

Approximate Revenue Maximization with Multiple Items*

Sergiu Hart[†] Noam Nisan[‡]

September 20, 2017

Abstract

Maximizing the revenue from selling *more than one* good (or item) to a single buyer is a notoriously difficult problem, in stark contrast to the one-good case. For two goods, we show that simple “one-dimensional” mechanisms, such as selling the goods separately, *guarantee at least 73%* of the optimal revenue when the valuations of the two goods are independent and identically distributed, and at least 50% when they are independent.

For the case of $k > 2$ independent goods, we show that selling them separately guarantees at least a $c/\log^2 k$ fraction of the optimal revenue; and, for independent and identically distributed goods, we show that selling them as one bundle guarantees at least a $c/\log k$ fraction of the optimal revenue.

*Previous versions: February 2012; April 2012 (*arXiv* 1204.1846 and Center for Rationality DP-606); May 2014; March 2017. Research partially supported by a European Research Council Advanced Investigator grant (Hart) and by an Israel Science Foundation grant (Nisan). We thank Motty Perry and Phil Reny for introducing us to the subject and for many helpful discussions, and the referees for their very careful reading and useful comments. A presentation that covers some of this work is available at <http://www.ma.huji.ac.il/hart/abs/2good-p.html>

[†]The Hebrew University of Jerusalem (Federmann Center for the Study of Rationality, Department of Economics, and Institute of Mathematics). *E-mail:* hart@huji.ac.il
Web site: <http://www.ma.huji.ac.il/hart>

[‡]The Hebrew University of Jerusalem (Federmann Center for the Study of Rationality, and School of Computer Science and Engineering), and Microsoft Research. *E-mail:* noam@cs.huji.ac.il *Web site:* <http://www.cs.huji.ac.il/~noam>

Additional results compare the revenues from the two simple mechanisms of selling the goods separately and bundled, identify situations where bundling is optimal, and extend the analysis to multiple buyers.

Contents

1	Introduction	3
1.1	Literature	8
2	Preliminaries	10
2.1	The Model	11
2.2	Guaranteed Fraction of Optimal Revenue (GFOR)	15
3	Two Independent Goods	16
4	The General Decomposition Result	18
5	Separate and Bundled Selling	21
6	k Independent Goods	25
7	Additional Results	27
7.1	Upper Bound on GFOR for Two Goods	27
7.2	When Bundling Is Optimal	28
7.3	Multiple Buyers	28
8	Open Problems	30
A	Appendix	31
A.1	Proof for Two I.I.D. Goods	31
A.2	Some Comments on Decomposition	35
A.3	Equal Revenue (ER) Goods	37
A.4	Separate vs. Bundled Selling	41
A.5	Many I.I.D. Goods	42
A.6	When Bundling Is Optimal	44

A.7 Multiple Buyers	46
A.8 Summary of Results	50

References	52
-------------------	-----------

1 Introduction

Suppose that a seller has one good (or “item”) to sell to a single buyer whose willingness to pay (or “value”) for the good is x . While x is known to the buyer, it is unknown to the seller, who knows only its distribution (given by a cumulative distribution function F). If the seller offers to sell the good for a price p then the probability that the buyer will buy is $1 - F(p)$, and the seller’s revenue will be $p \cdot (1 - F(p))$. The seller will choose a price p^* that maximizes this expression.

This problem is the classic monopolist-pricing problem. Looking at it from an auction point of view, one may ask whether there are mechanisms for selling the good that yield a higher revenue. Such mechanisms could be indirect, could offer different prices for different probabilities of getting the good, and so on. Yet, the characterization of optimal mechanisms of Myerson (1981) (see also Riley and Samuelson 1981 and Riley and Zeckhauser 1983) concludes that the take-it-or-leave-it offer at the above price p^* yields the optimal revenue among *all* mechanisms. Even more, Myerson’s result also applies when there are multiple buyers, in which case p^* would be the reserve price in a second-price auction.

Now suppose that the seller has two (different) goods that he wants to sell to a single buyer. Furthermore, consider the simplest case where the buyer’s values for the two goods are independently and identically distributed according to the distribution F (“i.i.d.- F ” for short), and where, furthermore, his valuation is additive: if the value of the first good is y and that of the second is z , then the value of the *bundle* consisting of both goods is¹ $y + z$. It would seem that since the two goods are completely independent of each

¹Our buyer’s demand is thus *not* limited to one good (as is the case in some of the existing literature; see “unit-demand” in Section 1.1).

other, then the best one should be able to do is to sell each of them separately in the optimal way, and thus extract exactly twice the revenue one would make from a single good. Yet this turns out to be false.

Example 1 Consider the one-good distribution F taking values 1 and 2, each with probability $1/2$. Let us first look at selling a single good optimally: the seller can either choose to price it at 1, selling always² and getting a revenue of 1, or choose to price the good at 2, selling it with probability $1/2$, again obtaining an expected revenue of 1, and so the optimal revenue from a single good is 1. Now consider the following mechanism for selling both goods: bundle them together, and sell the bundle for price 3. The probability that the sum of the buyer’s values for the two goods is at least 3 is $3/4$, and so the revenue is $3 \cdot 3/4 = 2.25$ —larger than the revenue of 2 that is obtained by selling them separately.

However, that is not always so: bundling may sometimes be worse than selling the goods separately.

Example 2 Consider the one-good distribution F taking values 0 and 1, each with probability $1/2$. Selling the two goods separately yields a revenue of $1/2$ from each good (set the price at 1), and so 1 in total, whereas the revenue from selling the bundle is only $3/4$ (the optimal price for the bundle is 1).

In other cases neither selling separately nor bundling is optimal.

Example 3 Consider the one-good distribution F taking the values 0, 1, and 2, each with probability $1/3$. The unique optimal mechanism for two such i.i.d. goods turns out to be³ to offer to the buyer the choice between

²Since we maximize revenue we can assume without loss of generality that ties are broken by the buyer in a way that maximizes the seller’s revenue. This “seller-favorable” property can always be achieved by appropriate small perturbations of the mechanism; for instance, by the seller giving a small fixed proportional discount on all payments. See Hart and Reny (2015a, Section 1.2, and Remark (a) after Corollary 18).

³For distributions with finite support, finding the optimal mechanism amounts to solving a linear programming problem.

any single good at price 2 and the bundle of both goods at a “discount” price of 3. This mechanism gets a revenue of $13/9 \approx 1.44$, which is larger than the revenue of $4/3 \approx 1.33$ obtained either from selling the two goods separately or from selling them as a single bundle.

A similar situation obtains for the uniform distribution on $[0, 1]$, for which neither bundling nor selling separately is optimal (Manelli and Vincent 2006). In still other cases the optimal mechanism is not even deterministic and must offer lotteries for the goods. This happens for instance in the following example, taken from Hart and Reny (2015a, Example 4).⁴

Example 4 Consider the distribution taking the values 1, 2, and 4, with probabilities $1/6$, $1/2$, and $1/3$, respectively. It turns out that the unique optimal mechanism for two such i.i.d. goods offers the buyer a choice of one of the following options: buying a lottery ticket that has price 1 and gives the first good *with probability* $1/2$, buying a similar lottery ticket for good 2, buying the bundle of both goods for a price of 4, and buying nothing (and paying nothing); indeed, any deterministic mechanism has a strictly lower revenue.

Thus, it is not clear what optimal mechanisms for selling two goods look like, and indeed characterizations of optimal mechanisms even for this simple case are not known (see Section 1.1). The two-dimensional problem is extremely difficult, and the simple mechanisms that amount to solving only one-dimensional problems—such as separate selling and bundling—do not maximize the revenue in general.

This leads to the following question: how good are such simple mechanisms for selling two goods? That is, how much of the optimal revenue is guaranteed when using them? Consider a class of mechanisms \mathcal{N} (such as separate selling or bundled selling) and a class of environments \mathbb{X} (such as two independent and identically distributed goods, or k independent goods);

⁴Examples in which randomization increases the revenue appear in the literature, starting with Thanassoulis (2004) in the somewhat different setup of unit demand, and Manelli and Vincent (2006, 2007, 2012). See Section 1.1.

we then define the *Guaranteed Fraction of Optimal Revenue* (GFOR) as that maximal fraction α between 0 and 1 such that for every environment in \mathbb{X} there is a mechanism in \mathcal{N} that yields a revenue of at least the fraction α of the optimal revenue (it is thus the reciprocal of the “competitive ratio” often used in the computer science literature; see Section 2.2 for the formal definition and a discussion of these concepts).

We start with two independent goods, and consider selling them separately. Our first result is:

Theorem A *For any two independent goods, selling each good separately at its optimal one-good price guarantees at least 50% of the optimal revenue; i.e.,*

$$\text{GFOR}(\text{SEPARATE}; 2 \text{ independent goods}) \geq \frac{1}{2}.$$

This result applies to any distribution of values (we make no assumptions, such as monotone hazard rate or increasing virtual values), and it holds also for any number of buyers (see Section 7.3).

When the two goods are identically distributed, separate selling is guaranteed to perform even better.

Theorem B *For any two independent and identically distributed goods, selling each one at the one-good optimal price guarantees at least 73% of the optimal revenue; i.e.,*

$$\text{GFOR}(\text{SEPARATE}; 2 \text{ i.i.d. goods}) \geq \frac{e}{e+1} \approx 0.73.$$

Thus, for two i.i.d. goods with distribution F , setting the price at p^* that maximizes the one-good revenue (i.e., $p^*(1 - F(p^*)) = \max_p p(1 - F(p))$) and allowing the buyer to buy any number of units—0, 1, or 2 units—at price p^* per unit guarantees at least 73% of the optimal revenue.

We next consider the case of more than two goods. It turns out that, as the number k of goods grows, the fraction of the optimal revenue that is obtainable from selling them separately may become arbitrarily small (specifically, of the order of $1/\log k$, cf. Corollary 26; the reader may refer to the

tables in Appendix A.8 that summarize all these comparisons). Our main positive result here is:

Theorem C *There exists a constant $c > 0$ such that for any $k \geq 2$ and any k independent goods, selling each good separately at its optimal one-good price guarantees at least $c/\log^2 k$ of the optimal revenue; i.e.,*

$$\text{GFOR}(\text{SEPARATE}; k \text{ independent goods}) \geq \frac{c}{\log^2 k}.$$

Finally, we move to the other simple one-dimensional mechanism, the *bundling* mechanism, which offers a single price for the bundle of all goods. We first show that for general independent goods, bundling may do much worse and yield only a $1/k$ -fraction of the optimal revenue (Example 27). However, when the goods are independent and *identically distributed*, then bundling does much better. It is well known (Armstrong 1999, Bakos and Brynjolfsson 1999) that for every fixed distribution F , as the number of goods distributed independently according to F increases, the bundling mechanisms become close to being optimal (for completeness we provide a short proof in Appendix A.5). This, however, requires k to grow as F remains fixed. On the other hand, we show that this is not true uniformly over F : for every large enough k , there are distributions where the bundling mechanism on k goods gives less than 57% of the optimal revenue (Example 32). Our main result for the bundling mechanism in the i.i.d. case is:

Theorem D *There exists a constant $c > 0$ such that for any $k \geq 2$ and any k independent and identically distributed goods, selling them as one bundle at the bundle-optimal price guarantees at least $c/\log k$ of the optimal revenue:*

$$\text{GFOR}(\text{BUNDLED}; k \text{ i.i.d. goods}) \geq \frac{c}{\log k}.$$

The paper is organized as follows. Section 1.1 presents a survey of the relevant literature, including work done following the circulation of the early versions of this paper in 2012. In Section 2 we present the model and the basic concepts, followed by a number of useful preliminary results. Section 3 deals

with the case of two independent goods, and provides the proof of Theorem A; the more complex proof of Theorem B is relegated to Appendix A.1. The main argument of these proofs is then extended to a general decomposition theorem in Section 4. Section 5 studies the relations between the revenues from separate and bundled selling (with some of the proofs and additional results in Appendices A.3 and A.4); these relations are not only interesting in their own right, but are also used as part of the general analysis, and provide us with most of the examples that we have of gaps in revenue. The results for more than two goods, Theorems C and D, are then proved in Section 6, making use of the decomposition of Section 4 and the comparisons of Section 5. Additional relevant results, namely, upper bounds on GFOR, a two-good setup where bundling is shown to be optimal, and the extension to multiple buyers, are stated in Section 7 (with proofs relegated to Appendices A.6 and A.7). In Section 8 we discuss open problems. Finally, Appendix A.8 provides two tables: the first summarizes the lower and upper bounds that we have obtained on the fraction of optimal revenue that is guaranteed for both separate and bundled selling, and the second summarizes the comparisons between separate and bundled selling.

