - Prove the unique class of payment rules p which makes a DSIC is given by: (**3 marks**)

$$p_1(x_1,x_2) = p_1(0,x_2) + a_1(x_1,x_2)x_1 - \int_0^{x_1} a_1(t_1,x_2)dt_1 \qquad \forall\ x_1 \in X_1, x_2 \in X_2$$
$$p_2(x_1,x_2) = p_2(x_1,0) + a_2(x_1,x_2)x_2 - \int_0^{x_2} a_2(x_1,t_2)dt_2 \qquad \forall\ x_1 \in X_1, x_2 \in X_2.$$

   - Determine the unique payment rule p which makes f DSIC, which is individually rational, and which is non-negative. (**2 marks**)
   - If a is the efficient allocation rule, then show using the above that the unique payment rule (which makes a DSIC, which individually rational, and which is non-negative) reduces to the second-price auction. (**3 marks**)

Answer: Consider agent i and fix the report of other agents at x_{-i}. Since a is DSIC, there exists a p such that (a,p) is DSIC. Consider a DSIC mechanism (a,p). Since a is DSIC, it is non-decreasing. Let $U_i(x_i,x_{-i}) = a_i(x_i,x_{-i})x_i - p_i(x_i,x_{-i})$. First step is to show that $a_i(x_i,x_{-i})$ is the derivative of $U_i(x_i,x_{-i})$ with respect to type of agent i at x_i for any p which makes f DSIC (done similar to BIC case discussed in class). With this observation, we use fundamental theorem of calculus to write the exact form of p as:

$$p_i(x_i,x_{-i}) = p_i(0,x_{-i}) + a_i(x_i,x_{-i})x_i - \int_0^{x_i} a_i(t_i,x_{-i})dt_i \qquad \forall\ x_i \in X_i,\ \forall\ x_{-i} \in X_{-i},\ \forall\ i \in N.$$

IR requires, utilities to be non-negative. Hence, $p_i(0,x_{-i}) \leq 0$ for all $i \in N$. But non-negative payments make $p_i(0,x_{-i}) = 0$ for all $i \in N$. Hence, the unique payment rule which is DSIC, IR, and non-negative is

$$p_i(x_i,x_{-i}) = a_i(x_i,x_{-i})x_i - \int_0^{x_i} a_i(t_i,x_{-i})dt_i \qquad \forall\ x_i \in X_i,\ \forall\ x_{-i} \in X_{-i},\ \forall\ i \in N.$$

If a is efficient, $a_i(x_i,x_{-i}) = 0$ if $x_i < \max_{j\neq i} x_j$ and $a_i(x_i,x_{-i}) = 1$ if $x_i > \max_{j \neq i} x_j$. Hence, if $a_i(x_i,x_{-i}) = 0$, we have $\int_0^{x_i} a_i(t_i,x_{-i})dt_i = 0$. If $a_i(x_i,x_{-i}) = 1$, we have $\int_0^{x_i} a_i(t_i,x_{-i})dt_i = [t_i]^{x_i}_{\max_{j\neq i} x_j} = x_i - \max_{j\neq i} x_j$. This reduces the payment scheme to $p_i(x_i,x_{-i}) = 0$ if $a_i(x_i,x_{-i}) = 0$ and $p_i(x_i,x_{-i}) = \max_{j\neq i} x_j$ if $a_i(x_i,x_{-i}) = 1$. This is the Vickrey auction.

4. Consider a setting with two buyers whose values are distributed uniformly in the intervals $X_1 = [0, 12]$ (buyer 1) and $X_2 = [0, 18]$ (buyer 2).

   - What are the reserve prices for buyer 1 and buyer 2 in the optimal mechanism, i.e., for what valuations do the buyers have zero virtual valuation? (**1 mark**)
   - Describe the allocation and payments of buyers in an optimal mechanism when valuations are as given in Table 3. (**10 marks**)

   Answer: The reserve price of buyer 1 is 6 and buyer 2 is 9. The optimal mechanism is described in Table 3 - calculations are as follows: in first line, both values are below respective reserve prices, so object unsold; in second line, value of buyer 1 is below his reserve price whereas buyer 2 has value higher than reserve price, so he wins and pays a price equal to his reserve price; in line 3 buyer 2 has value below his reserve price while buyer 1 has value greater than or equal to his reserve price, so he wins and pays his reserve price; in line 4, both buyers have value greater than or equal to their respective reserve prices, but buyer 1's virtual valuation is higher ($6 > 0$), so he wins and pays 6 (at which his virtual valuation beats the virtual valuation of buyer 2); in line 5 also, both buyers have value greater than their respective reserve prices but buyer 2 has higher virtual valuation ($12 > 4$), so he wins and pays an amount 11 (at valuation 11, buyer 2 has virtual valuation 4 which is equal to virtual valuation of buyer 1).

Valuations	Allocation (who gets object)	Payment of Buyer 1	Payment of Buyer 2
$(x_1 = 4, x_2 = 8)$	Object not sold	0	0
$(x_1 = 2, x_2 = 12)$	Buyer 2	0	9
$(x_1 = 6, x_2 = 6)$	Buyer 1	6	0
$(x_1 = 9, x_2 = 9)$	Buyer 1	6	0
$(x_1 = 8, x_2 = 15)$	Buyer 2	0	11

Table 3: Description of Optimal Mechanism

5. Suppose bidders are symmetric. The seller is debating the use of first-price auction and the second-price auction, both with a reserve price $r \geq 0$.

   - Which auction format will you recommend the seller and why? (**3 marks**)
   - Suppose bidders draw their valuations using uniform distribution with support $[0, h]$ where $h > 0$. In your recommended auction, what is the optimal value of reserve price r? (**2 marks**)

Answer: With symmetric bidders, both the auctions use the same allocation rule - when values of bidders are less than r object is not allocated whereas when it is above r it goes to the bidder with the highest *value*. Applying revenue equivalence, we get the same payments in both auctions (note: a bidder with value 0 pays zero in both the auctions). Hence, if a seller aims to maximize expected revenue, then either auction achieves it. With uniform distribution, the optimum reserve price is $\frac{h}{2}$.