1.1 Literature

We briefly describe some of the existing work on these issues. McAfee and McMillan (1988) identify cases where the optimal mechanism is deterministic. However, Thanassoulis (2004) and Manelli and Vincent (2006) found a technical error in the paper and present counterexamples. These last two papers contain good surveys of the work within economic theory, with more recent analysis by Fang and Norman (2006), Jehiel, Meyer-ter-Vehn, and Moldovanu (2007), Pycia (2006), Lev (2011), Pavlov (2011), and Hart and Reny (2015a). In the past few years algorithmic work on these types of topics was carried out. One line of work (e.g., Briest, Chawla, Kleinberg, and Weinberg 2015; Cai, Daskalakis, and Weinberg 2012a; Alaei, Fu, Haghpanah, Hartline, and Malekian 2012) shows that for discrete distributions the optimal mechanism can be found by linear programming in rather general

settings. This is certainly true in our simple setting where the direct representation of the mechanism constraints provides a polynomial-size linear program. Thus we emphasize that the difficulty in our case is *not computational*, but is rather one of characterizing and understanding the results of the explicit computations: this is certainly so for continuous distributions, but also for discrete ones.⁵ Another line of work in computer science (Chawla, Hartline, and Kleinberg 2007; Chawla, Hartline, Malec, and Sivan 2010; Chawla, Malec, and Sivan 2010; Alaei, Fu, Haghpanah, Hartline, and Malekian 2012; Cai, Daskalakis, and Weinberg 2012b) attempts to approximate the optimal revenue by simple mechanisms. This was done for various settings, especially unit-demand settings and some generalizations.⁶ One particular conclusion from this line of work is that for many subclasses of distributions (such as those with a monotone hazard rate) various simple mechanisms can extract a constant fraction of the expected value of the goods.⁷ This is true in our simple setting, where for such distributions selling the goods separately provides a constant fraction of the expected value and thus of the optimal revenue. The case of multiple goods with *correlated distributions* was studied by Briest, Chawla, Kleinberg, and Weinberg (2010) and Hart and Nisan (2013) and turns out to be quite different from the independently distributed case: classes of simple mechanisms (even the class of all deterministic mechanisms) may well yield only an arbitrarily small fraction of the optimal revenue.

⁵This suggests that a notion of “*conceptual complexity*” may be appropriate here. The usual computational complexity may not capture all the difficulty of a problem, since, even after computing the precise solution, one may not understand its structure, what it means and represents, and how it varies with the given parameters (e.g., Hart and Reny 2015a, where it is shown that the optimal revenue may *decrease* when the buyer’s valuations *increase*).

⁶“Unit demand” means that buyers are willing to buy *at most* one of the goods. The relations between the revenues in the additive setup and those in the unit-demand setup are discussed in our paper Hart and Nisan (2013, Appendix 1). It is thus possible that bounds obtained in the unit-demand literature lead, in the additive setup, to weaker versions of our Theorem A. Our direct approach is simpler—cf. Section 3—and generalizable—cf. Section 4.

⁷In our setting this is true even more generally, for instance, whenever the ratio between the median and the expectation is bounded (which happens in particular when the tail of the distribution is “thinner” than $x^{-\alpha}$ for $\alpha > 1$). Indeed, posting a price equal to the median yields a revenue of one-half of the median, and hence at least a constant fraction of the expectation (which is, by IR, the most that the seller can extract as expected revenue).

Since the circulation of early versions of this paper in 2012 (Hart and Nisan 2012), there has been a flurry of work on optimal mechanisms for multiple goods for various cases: Daskalakis, Deckelbaum, and Tzamos (2013, 2017), Giannakopoulos (2014), Giannakopoulos and Koutsoupias (2014), Menicucci, Hurkens, and Jeon (2015), and Tang and Wang (2017). The general problem was studied from a computational perspective in Daskalakis, Deckelbaum, and Tzamos (2014), where it is shown to be computationally intractable (formally, $\#P$ -hard). Several developments have occurred regarding GFOR for multiple goods. Li and Yao (2013) improved our lower bound on GFOR(SEPARATE) for k goods from $c/\log^2 k$ to the tight $c/\log k$. For the case of k independent and identically distributed goods, Li and Yao (2013) proved that GFOR(BUNDLED) is bounded from below by a constant that is independent of the number of goods k . Babaioff, Immorlica, Lucier, and Weinberg (2014) showed that, for k independent (but not necessarily identically distributed) goods, GFOR({SEPARATE, BUNDLED}) is bounded from below by a constant that is independent of k (that is, there is $c > 0$ such that for any number k and any k independent goods, either separate selling or bundling yields at least the fraction c of the optimal revenue). This was generalized by Yao (2014) to the case of multiple bidders and by Rubinfeld and Weinberg (2015) to buyers with submodular (rather than just additive) valuations for the goods. Measures quantifying how complex mechanisms need to be in order to yield a good proportion of the optimal revenue were studied in Hart and Nisan (2013), Dughmi, Han, and Nisan (2014), Morgenstern and Roughgarden (2016), and Babaioff, Gonczarowski, and Nisan (2017).

2 Preliminaries

In this section we present the model formally and define the concepts that we use, followed by a number of preliminary results.

2.1 The Model

One seller (or “monopolist”) is selling a number $k \geq 1$ of goods (or “items,” “objects,” etc.) to one buyer.

The goods have no value or cost to the seller. Let $x_1, x_2, \dots, x_k \geq 0$ be the buyer’s values for the goods. The value for getting a set of goods is *additive*: getting the subset $I \subseteq \{1, 2, \dots, k\}$ of goods is worth $\sum_{i \in I} x_i$ to the buyer (and so, in particular, the buyer’s demand is *not* restricted to one good only). The values are given by a random variable $X = (X_1, X_2, \dots, X_k)$ that takes values in \mathbb{R}_+^k , (we thus assume that valuations are always nonnegative); we will refer to X as a *k-good random valuation*. The realization $x = (x_1, x_2, \dots, x_k) \in \mathbb{R}_+^k$ of X is known to the buyer, but not to the seller, who knows only the distribution F of X (which may be viewed as the seller’s belief). The buyer and the seller are assumed to be risk neutral and to have quasi-linear utilities (i.e., the utility is additive with respect to monetary transfers; e.g., getting good 1 with probability 1/2 and paying 8 with probability 1/4 is worth $(1/2) \cdot x_1 - (1/4) \cdot 8$ to the buyer and $(1/4) \cdot 8$ to the seller).

The objective is to *maximize* the seller’s (expected) *revenue*.

As has been well established by the so-called “Revelation Principle” (starting with Myerson 1981; see for instance the book of Krishna 2010), we can restrict ourselves to “direct mechanisms” and “truthful equilibria.” A (direct)⁸ *mechanism* μ consists of a pair of functions (q, s) , where $q = (q_1, q_2, \dots, q_k) : \mathbb{R}_+^k \rightarrow [0, 1]^k$ and $s : \mathbb{R}_+^k \rightarrow \mathbb{R}$, which prescribe the *allocation* of goods and the *payment*, respectively. Specifically, if the buyer reports a value vector $x \in \mathbb{R}_+^k$, then $q_i(x) \in [0, 1]$ is the probability that the buyer receives good⁹ i (for $i = 1, 2, \dots, k$), and $s(x)$ is the payment that the seller receives from the buyer. When the buyer reports his value x truthfully, his payoff is¹⁰ $b(x) = \sum_{i=1}^k q_i(x)x_i - s(x) = q(x) \cdot x - s(x)$, and the seller’s payoff is¹¹ $s(x)$.

⁸“Mechanism” will henceforth always mean “direct mechanism.”

⁹When the goods are infinitely divisible and the valuations are linear in quantities (i.e., the value of a quantity λ of good i is λx_i), we may interpret q_i also as the *quantity* of good i that the buyer gets.

¹⁰When $y = (y_i)_{i=1, \dots, n}$ and $z = (z_i)_{i=1, \dots, n}$ are n -dimensional vectors, $y \cdot z$ denotes their scalar product $\sum_{i=1}^n y_i z_i$.

¹¹In the literature the payment to the seller is called transfer, cost, price, revenue, and

The mechanism $\mu = (q, s)$ satisfies *individual rationality* (**IR**) if $b(x) \geq 0$ for every $x \in \mathbb{R}_+^k$, and *incentive compatibility* (**IC**) if $b(x) \geq q(\tilde{x}) \cdot x - s(\tilde{x})$ for every alternative report $\tilde{x} \in \mathbb{R}_+^k$ of the buyer when his value is x , for every $x \in \mathbb{R}_+^k$. Let \mathcal{M} denote the class of all IC and IR mechanisms $\mu = (q, s)$. The expected revenue from a buyer with random valuation X using a mechanism $\mu = (q, s) \in \mathcal{M}$ is¹² $R(\mu; X) := \mathbb{E}[s(X)]$, and the *optimal revenue* from X is, by the Revelation Principle, $\text{REV}(X) := \sup_{\mu \in \mathcal{M}} R(\mu; X)$, the highest revenue that can be obtained by any IC and IR mechanism μ . The revenue can never exceed the expected valuation of all goods together: $\text{REV}(X) \leq \mathbb{E}[\sum_i X_i]$ (since $s(x) \leq q(x) \cdot x \leq \sum_i x_i$ by IR and $x \geq 0$).

When there is only one good, i.e., when $k = 1$, Myerson's (1981) result is that

$$\text{REV}(X) = \sup_{p \geq 0} p \cdot \mathbb{P}[X \geq p] = \sup_{p \geq 0} p \cdot \mathbb{P}[X > p] = \sup_{p \geq 0} p \cdot (1 - F(p)), \quad (1)$$

where F is the cumulative distribution function of X . Optimal mechanisms correspond to the seller "posting" a price p and the buyer buying the good for the price p whenever his value is at least p ; in other words, the seller makes the buyer a "take-it-or-leave-it" offer to buy the good at price p .

Besides the maximal revenue, we are also interested in what can be obtained from certain classes of mechanisms. Thus, given a class $\mathcal{N} \subset \mathcal{M}$ of IC and IR mechanisms, let $\mathcal{N}\text{-REV}(X) := \sup_{\nu \in \mathcal{N}} R(\nu; X)$ be the maximal revenue that can be extracted from a buyer with random valuation X when restricted to mechanisms ν in the class \mathcal{N} . Some classes of mechanisms are:

- **SEPARATE**: Each good i is sold separately. The maximal revenue from separate mechanisms is denoted by **SREV**, and so

$$\text{SREV}(X) := \text{REV}(X_1) + \text{REV}(X_2) + \dots + \text{REV}(X_k).$$

so on, and is denoted by t, c, p, \dots ; this plethora of names and notations applies to the buyer as well. We hope that using the mnemonic s for the seller's final payoff and b for the buyer's final payoff will avoid confusion.

¹²In Hart and Reny (2015a, Proposition 16) it is shown that for IC and IR mechanisms one may assume without loss of generality that s is measurable; since s is bounded from below by $s(0)$ (which follows from IC at 0), the expected revenue $\mathbb{E}[s(X)]$ is well defined (but may be infinite).

- **BUNDLED:** All goods are sold together in one “bundle.” The maximal revenue from bundled mechanisms is denoted by BREV , and so

$$\text{BREV}(X) := \text{REV}(X_1 + X_2 + \dots + X_k).$$

- **DETERMINISTIC:** Each good i is either fully allocated or not at all, i.e., $q_i(x) \in \{0, 1\}$ (rather than $q_i(x) \in [0, 1]$) for every $x \in \mathbb{R}_+^k$ and $1 \leq i \leq k$. The maximal revenue from deterministic mechanisms is denoted by DREV .

The separate and the bundled revenues are obtained by solving one-dimensional problems (where one uses (1)), whereas the deterministic revenue is a multi-dimensional problem. Examples where these classes yield revenues that are smaller than the maximal revenue are well known (see also the examples in the Introduction, and the references in Section 1.1).

The following is a useful characterization of incentive compatibility that is well known (starting with Rochet 1985).

Proposition 5 *Let $\mu = (q, s)$ be a mechanism for k goods with buyer payoff function b . Then $\mu = (q, s)$ satisfies IC if and only if b is a convex function and for all x the vector $q(x)$ is a subgradient of b at x (i.e., $b(\tilde{x}) - b(x) \geq q(x) \cdot (\tilde{x} - x)$ for all \tilde{x}).*

Proof. μ is IC if and only if $b(x) = q(x) \cdot x - s(x) = \max_{\tilde{x} \in \mathbb{R}_+^k} (q(\tilde{x}) \cdot x - s(\tilde{x}))$ for every x , which implies that b is a convex function of x (as the maximum of a collection of affine functions of x). Moreover, for every x and \tilde{x} we have $b(\tilde{x}) - b(x) - q(x) \cdot (\tilde{x} - x) = b(\tilde{x}) - (q(x) \cdot \tilde{x} - s(x))$, and so the subgradient inequalities are precisely the IC inequalities. ■

Thus $s(x) = q(x) \cdot x - b(x) = \nabla b(x) \cdot x - b(x)$, where $\nabla b(x)$ stands for a (sub)gradient of b at x , and so the revenue can be expressed in terms of the buyer payoff function¹³ b . We also note that there is no loss of generality

¹³The function b , being convex, is differentiable almost everywhere, and so $\nabla b(x)$ is the gradient $(\partial b(x)/\partial x_i)_{i=1, \dots, k}$ for almost every x . As pointed out in Hart and Reny (2015a, Appendix A.1), when maximizing revenue one may use “seller-favorable” mechanisms and replace the term $\nabla b(x) \cdot x$ with $b'(x; x)$, the directional derivative of b at x in the direction x , which is well defined for every x .

in assuming that the mechanism μ is defined and satisfies IC and IR on the whole space \mathbb{R}_+^k , rather than just on some domain $D \subset \mathbb{R}_+^k$, such as the set of possible values of X ; see Hart and Reny (2015a, Appendix A.1).

We conclude with a useful property: a mechanism $\mu = (q, s)$ satisfies the *no positive transfer*¹⁴ (**NPT**) property if $s(x) \geq 0$ for every $x \in \mathbb{R}_+^k$. Proposition 6 below shows that NPT can always be assumed without loss of generality when maximizing revenue.¹⁵ Moreover, the revenue from any IC and IR mechanism on a *subdomain* of valuations cannot exceed the overall maximal revenue—even when the mechanism does not satisfy NPT.¹⁶

Proposition 6 *Let $\mu = (q, s)$ be an IC and IR mechanism, and let X be a k -good random valuation in \mathbb{R}_+^k , where $k \geq 1$. Then:*

- (i) μ satisfies NPT if and only if $s(0) = 0$, which occurs if and only if $b(0) = 0$.
- (ii) There is a mechanism $\hat{\mu} = (q, \hat{s})$ with the same q and with $\hat{s}(x) \geq s(x)$ for all $x \in \mathbb{R}_+^k$, such that $\hat{\mu}$ satisfies IC, IR, and NPT.
- (iii) $\text{REV}(X) = \sup_{\mu} R(\mu; X)$ where the supremum is taken over all IC, IR, and NPT mechanisms μ .
- (iv) Let $A \subseteq \mathbb{R}_+^k$ be a set of values of X ; then¹⁷

$$\mathbb{E}[s(X) \mathbf{1}_{X \in A}] \leq \text{REV}(X \mathbf{1}_{X \in A}) \leq \text{REV}(X).$$

Proof. (i) IC at 0 yields $s(x) \geq s(0)$ for all x , and IR at 0 yields $s(0) \leq 0$; the minimal payment is thus $s(0)$, which cannot be positive. Therefore, $s(x) \geq 0$ for all x if and only if $s(0) = 0$. Now $s(0) + b(0) = q(0) \cdot 0 = 0$, and so $s(0) = 0$ if and only if $b(0) = 0$.

¹⁴The “transfer” is from the seller to the buyer, i.e., $-s(x)$.

¹⁵This is not true in more general setups; for instance, when there are multiple buyers that are correlated, Bayesian Nash implementation may require using positive transfers, i.e., $s(x) < 0$ (cf. Crémer and McLean 1988; see also Appendix A.7 below).

¹⁶This is used in our proofs, e.g., in Section 3, where we construct mechanisms for which s may take negative values.

¹⁷We write $\mathbf{1}_W$ for the indicator of the event W : it takes the value 1 when W occurs and the value 0 otherwise.

(ii) Put $\hat{s}(x) := s(x) - s(0) \geq s(x)$ for all x (recall that $s(0) \leq 0$ by IR at 0). Then $\hat{\mu} = (q, \hat{s})$ satisfies IC since the payment differences have not changed (i.e., $\hat{s}(x) - \hat{s}(\tilde{x}) = s(x) - s(\tilde{x})$ for all x, \tilde{x}); it satisfies IR since $q(x) \cdot x - s(x) \geq q(0) \cdot x - s(0) \geq -s(0)$ (by IC); and it satisfies NPT since $\hat{s}(0) = 0$.

(iii) Follows from (ii) since $\hat{\mu}$ yields at least as much revenue as μ (because $\hat{s}(x) \geq s(x)$ for all x).

(iv) For the first inequality, use (ii) to get $\mathbb{E}[s(X) \mathbf{1}_{X \in A}] \leq \mathbb{E}[\hat{s}(X) \mathbf{1}_{X \in A}] = \mathbb{E}[\hat{s}(X) \mathbf{1}_{X \in A}] \leq \text{REV}(X \mathbf{1}_{X \in A})$ (the equality since $\hat{s}(0) = 0$ by (i)). For the second inequality, $\mathbb{E}[s(X) \mathbf{1}_{X \in A}] = \mathbb{E}[s(X) \mathbf{1}_{X \in A}] \leq \mathbb{E}[s(X)]$ for any μ that satisfies NPT; apply (iii). ■

2.2 Guaranteed Fraction of Optimal Revenue (GFOR)

Let \mathbb{X} be a class of random valuations (such as two independent goods, or k i.i.d. goods; formally, it is a class of random variables X with values in \mathbb{R}_+^k spaces), and let \mathcal{N} be a class of IC and IR mechanisms (such as separate selling, or deterministic mechanisms; formally, \mathcal{N} is a subset of the class \mathcal{M} of all IC and IR mechanisms). The *Guaranteed Fraction of Optimal Revenue* (GFOR) for the class of random valuations \mathbb{X} and the class of mechanisms \mathcal{N} is defined as the maximal fraction α such that, for any random valuation X in \mathbb{X} , there are mechanisms in the class \mathcal{N} that yield at least the fraction α of the optimal revenue. Formally,¹⁸

$$\text{GFOR} \equiv \text{GFOR}(\mathcal{N}; \mathbb{X}) := \inf_{X \in \mathbb{X}} \frac{\mathcal{N}\text{-REV}(X)}{\text{REV}(X)} = \inf_{X \in \mathbb{X}} \frac{\sup_{\nu \in \mathcal{N}} R(\nu; X)}{\sup_{\mu \in \mathcal{M}} R(\mu; X)}.$$

Thus $\text{GFOR} \geq \alpha$ if and only if for every random valuation X in \mathbb{X} there is a mechanism ν in \mathcal{N} such that its revenue is $R(\nu; X) \geq \alpha \cdot \text{REV}(X)$ (we are ignoring here the trivial issues of “max” vs. “sup”), and $\text{GFOR} \leq \alpha$ if there exists a random valuation X in \mathbb{X} such that for every mechanism ν in \mathcal{N} its revenue is $R(\nu; X) \leq \alpha \cdot \text{REV}(X)$.

Remarks. (a) One may argue that there is no need for results that are *uni-*

¹⁸Put $0/0 = 1$.

form with respect to the values’ distributions, on the grounds that the seller knows that distribution. However, in the case of multiple goods, knowing the distribution does not help find the optimal mechanism (even for simple distributions), whereas simple mechanisms, such as separate selling, are always easy to compute (as they use only optimal prices for one-dimensional distributions). It is thus important to know how far from optimal these mechanisms are guaranteed to be, particularly when one does not know what that optimum is or how to find it.

(b) *Ratios.* Why are we considering ratios? The reason is that the revenue is covariant with rescalings, but not with translations. Indeed, $\text{REV}(\lambda X) = \lambda \cdot \text{REV}(X)$ for any $\lambda > 0$, but $\text{REV}(X + c)$ is in general different from $\text{REV}(X) + c$ for constant $c > 0$ (this happens already in the one-good case; see (1)).

(c) *Competitive ratio.* The computer science literature uses the concepts of “competitive ratio” and “approximation ratio,” which are just the reciprocal $1/\text{GFOR}$ of GFOR. While the two notions are clearly equivalent, using the optimal revenue as the benchmark (i.e., 100%) and measuring everything relative to this basis—as GFOR does—seems to come more naturally.

3 Two Independent Goods

We start by proving our first result, Theorem A, stated in the Introduction. Its proof forms the basis of more complex proofs later—including Theorem B, whose significantly more intricate proof is relegated to Appendix A.1. Theorem A can be restated as follows: for every two-good random valuation $X = (Y, Z)$ with Y, Z independent goods (i.e., one-dimensional nonnegative random variables),

$$\text{REV}(X) \leq 2 \cdot \text{SREV}(X) = 2(\text{REV}(Y) + \text{REV}(Z)). \quad (2)$$

Proof of Theorem A. Let $\mu = (q, s)$ be a two-good IC, IR, and NPT mechanism (recall Proposition 6 (iii)); we will prove that its revenue from

X satisfies $R(\mu; X) \leq 2\text{REV}(Y) + 2\text{REV}(Z)$. To do so, we split the revenue into two parts, according to which one of Y and Z is higher, and show that

$$\mathbb{E}[s(Y, Z) \mathbf{1}_{Y \geq Z}] \leq 2\text{REV}(Y), \quad \text{and} \quad (3)$$

$$\mathbb{E}[s(Y, Z) \mathbf{1}_{Z \geq Y}] \leq 2\text{REV}(Z). \quad (4)$$

Since $R(\mu; X) = \mathbb{E}[s(Y, Z)] \leq \mathbb{E}[s(Y, Z) \mathbf{1}_{Y \geq Z}] + \mathbb{E}[s(Y, Z) \mathbf{1}_{Z \geq Y}]$ (the inequality is due to the diagonal $Y = Z$ being counted twice; recall that $s \geq 0$ by NPT), adding (3) and (4) gives (2).

We now prove (3) (which then yields (4) by interchanging Y and Z). For every fixed value $z \geq 0$ of the second good define a mechanism $\mu^z = (q^z, s^z)$ for the first good by replacing the allocation of the second good with an equivalent decrease in payment; that is, the allocation of the first good is unchanged, i.e., $q^z(y) := q_1(y, z)$, and the payment is $s^z(y) := s(y, z) - q_2(y, z) \cdot z$, for every $y \geq 0$. The one-good mechanism μ^z is IC and IR for y , since $\mu = (q, s)$ was IC and IR for (y, z) (for IC, only the constraints (\tilde{y}, z) vs. (y, z) matter; for IR, the buyer payoff function of μ^z is $b^z(y) = b(y, z)$). Now $s(y, z) = s^z(y) + q_2(y, z) \cdot z \leq s^z(y) + z$ (because $z \geq 0$ and $q_2 \leq 1$), and so

$$\mathbb{E}[s(Y, Z) \mathbf{1}_{Y \geq Z} \mid Z = z] = \mathbb{E}[s(Y, z) \mathbf{1}_{Y \geq z}] \leq \mathbb{E}[s^z(Y) \mathbf{1}_{Y \geq z}] + \mathbb{E}[z \mathbf{1}_{Y \geq z}]$$

(the equality uses the independence of Y and Z). The first term is the revenue from a subdomain of values of Y , and so is at most its maximal revenue $\text{REV}(Y)$ by Proposition 6 (iv).¹⁹ As for the second term, we have

$$\mathbb{E}[z \mathbf{1}_{Y \geq z}] = z \cdot \mathbb{P}[Y \geq z] \leq \text{REV}(Y), \quad (5)$$

since posting a price of z , and the buyer buying when $Y \geq z$, constitutes an

¹⁹ μ^z need not satisfy NPT, as s^z may take negative values.

IC and IR mechanism for²⁰ y . Thus

$$\mathbb{E}[s(Y, Z) \mathbf{1}_{Y \geq Z} \mid Z = z] \leq 2\text{REV}(Y)$$

holds for every value z of Z ; taking expectation yields (3), completing the proof. ■

In Appendix A.2 we provide a number of observations arising from this proof.

4 The General Decomposition Result

We generalize the decomposition of the previous section from two goods to two *sets* of goods. Let now Y be a k_1 -dimensional nonnegative random variable, and Z a k_2 -dimensional nonnegative random variable (with $k_1, k_2 \geq 1$). While we assume that the vectors Y and Z are *independent*, we allow for arbitrary interdependence among the coordinates of Y , and likewise for the coordinates of Z .

The main decomposition result is:

Theorem 7 *Let Y and Z be multi-dimensional nonnegative random variables. If Y and Z are independent then*

$$\text{REV}(Y, Z) \leq \text{REV}(Y) + \text{REV}(Z) + \text{BREV}(Y) + \text{BREV}(Z) \quad (6)$$

$$\leq 2(\text{REV}(Y) + \text{REV}(Z)). \quad (7)$$

The second inequality (7) follows immediately from the first (6), because $\text{BREV} \leq \text{REV}$. When Y and Z are one-dimensional, both inequalities become (2) of Theorem A.

We start with the basic argument that uses the “marginal” mechanism on y generated from a mechanism on (y, z) (as in the previous section). For

²⁰We are *not* using here the characterization (1) of optimal one-good mechanisms as posting-price mechanisms, but only the simple fact that these mechanisms are IC and IR.

a k -dimensional random valuation $X = (X_1, \dots, X_k)$, we use the notation

$$\text{Val}(X) := \mathbb{E} \left[\sum_{i=1}^k X_i \right] = \sum_{i=1}^k \mathbb{E}[X_i]$$

for the expected total sum of values (for one-dimensional X we have $\text{Val}(X) = \mathbb{E}[X]$).

Lemma 8 (Marginal Mechanism on Subdomain) *Let Y and Z be multi-dimensional nonnegative random variables, and let $A \subseteq \mathbb{R}^{k_1+k_2}$ be a set of values of (Y, Z) . If Y and Z are independent then*

$$\text{REV}((Y, Z) \mathbf{1}_{(Y,Z) \in A}) \leq \text{REV}(Y) + \text{Val}(Z \mathbf{1}_{(Y,Z) \in A}).$$

Proof. For every z put $A_z := \{y | (y, z) \in A\}$. Take an IC and IR mechanism (q, s) for (y, z) , and fix some value of $z = (z_1, \dots, z_{k_2})$. The induced mechanism on the y goods is IC and IR, but it also hands out quantities of the z goods. If we modify it so that instead of allocating z_j with probability $q_j = q_j(y, z)$, it reduces the buyer's payment by the amount of $q_j z_j$, we are left with an IC and IR mechanism, call it (q^z, s^z) , for the y goods. Now $s(y, z) = s^z(y) + \sum_j q_j z_j \leq s^z(y) + \sum_j z_j$, and so, conditioning on $Z = z$,

$$\begin{aligned} \mathbb{E} [s(Y, Z) \mathbf{1}_{(Y,Z) \in A} | Z = z] &= \mathbb{E} [s(Y, z) \mathbf{1}_{Y \in A_z}] \\ &\leq \mathbb{E} [s^z(Y) \mathbf{1}_{Y \in A_z}] + \mathbb{E} \left[\left(\sum_j z_j \right) \mathbf{1}_{Y \in A_z} \right] \end{aligned}$$

(the equality in the first line is because Y is independent of Z). The first term in the second line is bounded from above by $\text{REV}(Y)$ by Proposition 6 (iv), and the second term is $\mathbb{E} \left[\sum_j (Z_j \mathbf{1}_{(Y,Z) \in A}) | Z = z \right]$; taking expectation over the values z of Z completes the proof. ■

In the case of two goods, i.e., one-dimensional Y and Z , the set of values A for which we bound $\text{Val}(Z \mathbf{1}_{(Y,Z) \in A})$ is the set $A = \{(y, z) : y \geq z\}$.

Lemma 9 (Smaller Value) *Let Y and Z be one-dimensional nonnegative random variables. If Y and Z are independent then*

$$\text{Val}(Z \mathbf{1}_{Y \geq Z}) \leq \text{REV}(Y).$$

Proof. For every value z of Z , setting the price for Y at z yields a revenue of $z \cdot \mathbb{P}[Y \geq z]$, which is therefore at most $\text{REV}(Y)$. Thus $\text{Val}(Z \mathbf{1}_{Y \geq Z}) = \mathbb{E}_{z \sim Z}[\mathbb{E}[Z \mathbf{1}_{Y \geq Z} \mid Z = z]] = \mathbb{E}_{z \sim Z}[z \cdot \mathbb{P}[Y \geq z]] \leq \mathbb{E}_{z \sim Z}[\text{REV}(Y)] = \text{REV}(Y)$.

■

In the multi-dimensional case we take $A = \{(y, z) : \sum_i y_i \geq \sum_j z_j\}$ (where y_i and z_j are the coordinates of y and z , respectively), and get:

Lemma 10 (Smaller Value for Multiple Goods) *Let Y and Z be multi-dimensional nonnegative random variables. If Y and Z are independent then*

$$\text{Val}(Z \mathbf{1}_{\sum_i Y_i \geq \sum_j Z_j}) \leq \text{BREV}(Y).$$

Proof. Apply Lemma 9 to the one-dimensional random variables $\sum_i Y_i$ and $\sum_j Z_j$, and use $\text{REV}(\sum_i Y_i) = \text{BREV}(Y)$. ■

We can now prove our result.

Proof of Theorem 7. We divide the space as follows:

$$\text{REV}(Y, Z) \leq \text{REV}\left((Y, Z) \mathbf{1}_{\sum_i Y_i \geq \sum_j Z_j}\right) + \text{REV}\left((Y, Z) \mathbf{1}_{\sum_j Z_j \geq \sum_i Y_i}\right)$$

(the inequality by NPT; see Proposition 6 (iii)). The first term is at most

$$\text{REV}(Y) + \text{Val}\left(Z \mathbf{1}_{\sum_i Y_i \geq \sum_j Z_j}\right) \leq \text{REV}(Y) + \text{BREV}(Y)$$

by Lemmas 8 and 10. The second term is bounded similarly. ■

See Appendix A.2 for some comments on possible generalizations of this decomposition approach.

5 Separate and Bundled Selling

In this section we compare the revenue obtainable from the two simple mechanisms of selling the goods separately and selling them as one bundle. These mechanisms are simple as they reduce to one-good mechanisms, for which the Myerson (1981) characterization (1) applies. The results below are not only interesting in their own right, but also useful when we make comparisons to the optimal revenue (Theorems C and D; cf. Section 6 below).

One advantage of one-good mechanisms is that revenue is *monotonic* with respect to valuation: increasing the buyer’s values can only increase the seller’s revenue. As natural and appealing as this may sound, monotonicity does *not* extend to the multiple good case; see Hart and Reny (2015a).

Formally, for X and Y real random variables, X is (*first-order*) *stochastically dominated* by Y if for every real p we have $\mathbb{P}[X \geq p] \leq \mathbb{P}[Y \geq p]$; essentially,²¹ what this says is that Y gets higher values than X . We have:

Proposition 11 (Monotonicity for One Good) *Let X and Y be one-good random valuations. If X is stochastically dominated by Y then $\text{REV}(X) \leq \text{REV}(Y)$.*

Proof. $\text{REV}(X) = \sup_p p \cdot \mathbb{P}[X \geq p] \leq \sup_p p \cdot \mathbb{P}[Y \geq p] = \text{REV}(Y)$ by (1). ■

This monotonicity property leads one to consider the highest one-good random valuation with a given revenue. Normalizing the revenue at 1, this is the real random variable V that takes values $V \geq 1$ with probabilities $\mathbb{P}[V \geq p] = 1/p$ for every $p \geq 1$. We refer to a good with random valuation V as an *equal-revenue (ER)* good, and to its distribution, i.e., $F_V(p) = 1 - 1/p$ and $f_V(p) = 1/p^2$ for $p \geq 1$, as the equal-revenue (ER) distribution.²² Indeed, the revenue $\text{REV}(V) = 1$ of an ER good is obtained at *any* posted price $p \geq 1$ (recall (1)).²³ Note that while the revenue of V is finite, its expected value is infinite: $\mathbb{E}[V] = \int_1^\infty p \cdot (1/p^2) dp = \infty$.

²¹One may indeed take X and Y to be defined on the same probability space Ω and to satisfy $X \leq Y$ pointwise, i.e., $X(\omega) \leq Y(\omega)$ for almost every realization $\omega \in \Omega$ (this is called “coupling” of X and Y). See, e.g., Shaked and Shantikumar (2010, Theorem 1.A.1).

²²Also known as the Pareto distribution with index 1 and scale 1; interestingly, V is ER if and only if $1/V$ is Uniform on $(0, 1]$.

²³Moreover, an IC and IR mechanism $\mu = (q, s)$ is optimal for V if and only if it does not sell the good for values below 1, i.e., $q(x) = s(x) = 0$ for all $x < 1$, and

The result (1) for one good may now be restated as follows: $\text{REV}(X) \leq 1$ if and only if X is stochastically dominated by an ER good V (indeed, $\text{REV}(X) \leq 1$ if and only if $1 - F_X(p) \leq 1/p = 1 - F_V(p)$ for all $p \geq 1$). That is, the revenue from a one-good random valuation X is at most 1 if and only if one can increase the values of X and obtain a new random valuation V that is ER-distributed.

The following proposition collects the above observation together with a number of useful results on the ER distribution; the proofs are relegated to Appendix A.3. From now on we use the constant $w \approx 0.278$ to denote the solution of the equation²⁴

$$we^{w+1} = 1.$$

Proposition 12 (i) *Let X be a one-good random valuation, and let $r \geq 0$.*

Then $\text{REV}(X) \leq r$ if and only if X is stochastically dominated by rV where V is an ER valuation.

(ii) *Let V_1, V_2, \dots, V_k be i.i.d.-ER, let $r_1, r_2, \dots, r_k \geq 0$, and put $\bar{r} := (1/k) \sum_{i=1}^k r_i$ for the average of the r_i . Then $\sum_{i=1}^k r_i V_i$ is stochastically dominated by $\sum_{i=1}^k \bar{r} V_i$.*

(iii) *Let V_1 and V_2 be i.i.d.-ER. Then*

$$\text{BREV}(V_1, V_2) = 2(w + 1) \approx 2.56.$$

(iv) *There exist constants $c_1 > 0$ and $c_2 < \infty$ such that for all $k \geq 2$ and V_1, V_2, \dots, V_k i.i.d.-ER,*

$$c_1 k \log k \leq \text{BREV}(V_1, V_2, \dots, V_k) \leq c_2 k \log k.$$

Remarks. (a) We will see below (Corollary 17) that bundling is in fact optimal for two i.i.d.-ER goods, and so (iii) will become $\text{REV}(V_1, V_2) = \text{BREV}(V_1, V_2)$

$\sup_{x \geq 1} q(x) = \lim_{x \rightarrow \infty} q(x) = 1$ (but is otherwise arbitrary for $x \geq 1$). Also, the ER distribution is the only distribution (up to rescaling) for which the “virtual valuation” of Myerson (1981) (used, for instance, when there are multiple buyers) vanishes everywhere in the support: $x - (1 - F(x))/f(x) = 0$ for all $x \geq 1$.

²⁴Thus $we^w = 1/e$, and so $w = W(1/e)$ where W is the so-called “Lambert- W ” function.

$$=\text{REV}(V_1 + V_2) = 2(w + 1).$$

(b) The fact that the revenue is *not* monotonic for multiple goods (Hart and Reny 2015a) foils the following natural attempt to estimate GFOR for separate selling. For concreteness, consider two i.i.d. goods X_1 and X_2 , without loss of generality normalized so that $\text{REV}(X_1) = \text{REV}(X_2) = 1$. Let V_1, V_2 be i.i.d.-ER. Then each X_i is stochastically dominated by V_i , and so $X = (X_1, X_2)$ is stochastically dominated by $V = (V_1, V_2)$. However, we *cannot* deduce from this that $\text{REV}(X) \leq \text{REV}(V) = 2(w + 1)$ (see Remark (a) above)—which would have given a better bound of $1/(w + 1) \approx 0.78$, and with a much simpler proof, for GFOR(SEPARATE) in this case (cf. Theorem B and its proof in Appendix A.1).

Using ER goods allows us to compare the separate selling revenue to the bundling revenue.

Proposition 13 (i) *For any two independent goods X_1, X_2 ,*

$$\text{SREV}(X_1, X_2) \geq \frac{1}{w + 1} \text{BREV}(X_1, X_2) \approx 0.78 \cdot \text{BREV}(X_1, X_2).$$

(ii) *There exists a constant $c > 0$ such that for any $k \geq 2$ and any k independent goods X_1, X_2, \dots, X_k ,*

$$\text{SREV}(X_1, X_2, \dots, X_k) \geq \frac{c}{\log k} \text{BREV}(X_1, X_2, \dots, X_k).$$

Proof. Put $r_i := \text{REV}(X_i)$ and $\bar{r} := (1/k) \sum_i r_i = (1/k) \text{SREV}(X)$, and let V_1, \dots, V_k be k i.i.d.-ER goods. Using Proposition 12 (i) and (ii): each X_i is stochastically dominated by $r_i V_i$, hence $\sum_i X_i$ is stochastically dominated

by²⁵ $\sum_i r_i V_i$, which is in turn dominated by $\bar{r} \sum_i V_i$. Therefore

$$\begin{aligned} \text{BREV}(X_1, \dots, X_k) &= \text{REV} \left(\sum_{i=1}^k X_i \right) \leq \text{REV} \left(\bar{r} \sum_{i=1}^k V_i \right) \\ &= \bar{r} \text{REV} \left(\sum_{i=1}^k V_i \right) = \bar{r} \text{BREV}(V_1, \dots, V_k) \end{aligned}$$

(the inequality is by monotonicity for one good, Proposition 11), and then the two results follow from Proposition 12 (iii) and (iv), respectively. ■

Taking the goods X_i to be ER goods shows that $1/(w+1)$ and $c/\log k$ above are both tight (cf. Proposition 12 (iii) and (iv)).

We conclude with comparisons in the other direction: the bundled revenue as a fraction of the separate revenue; see Appendix A.4 for additional results.

Proposition 14 (i) *For any $k \geq 1$ and any k independent goods X_1, X_2, \dots, X_k ,*

$$\text{BREV}(X_1, X_2, \dots, X_k) \geq \frac{1}{k} \text{SREV}(X_1, X_2, \dots, X_k).$$

(ii) *For any $k \geq 1$ and any k i.i.d. goods X_1, X_2, \dots, X_k ,*

$$\text{BREV}(X_1, X_2, \dots, X_k) \geq \frac{1}{4} \text{SREV}(X_1, X_2, \dots, X_k).$$

Proof. **(i)** For every i we have $X_i \leq \sum_j X_j$ and so $\text{REV}(X_i) \leq \text{REV}(\sum_j X_j) = \text{BREV}(X_1, \dots, X_k)$; summing over j yields $\sum_j \text{REV}(X_j) \leq k \text{BREV}(X_1, \dots, X_k)$.

(ii) Let p be an optimal one-good price for each X_i and put $\alpha := \mathbb{P}[X_i \geq p]$; thus $\text{REV}(X_i) = p\alpha$. We separate between two cases. If $k\alpha \leq 1$ then consider setting the bundle price at p ; the probability that the buyer will

²⁵We use here the following fact: if X_i is stochastically dominated by Y_i for every i , then $X_1 + \dots + X_k$ is stochastically dominated by $Y_1 + \dots + Y_k$ (this is immediate when all the random variables are defined on the same probability space and $X_i \leq Y_i$ pointwise for every i —cf. the coupling in footnote 21—because then $\sum X_i \leq \sum Y_i$); see, e.g., Shaked and Shantikumar (2010, Theorem 1.A.3.(b)).

buy is

$$\begin{aligned}
\mathbb{P} \left[\sum_i X_i \geq p \right] &\geq \mathbb{P} [X_i \geq p \text{ for some } i] = \mathbb{P} \left[\bigcup_i [X_i \geq p] \right] \\
&\geq \sum_i \mathbb{P} [X_i \geq p] - \sum_{i < j} \mathbb{P} [X_i \geq p, X_j \geq p] \\
&= k\alpha - \binom{k}{2} \alpha^2 \geq \frac{1}{2} k\alpha,
\end{aligned}$$

and so the revenue will be at least $pk\alpha/2 \geq k \text{REV}(X_i)/2$. If $k\alpha \geq 1$ then consider setting the bundle price at $p \lfloor k\alpha \rfloor$. Since the median in the Binomial(k, α) distribution is at least $\lfloor k\alpha \rfloor$, the probability that the buyer will buy is at least $1/2$, and so the revenue will be at least $p \lfloor k\alpha \rfloor / 2 \geq pk\alpha/4 = k \text{REV}(X_i)/4$. ■

While the constant $1/k$ in (i) is tight, the $1/4$ in (ii) is not (we have not attempted to optimize it); see Example 27 in Appendix A.4 and Example 32 in Appendix A.5.

6 k Independent Goods

We now prove the two main results on $k \geq 2$ goods, Theorems C and D stated in the Introduction, using our general decomposition result of Theorem 7.

We start with separate selling. Viewing $2k$ goods as two sets of k goods each and using (7) one can easily get by induction that $\sum_{i=1}^k \text{REV}(X_i) \geq (1/k) \text{REV}(X_1, \dots, X_k)$, as follows:

$$\begin{aligned}
\text{REV}(X_1, \dots, X_{2k}) &\leq 2(\text{REV}(X_1, \dots, X_k) + \text{REV}(X_{k+1}, \dots, X_{2k})) \\
&\leq 2 \left(k \sum_{i=1}^k \text{REV}(X_i) + k \sum_{i=k+1}^{2k} \text{REV}(X_i) \right) \\
&= 2k \sum_{i=1}^{2k} \text{REV}(X_i).
\end{aligned}$$

However, using the stronger inequality (6), together with the relations we

have shown in the previous section between the bundling and the separate revenues, gives us the better bound of $c/\log^2 k$ (instead of $1/k$) of Theorem C.

Proof of Theorem C. We will first prove by induction that $\text{REV}(X_1, \dots, X_k) \leq (1/c') \log_2^2 k \sum_{i=1}^k \text{REV}(X_i)$ for every $k \geq 2$ that is a power of 2, where $c' := \min\{c, 1/2\} > 0$ with $c > 0$ given by Proposition 13 (ii). This inequality holds for $k = 2$ by Theorem A (since $c' \leq 1/2$). For $k \geq 4$ we apply Theorem 7 to $Y = (X_1, \dots, X_k)$ and $Z = (X_{k+1}, \dots, X_{2k})$, to get

$$\begin{aligned} \text{REV}(X_1, \dots, X_{2k}) &\leq \text{REV}(X_1, \dots, X_k) + \text{REV}(X_{k+1}, \dots, X_{2k}) \\ &\quad + \text{BREV}(X_1, \dots, X_k) + \text{BREV}(X_{k+1}, \dots, X_{2k}). \end{aligned} \quad (8)$$

First, using Proposition 13 (ii) (and $c' \leq c$) on each of the BREV terms shows that their sum is bounded by $(1/c') \log_2^2 k \sum_{i=1}^{2k} \text{REV}(X_i)$. Second, using the induction hypothesis on each of the REV terms shows that their sum is bounded by $(1/c') \log_2^2 k \sum_{i=1}^{2k} \text{REV}(X_i)$. Now $\log_2 k + \log_2^2 k \leq \log_2^2(2k)$, and so adding the two bounds gives the result.

Next, when $2^{m-1} < k < 2^m$ we can “pad” to 2^m goods by adding goods that have value identically zero, and so do not contribute anything to the revenue; this at most doubles k . ■

In Appendix 5 we show that bundling may, by contrast, extract only a $1/k$ fraction of the optimal revenue. However, bundling does much better for identically distributed goods, and in fact we have a tighter result, Theorem D, with $\log k$ instead of k .

Proof of Theorem D. Let X_i be i.i.d., and put $R_k := \text{REV}(X_1, \dots, X_k)$ and $B_k := \text{BREV}(X_1, \dots, X_k)$. We want to show that there is a finite $c > 0$ such that $R_k \leq c \log k B_k$ for all $k \geq 2$. If $k \geq 2$ is a power of 2 we apply Theorem 7 inductively to obtain $R_k \leq 2B_{k/2} + 4B_{k/4} + \dots + (k/2)B_2 + kB_1 + kR_1$. Each of the $\log_2 k + 1$ terms in this sum is of the form $(k/\ell)B_\ell = (k/\ell)\text{REV}(X_1 + \dots + X_\ell)$, and is thus bounded from above by $4B_k$ (apply Proposition 14 (ii) to k/ℓ i.i.d. random variables each distributed as $X_1 + \dots + X_\ell$). Altogether we have $R_k \leq 4(\log_2 k + 1)B_k$.

When $2^{m-1} < k < 2^m$ we have $R_k \leq R_{2^m}$ and $B_k \geq B_{2^{m-1}}$ (adding goods can only increase the revenue: take the optimal mechanism for the original set of goods and extend it so that it ignores the additional goods), and $B_{2^m} \leq 2(w+1)B_{2^{m-1}} \leq 2(w+1)B_k$ (apply Proposition 13 (i) to the two i.i.d. random variables $X_1 + \dots + X_{2^{m-1}}$ and $X_{2^{m-1}+1} + \dots + X_{2^m}$), which together with the above inequality for 2^m goods yields $R_k \leq 8(w+1)(\log_2 k + 2)B_k$. ■

7 Additional Results

7.1 Upper Bound on GFOR for Two Goods

Our results for two goods give lower bounds on GFOR (50% and 73% for selling separately two independent goods and two i.i.d. goods, respectively). Now what about upper bounds? That is, how high can GFOR(SEPARATE) actually be? The best estimate we have is that it cannot exceed approximately 78%: there exist two i.i.d. goods where selling separately yields only that fraction of the optimal revenue (while GFOR may well be lower for independent goods than for the more restricted i.i.d. goods, we have not found a better example—i.e., with a lower fraction—in the former class).

Proposition 15 *In the case of two independent goods, selling separately cannot guarantee more than 78% of the optimal revenue; i.e.,*

$$\begin{aligned} & \text{GFOR}(\text{SEPARATE}; 2 \text{ independent goods}) \\ & \leq \text{GFOR}(\text{SEPARATE}; 2 \text{ i.i.d. goods}) \leq \frac{1}{w+1} \approx 0.78. \end{aligned}$$

Proof. Let $V = (V_1, V_2)$ with V_1 and V_2 two i.i.d.-ER goods. The revenue from selling separately is $\text{SREV}(V) = \text{REV}(V_1) + \text{REV}(V_2) = 2$, whereas, as shown in the next section (Corollary 17), the optimal revenue is $\text{REV}(V) = 2(w+1)$ (obtained by bundling). ■

7.2 When Bundling Is Optimal

Interestingly, we have identified a class of two-good i.i.d. distributions for which bundling is optimal.

Theorem 16 *Let F be a continuous one-good distribution with values in $[a, \infty)$ for some $a > 0$, and density function f that is differentiable and satisfies*

$$xf'(x) + \frac{3}{2}f(x) \leq 0 \quad (9)$$

for every $x > a$. Then bundling is optimal for two i.i.d.- F goods X_1, X_2 :

$$\text{REV}(X_1, X_2) = \text{BREV}(X_1, X_2) = \text{REV}(X_1 + X_2).$$

Theorem 16 is proved in Appendix A.6. Condition (9) is equivalent to $(x^{3/2}f(x))' \leq 0$, i.e., $x^{3/2}f(x)$ is nonincreasing in x (the support of f is thus either some finite interval $[a, b]$ or the half-line $[a, \infty)$). When $f(x) = cx^{-\gamma}$, (9) holds whenever $\gamma \geq 3/2$. In particular, the ER distribution (where $\gamma = 2$) satisfies (9), and so does any general Pareto distribution with index $\alpha \geq 1/2$. Together with Proposition 12 (iii) we thus get:

Corollary 17 *Let V_1, V_2 be two i.i.d.-ER goods. Then*

$$\text{REV}(V_1, V_2) = \text{BREV}(V_1, V_2) = 2(w + 1) \approx 2.56.$$

7.3 Multiple Buyers

Up to now we have been dealing with a single buyer, but our result for two independent goods turns out to hold also when there are multiple buyers. Unlike the simple decision-theoretic problem facing a single buyer, we now have a multi-person game among the buyers. Two main notions of equilibrium are considered: *dominant strategy* equilibrium and *Bayesian Nash* equilibrium (corresponding to “ex-post” and “interim” implementations, respectively); see Appendix A.7 for details. Our result holds for both concepts.

Theorem 18 *In the case of n independent buyers and two goods, if the random valuations of the two goods are independent, then selling each good separately using its optimal one-good mechanism guarantees at least 50% of the optimal revenue:*

$$\text{GFOR}(\text{SEPARATE}) \geq \frac{1}{2};$$

this holds when the optimal revenue is taken throughout²⁶ with respect to either dominant strategy implementation or Bayesian Nash implementation.

That is, let the one-dimensional random variable $X_i^j \geq 0$ denote the value of good i to buyer j , for $i = 1, 2$ and $j = 1, \dots, n$. Write $X^j = (X_1^j, X_2^j) \in \mathbb{R}_+^2$ for the random valuation vector of buyer j for both goods, and $X_i = (X_i^j)_{j=1, \dots, n} \in \mathbb{R}_+^n$ for the vector of values of all buyers for good i . *Independent buyers* means that the random vectors X^1, X^2, \dots, X^n are independent; *independent goods* means that the random vectors X_1 and X_2 are independent.²⁷ Theorem 18 is proved in Appendix A.7, which also contains the precise notations and statements; the proof is again a generalization of the proof of Theorem A for one buyer (Section 3).

Remarks. (a) *Dependent buyers and dominant strategy implementation.* In the dominant strategy case, our proof does not use the independence between the buyers' random valuations; thus $\text{GFOR}(\text{SEPARATE}) \geq 1/2$ holds under dominant strategy implementation for two independent goods and any number of buyers, whether independent or not; see Theorem 33 in Appendix A.7.

(b) *Dependent buyers and Bayesian Nash implementation.* In the Bayesian Nash case our proof does not extend when the buyers are not independent (see Appendix A.7). However, for this case Crémer and McLean (1988) show that, under a certain general “correlation-between-buyers” condition, the seller can extract all the surplus from any single good: $\text{REV}(X_i) = \mathbb{E} [\max_{1 \leq j \leq n} X_i^j]$. Since the most that the seller can extract from the two

²⁶I.e., for the two goods, as well as for each good separately.

²⁷Independent buyers together with independent goods means that the $2n$ random variables X_i^j are all independent.

goods is, by IR,²⁸ $\mathbb{E} [\max_j X_1^j] + \mathbb{E} [\max_j X_2^j]$, it follows that in this case $\text{REV}(X_1, X_2) = \text{REV}(X_1) + \text{REV}(X_2)$, and so $\text{GFOR}(\text{SEPARATE}) = 1$. We do not know what $\text{GFOR}(\text{SEPARATE})$ is when the buyers are neither independent nor satisfy the Crémer–McLean condition.

8 Open Problems

Many interesting problems remain open. As attested by the long time that has passed since Myerson’s (1981) work in the one-good case, characterizing the optimal mechanisms in the multiple-goods case—even when there are just two goods—is an extremely difficult problem.²⁹ While the general problem appears very complex, one may well be able to obtain results for certain useful classes of random valuations and mechanisms. Following are some specific questions that arise from our study:

1. Characterize distributions where separate selling is optimal (cf. Theorem 16 for bundling).
2. Provide bounds for $\text{GFOR}(\text{DETERMINISTIC})$, the fraction of optimal revenue that is guaranteed by mechanisms that do not use randomizations. In addition, characterize distributions where deterministic mechanisms are optimal.³⁰
3. Tighten the bounds on $\text{GFOR}(\text{SEPARATE})$. While the gap in the i.i.d. case (73% vs. 78%) is quite small, we do not know what the right value is; also, is GFOR in the independent case in fact lower than in the i.i.d. case? Is 50% the right bound?
4. Evaluate $\text{GFOR}(\text{SEPARATE})$ for Bayesian Nash implementation when there are multiple buyers that are neither independent nor satisfy the

²⁸Indeed (see Appendix A.7 for notations), $b^j(x) \geq 0$ implies that $s^j(x) \leq q^j(x) \cdot x^j$, and so $\sum_j s^j(x) \leq \sum_j q^j(x) \cdot x^j = \sum_i \sum_j q_i^j(x) x_i^j \leq \sum_i \max_j x_i^j$.

²⁹Recall footnote 5 on *conceptual complexity*.

³⁰The recent work of Babaioff, Immorlica, Lucier, and Weinberg (2014) implies in particular that $\text{GFOR}(\text{DETERMINISTIC})$ is bounded from below by a constant that is independent of the number of goods.

Crémer–McLean condition (see Remark (b) in Section 7.3).

5. Find simple mechanisms different from separate selling that can guarantee a larger fraction of the optimal revenue.
6. Study the case of two or more goods that are not necessarily independent (see Hart and Nisan 2013).³¹
7. Obtain useful ways to quantify the complexity (vs. simplicity) of mechanisms, and analyze the tradeoffs between complexity and revenue (see Hart and Nisan 2013 for such an approach: “menu complexity”).

A Appendix

A.1 Proof for Two I.I.D. Goods

In this appendix we prove Theorem B, stated in the Introduction, which says that selling two i.i.d. goods separately yields at least $e/(e+1)$ of the optimal revenue. The proof follows a line of argument similar to that of the proof of Theorem A in Section 3, but is more intricate as we make all the estimates much tighter.

Proof of Theorem B. Let $X = (Y, Z)$, where Y and Z are i.i.d. nonnegative one-dimensional random variables, and let $r := \text{REV}(Y) = \text{REV}(Z) = \sup_{t \geq 0} t \cdot G(t)$ be the revenue from each good separately, where $G(t) := \mathbb{P}[Y \geq t]$. We want to prove that

$$\text{REV}(Y, Z) \leq \frac{e+1}{e}(\text{REV}(Y) + \text{REV}(Z)) = 2 \left(1 + \frac{1}{e}\right) r.$$

Take a two-good IC and IR mechanism $\mu = (q, s)$ with buyer payoff function b (i.e., $b(x) = q(x) \cdot x - s(x)$ for all x). Without loss of generality assume that it also satisfies NPT, i.e., $s(x) \geq 0$ for all x , and $b(0, 0) = s(0, 0) = 0$ (recall Proposition 6 (iii)). Since X is symmetric we will also

³¹Where it is shown that no simple class of mechanisms can guarantee any positive fraction of the optimal revenue; i.e., GFOR=0.

assume that μ is symmetric, i.e., $q_1(y, z) = q_2(z, y)$ and $s(y, z) = s(z, y)$ —and thus $b(y, z) = b(z, y)$ —for all $y, z \geq 0$. Indeed, μ can be replaced by its “symmetrization” $\bar{\mu} = (\bar{q}, \bar{s})$ given by $\bar{q}_1(y, z) = \bar{q}_2(z, y) := (q_1(y, z) + q_2(z, y))/2$ and $\bar{s}(y, z) = \bar{s}(z, y) := (s(y, z) + s(z, y))/2$ for all $y, z \geq 0$, which also satisfies IC, IR, and NPT, and yields the same revenue, $\mathbb{E}[\bar{s}(Y, Z)] = \mathbb{E}[s(Y, Z)] = \mathbb{E}[s(Z, Y)]$ (because Y, Z are i.i.d.).

For every $t \geq 0$ put $\Phi(t) := b(t, t)/2$ and $\varphi(t) := q_1(t, t) = q_2(t, t)$; Proposition 5 implies that Φ is a convex function, $\varphi(t) = \Phi'(t)$ almost everywhere, and $\Phi(u) = \int_0^u \varphi(t) dt$ (formally, use Corollary 24.2.1 in Rockafellar 1970 and $\Phi(0) = b(0, 0) = 0$).

Consider first the region $Y \geq Z$. For each fixed $z \geq 0$ such that $\mathbb{P}[Y \geq z] > 0$ define a mechanism $\mu^z = (q^z, s^z)$ for the first good by $q^z(y) := q_1(y, z)$ and $s^z(y) := s(y, z) - q_2(y, z) \cdot z$ for every $y \geq 0$; the buyer’s payoff remains the same: $b^z(y) = b(y, z)$. The mechanism μ^z is IC and IR for y , since μ is IC and IR for (y, z) . Let Y^z denote the random variable Y conditional on the event $Y \geq z$, and consider the revenue $R(\mu^z; Y^z) = \mathbb{E}[s^z(Y^z)] = \mathbb{E}[s^z(Y)|Y \geq z]$ of μ^z from Y^z . We have $Y^z \geq z$, $q^z(z) = \varphi(z)$, and $s^z(z) = s(z, z) - q_2(z, z) \cdot z = q_1(z, z) \cdot z - b(z, z) = z\varphi(z) - 2\Phi(z)$, and so applying Lemma 19 below to Y^z yields

$$\mathbb{E}[s^z(Y)|Y \geq z] \leq (1 - \varphi(z))\text{REV}(Y^z) + z\varphi(z) - 2\Phi(z). \quad (10)$$

Since $\mathbb{P}[Y^z \geq t] = \mathbb{P}[Y \geq t] / \mathbb{P}[Y \geq z] = G(t) / \mathbb{P}[Y \geq z]$ for all $t \geq z$, we get from (1) that

$$\text{REV}(Y^z) = \sup_{t \geq 0} t \cdot \mathbb{P}[Y^z \geq t] = \sup_{t \geq z} t \cdot \frac{G(t)}{\mathbb{P}[Y \geq z]} \leq \frac{\sup_{t \geq 0} t \cdot G(t)}{\mathbb{P}[Y \geq z]} = \frac{r}{\mathbb{P}[Y \geq z]}$$

(recall that $r = \text{REV}(Y)$). Substitute this in (10), and multiply it by $\mathbb{P}[Y \geq z]$, to get

$$\mathbb{E}[s^z(Y)\mathbf{1}_{Y \geq z}] \leq r(1 - \varphi(z)) + (z\varphi(z) - 2\Phi(z))\mathbb{P}[Y \geq z]$$

for all $z \geq 0$ (trivially including those where $\mathbb{P}[Y \geq z] = 0$). Taking expect-

tation over the values z of Z :

$$\mathbb{E} [s^Z(Y)\mathbf{1}_{Y \geq Z}] \leq r(1 - \mathbb{E} [\varphi(Z)]) + \mathbb{E} [(Z\varphi(Z) - 2\Phi(Z))\mathbf{1}_{Y \geq Z}]. \quad (11)$$

Now $s(y, z) = s^z(y) + q_2(y, z)z \leq s^z(y) + q_2(y, y)z = s^z(y) + z\varphi(y)$ (use $z \geq 0$ and the monotonicity of $q_2(y, z) = b_z(y, z)$ in z , again from the convexity of b), which together with (11) yields

$$\begin{aligned} \mathbb{E} [s(Y, Z)\mathbf{1}_{Y \geq Z}] &\leq \mathbb{E} [s^Z(Y)\mathbf{1}_{Y \geq Z}] + \mathbb{E} [Z\varphi(Y)\mathbf{1}_{Y \geq Z}] \\ &\leq r(1 - \mathbb{E} [\varphi(Z)]) + \mathbb{E} [(Z\varphi(Y) + Z\varphi(Z) - 2\Phi(Z))\mathbf{1}_{Y \geq Z}] \\ &= r(1 - \mathbb{E} [\varphi(Z)]) + \mathbb{E} [(\Lambda\varphi(Y) + \Lambda\varphi(Z) - 2\Phi(\Lambda))\mathbf{1}_{Y \geq Z}] \end{aligned}$$

where we put $\Lambda := \min\{Y, Z\}$.

Consider next the region $Z > Y$. Interchanging Y and Z and using $Z > y$ instead of $Z \geq y$ throughout gives

$$\mathbb{E} [s(Y, Z)\mathbf{1}_{Z > Y}] \leq r(1 - \mathbb{E} [\varphi(Y)]) + \mathbb{E} [(\Lambda\varphi(Z) + \Lambda\varphi(Y) - 2\Phi(\Lambda))\mathbf{1}_{Z > Y}].$$

Adding the last two inequalities yields

$$\begin{aligned} \mathbb{E} [s(Y, Z)] &\leq r(2 - \mathbb{E} [\varphi(Y)] - \mathbb{E} [\varphi(Z)]) \\ &\quad + \mathbb{E} [\Lambda\varphi(Y) + \Lambda\varphi(Z) - 2\Phi(\Lambda)]. \end{aligned}$$

Because Y and Z are i.i.d. we have $\mathbb{E} [\varphi(Y)] = \mathbb{E} [\varphi(Z)]$ and $\mathbb{E} [\Lambda\varphi(Y)] = \mathbb{E} [\Lambda\varphi(Z)]$, and so

$$\mathbb{E} [s(Y, Z)] \leq 2r - 2r\mathbb{E} [\varphi(Y)] + 2\mathbb{E} [W] \quad (12)$$

where $W := \Lambda\varphi(Y) - \Phi(\Lambda)$.

We want to bound (12) from above. This expression is affine in φ (recall that $\Phi(u) = \int_0^u \varphi(t) dt$), which is a real nondecreasing function with values in $[0, 1]$. Since every such function lies in the closed convex hull of the extreme functions $\varphi = \mathbf{1}_{[p, \infty)}$ for all³² $0 \leq p \leq \infty$, it suffices to bound (12) for these

³²Put weight $\theta_p = \varphi'(p) \geq 0$ on $\mathbf{1}_{[p, \infty)}$ for (almost) every $p > 0$, weight $\theta_0 = \varphi(0)$ on

extreme functions.

Consider such an extreme $\varphi = \mathbf{1}_{[p, \infty)}$ with $p \geq 0$; then $\Phi(u) = \int_0^u \varphi(t) dt = \max\{u - p, 0\}$. Substituting in the definition of W yields

$$W = \begin{cases} \Lambda - (\Lambda - p) = p, & \text{if } Y \geq p \text{ and } Z \geq p, \\ Z - 0 = Z, & \text{if } Y \geq p \text{ and } Z < p, \\ 0 - 0 = 0, & \text{if } Y < p. \end{cases}$$

Thus

$$\begin{aligned} \mathbb{E}[W] &= p \mathbb{P}[Y \geq p] \mathbb{P}[Z \geq p] + \mathbb{P}[Y \geq p] \mathbb{E}[Z \mathbf{1}_{Z < p}] \\ &= \mathbb{P}[Y \geq p] (\mathbb{E}[p \mathbf{1}_{Z \geq p}] + \mathbb{E}[Z \mathbf{1}_{Z < p}]) \\ &= G(p) \mathbb{E}[\min\{Z, p\}] \end{aligned}$$

(we have used the fact that Y and Z are independent and $\min\{Z, p\} = p \mathbf{1}_{Z \geq p} + Z \mathbf{1}_{Z < p}$). Together with $\mathbb{E}[\varphi(Y)] = \mathbb{E}[\mathbf{1}_{Y \in [p, \infty)}] = \mathbb{P}[Y \geq p] = G(p)$, (12) becomes

$$\mathbb{E}[s(Y, Z)] \leq 2r - 2rG(p) + 2G(p) \mathbb{E}[\min\{Z, p\}] = 2(r + \zeta(p)) \quad (13)$$

where we put $\zeta(p) := G(p) (\mathbb{E}[\min\{Z, p\}] - r)$. If $p \geq r$ then

$$\begin{aligned} \mathbb{E}[\min\{Z, p\}] &= \int_0^\infty \mathbb{P}[\min\{Z, p\} \geq u] du = \int_0^p \mathbb{P}[Z \geq u] du \\ &= \int_0^p G(u) du \leq \int_0^r 1 du + \int_r^p \frac{r}{u} du = r + r \ln\left(\frac{p}{r}\right), \end{aligned}$$

where the inequality follows from $G(u) \leq 1$ and $G(u) \leq r/u$ (because $r = \sup_{u \geq 0} u \cdot G(u)$). Therefore

$$\zeta(p) \leq G(p) r \ln\left(\frac{p}{r}\right) \leq \frac{r}{p} r \ln\left(\frac{p}{r}\right) = r \frac{\ln q}{q},$$

$\mathbf{1}_{[0, \infty)} \equiv 1$, and the remaining weight $\theta_\infty = 1 - \int_0^\infty \varphi'(p) dp - \varphi(0) = 1 - \varphi(\infty) \geq 0$ on $\mathbf{1}_{[\infty, \infty)} \equiv 0$; cf. Manelli and Vincent (2007, Lemma 4), where it is also shown how the one-good result (1) easily follows from this claim.

where $q := p/r \geq 1$. Since $\max_q (\ln q)/q = 1/e$ (attained at $q = e$), it follows that $\zeta(p) \leq r/e$ for all $p \geq r$. If $p \leq r$ then $\zeta(p) \leq 0$ (since $\mathbb{E}[\min\{Z, p\}] \leq p \leq r$), and so altogether $\zeta(p) \leq r/e$ for all $p \geq 0$. Therefore $\mathbb{E}[s(Y, Z)] \leq 2r(1 + 1/e)$ (recall (13)), which completes the proof. ■

The auxiliary result that we have used is:

Lemma 19 *Let X be a one-good random valuation that takes values $X \geq x_0$ for some $x_0 \geq 0$. Then for every IC and IR mechanism $\mu = (q, s)$ we have*

$$R(\mu; X) = \mathbb{E}[s(X)] \leq (1 - q(x_0)) \text{REV}(X) + s(x_0). \quad (14)$$

Proof. The function q is nondecreasing (because q is the derivative of the buyer payoff function b , which is convex), and so $q(x) \geq q(x_0)$ for all $x \geq x_0$.

If $q(x_0) = 1$ then $q(x) = 1$ for all $x \geq x_0$, hence $s(x) = s(x_0)$ for all $x \geq x_0$ by IC; therefore $\mathbb{E}[s(X)] = s(x_0)$ and (14) holds as an equality.

If $q(x_0) < 1$ then we define a new mechanism by rescaling q so that it uses the full range from 0 to 1 (instead of $q(x_0)$ to 1). Specifically, define $\hat{\mu} = (\hat{q}, \hat{s})$ by $\hat{q}(x) := (q(x) - q(x_0))/\lambda$ and $\hat{s}(x) := (s(x) - s(x_0))/\lambda$, where $\lambda := 1 - q(x_0) > 0$. It is immediate to verify that $\hat{\mu}$ is an IC and IR mechanism (for IC, $[\hat{q}(x) \cdot x - \hat{s}(x)] - [\hat{q}(\tilde{x}) \cdot \tilde{x} - \hat{s}(\tilde{x})] = ([q(x) \cdot x - s(x)] - [q(\tilde{x}) \cdot \tilde{x} - s(\tilde{x})]) / \lambda \geq 0$; for IR, the resulting buyer payoff function \hat{b} satisfies $\hat{b}(x_0) = \hat{q}(x_0) \cdot x_0 - \hat{s}(x_0) = 0$). Therefore $\text{REV}(X) \geq \mathbb{E}[\hat{s}(X)] = (\mathbb{E}[s(X)] - s(x_0))/\lambda$; multiplying by λ yields (14). ■

A.2 Some Comments on Decomposition

We provide here a number of remarks related to the decompositions of Theorems A, B, and 7.

Remarks. (a) In the proof of Theorem A in Section 3: For every fixed z , applying the one-dimensional mechanism μ^z to the whole range of Y , rather than to $Y \geq z$, yields $\mathbb{E}[s(Y, z)] \leq \text{REV}(Y) + z$ (recall that $s(y, z) \leq s^z(y) + z$), and so, taking expectation over the values z of Z , and then maximizing over

the mechanisms μ , we get³³ $\text{REV}(Y, Z) \leq \text{REV}(Y) + \mathbb{E}[Z]$. Unfortunately, this inequality does not suffice: $\mathbb{E}[Z]$ may well be infinite, even when $\text{REV}(Z)$ is finite (as is the case, e.g., for the Equal-Revenue (**ER**) distribution (defined in Section 5)). This explains the need to split the domain into the two regions, $Y \geq Z$ and $Y \leq Z$, which allows us to bound the resulting expectation terms (see (5)).

(b) The proof of Theorem A also implies that $\text{REV}(Y) + \text{REV}(Z) \geq \mathbb{E}[\min\{Y, Z\}]$ (take expectation of (5) over the values z of Z , interchange Y and Z , and add the two resulting inequalities). Thus, while in the single-good case one *cannot* guarantee any positive fraction of the expected value as revenue (take again the **ER** distribution, with infinite expectation and revenue 1), in the case of two independent goods one can at least guarantee the expectation of the minimum of the values of the two goods. A mechanism that yields a revenue of $\mathbb{E}[\min\{Y, Z\}]$ consists of posting the random prices p_1 for the y good and p_2 for the z good, where p_1 and p_2 are independent random variables, p_1 is distributed like Z , and p_2 is distributed like Y ; this is a randomized separate mechanism.³⁴

(c) The decomposition of Section 4 holds in more general setups than the totally additive valuation of this paper (where the value to the buyer of the outcome $q \in [0, 1]^k$ is $\sum_i q_i x_i$). Indeed, consider an abstract mechanism-design problem with a set of alternatives A , valued by the buyer according to a function $w : A \rightarrow \mathbb{R}_+^k$ (that he knows, whereas the seller knows only that the function w is drawn from a certain distribution); assume also that results such as those in Proposition 6 hold. If the set of alternatives A is in fact a product $A = A_1 \times A_2$ with the valuation additive between the two

³³When Y and Z are not necessarily independent, this becomes $\text{REV}(Y, Z) \leq \mathbb{E}[\text{REV}(Y|Z)] + \mathbb{E}[Z]$, where $(Y|Z)$ is the random variable Y conditional on the value of Z , and the expectation is over (the values of) Z .

³⁴The inequality $\text{REV}(Y) + \text{REV}(Z) \geq \mathbb{E}[\min\{Y, Z\}]$ is tight, as it becomes an equality when Y, Z are i.i.d.-**ER** goods. It does not hold when Y and Z are not independent (for an extreme case take the fully correlated case with $Y = Z$ being **ER**); the correct inequality here is $\mathbb{E}[\text{REV}(Y|Z)] + \mathbb{E}[\text{REV}(Z|Y)] \geq \mathbb{E}[\min\{Y, Z\}]$. All this generalizes to any $k \geq 2$ independent goods, where we obtain $\sum_i \text{REV}(X_i) \geq \mathbb{E}[(m-1)X^{(m)}]$ for every $m = 1, 2, \dots, k$ (of course, only $m \geq 2$ matters) with $X^{(m)}$ denoting the m -th order statistic of X_1, \dots, X_k (thus $X^{(1)} = \max_i X_i$ and $X^{(k)} = \min_i X_i$).

sets, i.e., $w(a_1, a_2) = w_1(a_1) + w_2(a_2)$, with w_1 distributed according to Y and w_2 according to Z , then Theorem 7 holds as stated. The proof now uses $\text{Val}(Z) = \mathbb{E}[\sup_{a_2 \in A_2} w_2(a_2)]$ (which, in our case, where $A_2 = [0, 1]^{k_2}$ and $w_2(q) = \sum_j q_j z_j$, is indeed $\text{Val}(Z) = \mathbb{E}(\sum_j Z_j)$ since $\sup_q w_2(q) = \sum_j z_j$).

A.3 Equal Revenue (ER) Goods

In this appendix we prove the claims of Proposition 12 concerning ER goods: Lemma 20, Propositions 24 and 25, and Corollary 23.

Lemma 20 *Let X be a one-good random valuation. Then $\text{REV}(X) \leq r$ if and only if X is stochastically dominated by rV where V is an ER valuation.*

Proof. By (1), $\text{REV}(X) \leq r$ if and only if $\mathbb{P}[X \geq p] \leq r/p$ for every $p \geq 0$; this inequality matters only for $p > r$, for which $r/p = \mathbb{P}[rV \geq p]$. ■

Next we compute the distribution of a weighted sum of two independent ER distributions.

Lemma 21 *Let V_1, V_2 be i.i.d.-ER and let $\alpha, \beta > 0$. Then*

$$\mathbb{P}[\alpha V_1 + \beta V_2 \geq z] = \frac{\alpha\beta}{z^2} \ln \left(1 + \frac{z^2 - (\alpha + \beta)z}{\alpha\beta} \right) + \frac{\alpha + \beta}{z}$$

for $z \geq \alpha + \beta$, and $\mathbb{P}[\alpha V_1 + \beta V_2 \geq z] = 1$ for $z \leq \alpha + \beta$.

Proof. Let $Z = \alpha V_1 + \beta V_2$. For $z \leq \alpha + \beta$ we have $\mathbb{P}[Z \geq z] = 1$ since $V_i \geq 1$. For $z > \alpha + \beta$ we get

$$\begin{aligned} \mathbb{P}[Z \geq z] &= \int f(x) \left(1 - F \left(\frac{z - \alpha x}{\beta} \right) \right) dx \\ &= \int_1^{(z-\beta)/\alpha} \frac{1}{x^2} \frac{\beta}{z - \alpha x} dx + \int_{(z-\beta)/\alpha}^{\infty} \frac{1}{x^2} dx \\ &= \frac{\beta}{z} \left[\frac{\alpha}{z} \ln x - \frac{\alpha}{z} \ln \left(\frac{z}{\alpha} - x \right) - \frac{1}{x} \right]_1^{(z-\beta)/\alpha} + \frac{\alpha}{z - \beta} \\ &= \frac{\alpha\beta}{z^2} \left(\ln \left(\frac{z}{\beta} - 1 \right) + \ln \left(\frac{z}{\alpha} - 1 \right) \right) - \frac{\alpha\beta}{z(z - \beta)} + \frac{\beta}{z} + \frac{\alpha}{z - \beta} \\ &= \frac{\alpha\beta}{z^2} \ln \left(1 + \frac{z^2 - (\alpha + \beta)z}{\alpha\beta} \right) + \frac{\alpha + \beta}{z}, \end{aligned}$$

completing the proof. ■

Weighted sums of independent ER distributions are used in the proof of Proposition 13 (see Section 5 and recall Lemma 20). What we will show now (Lemma 22 and Corollary 23) is that moving the weights in the direction of equalizing them yields stochastic domination.

Lemma 22 *Let V_1, V_2 be i.i.d.-ER and let $\alpha, \beta, \alpha', \beta' > 0$. If $\alpha + \beta = \alpha' + \beta'$ and³⁵ $\alpha\beta \leq \alpha'\beta'$ then $\alpha V_1 + \beta V_2$ is stochastically dominated by $\alpha' V_1 + \beta' V_2$.*

Proof. Let $Z = \alpha V_1 + \beta V_2$ and $Z' = \alpha' V_1 + \beta' V_2$, and put $\gamma = \alpha + \beta = \alpha' + \beta'$. Using Lemma 21, for $z \leq \gamma$ we have $\mathbb{P}[Z \geq z] = \mathbb{P}[Z' \geq z] = 1$, and for $z > \gamma$ we get

$$\begin{aligned} \mathbb{P}[Z \geq z] &= \frac{\alpha\beta}{z^2} \ln \left(1 + \frac{z^2 - \gamma z}{\alpha\beta} \right) + \frac{\gamma}{z} \\ &\leq \frac{\alpha'\beta'}{z^2} \ln \left(1 + \frac{z^2 - \gamma z}{\alpha'\beta'} \right) + \frac{\gamma}{z} = \mathbb{P}[Z' \geq z], \end{aligned}$$

since $t \ln(1 + 1/t)$ is increasing in $t > 0$, and $\alpha\beta/(z^2 - \gamma z) \leq \alpha'\beta'/(z^2 - \gamma z)$ by our assumption that $\alpha\beta \leq \alpha'\beta'$ together with $z > \gamma$. ■

Corollary 23 *Let V_1, V_2, \dots, V_k be i.i.d.-ER, let $r_1, r_2, \dots, r_k \geq 0$, and put $\bar{r} = (1/k) \sum_{i=1}^k r_i$ for the average of the r_i . Then $\sum_{i=1}^k r_i V_i$ is stochastically dominated by $\sum_{i=1}^k \bar{r} V_i$.*

Proof. If, say, $r_1 < \bar{r} < r_2$, then Lemma 22 above implies that $r_1 V_1 + r_2 V_2$ is stochastically dominated by $\bar{r} V_1 + r'_2 V_2$, where $r'_2 = r_1 + r_2 - \bar{r} > 0$, and so³⁶ $\sum_{i=1}^k r_i V_i$ is stochastically dominated by $\bar{r} V_1 + r'_2 V_2 + \sum_{i=3}^k r_i V_i$. Continue this way until all coefficients become \bar{r} . ■

We now calculate the revenue obtainable from bundling two independent ER goods. Recall that $w \approx 0.278$ is the solution of the equation $w e^{w+1} = 1$, or $\ln w + w = -1$.

³⁵Equivalently, α', β' are closer to one another than α, β are; i.e., $|\alpha' - \beta'| \leq |\alpha - \beta|$.

³⁶Recall footnote 25: stochastic dominance is closed under convolutions.

Proposition 24 *Let V_1, V_2 be i.i.d.-ER. Then*

$$\text{BREV}(V_1, V_2) = \text{REV}(V_1 + V_2) = 2(w + 1) \approx 2.56.$$

Proof. Using Lemma 21 with $\alpha = \beta = 1$ yields $p \mathbb{P}[V_1 + V_2 \geq p] = p^{-1} \ln(1 + p^2 - 2p) + 2 = 2p^{-1} \ln(p - 1) + 2$, which attains its maximum of $2w + 2$ at $p = 1 + 1/w$ (i.e., $1/(p - 1) = w$). ■

We estimate the bundling revenue from k independent ER goods.

Proposition 25 *There exist constants $c_1 > 0$ and $c_2 < \infty$ such that for any $k \geq 2$ and k i.i.d.-ER goods V_1, V_2, \dots, V_k ,*

$$c_1 k \log k \leq \text{BREV}(V_1, V_2, \dots, V_k) = \text{REV}(V_1 + \dots + V_k) \leq c_2 k \log k.$$

Proof. For a one-dimensional random variable X and a constant M , write $X^M := \min\{X, M\}$ for X truncated at M . When V is ER and $M \geq 1$ it is immediate to compute $\mathbb{E}[V^M] = \ln M + 1$ and $\text{Var}(V^M) \leq 2M$.

• *Lower bound:* For every $p, M > 0$ we have $\text{REV}(\sum_i V_i) \geq p \cdot \mathbb{P}[\sum_i V_i \geq p] \geq p \cdot \mathbb{P}[\sum_i V_i^M \geq p]$.

When $M = k \ln k$ and $p = (k \ln k)/2$ we get $(k \mathbb{E}[V^M] - p)/\sqrt{k \text{Var}(V^M)} \geq \sqrt{\ln k/8}$, and so p is at least $\sqrt{\ln k/8}$ standard deviations below the mean of $\sum_{i=1}^k V_i^M$. Therefore, by Chebyshev's inequality, $\mathbb{P}[\sum_{i=1}^k V_i^M \geq p] \geq 1 - 8/\ln k \geq 1/2$ for all k large enough, and then $\text{REV}(\sum_{i=1}^k V_i) \geq p \cdot 1/2 = k \ln k/4$.

• *Upper bound:* We need to bound $\sup_{p \geq 0} p \cdot \mathbb{P}[\sum_{i=1}^k V_i \geq p]$.

Consider two cases for p . If $p \leq 6k \ln k$ then $p \cdot \mathbb{P}[\sum_{i=1}^k V_i \geq p] \leq p \leq 6k \ln k$.

If $p \geq 6k \ln k$, then, taking $M = p$, we have

$$p \cdot \mathbb{P}\left[\sum_{i=1}^k V_i \geq p\right] \leq p \cdot \mathbb{P}\left[\sum_{i=1}^k V_i^p \geq p\right] + p \cdot \mathbb{P}[V_i > p \text{ for some } 1 \leq i \leq k]. \quad (15)$$

The second term on the right-hand side is at most $p \cdot k \cdot (1 - F_V(p)) = k$ (since $F_V(p) = 1 - 1/p$). To estimate the first term, we again use Chebyshev's

inequality: $\sum_{i=1}^k V_i^p$ has mean $k(\ln p + 1)$ and standard deviation $\sqrt{2kp}$. When k is large enough we have $p/(k(\ln p + 1)) \geq 2$ (recall that $p \geq 6k \ln k$), hence $(p - k(\ln p + 1))/\sqrt{2kp} \geq (p/2)/\sqrt{2kp} = \sqrt{p/(8k)}$, and so p is at least $\sqrt{p/(8k)}$ standard deviations above the mean of $\sum_{i=1}^k V_i^p$. Therefore $p \cdot \mathbb{P} \left[\sum_{i=1}^k V_i^p \geq p \right] \leq p \cdot (8k)/p = 8k$, which implies $p \cdot \mathbb{P} \left[\sum_{i=1}^k V_i \geq p \right] \leq 9k$ by (15).

Altogether, $\text{REV}(\sum_{i=1}^k V_i) \leq \max\{6k \ln k, 9k\} = 6k \ln k$ for all k large enough. ■

Remark. A more precise analysis, based on a Generalized Central Limit Theorem (see, e.g., Zaliapin, Kagan, and Schoenberg 2005), shows that $\text{REV}(\sum_{i=1}^k V_i)/(k \ln k)$ converges to 1 as $k \rightarrow \infty$. Indeed, the sequence $(\sum_{i=1}^k V_i - b_k)/a_k$ with $a_k = k\pi/2$ and³⁷ $b_k = k \ln k + \Theta(k)$ converges in distribution to the Cauchy distribution as $k \rightarrow \infty$. The revenue from a Cauchy distribution can easily be shown to be bounded (by $1/\pi$; use (1)), and so it follows that $\text{REV}(\sum_{i=1}^k V_i) = k \ln k + \Theta(k)$.

As a corollary, we get that separate selling may yield no more than a fraction of the order of $1/\log k$ of the optimal revenue.

Corollary 26 *There exists a constant $c < \infty$ such that for any $k \geq 2$ and k i.i.d.-ER goods V_1, V_2, \dots, V_k*

$$\text{SREV}(V_1, V_2, \dots, V_k) \leq \frac{c}{\log k} \text{REV}(V_1, V_2, \dots, V_k).$$

Proof. We have $\text{REV}(V_1, \dots, V_k) \geq \text{BREV}(V_1, \dots, V_k) \geq c_1 \log k \cdot k = c_1 \log k \cdot \text{SREV}(V_1, \dots, V_k)$ by Proposition 25, and $\text{SREV}(V_1, \dots, V_k) = k$ (because $\text{REV}(V_i) = 1$). ■

³⁷We use the standard computer science notations: $f(k) = O(g(k))$ means that there exists a constant $c < \infty$ such that $f(k) \leq cg(k)$ for all k , and $f(k) = \Omega(g(k))$ means that there exists a constant $c > 0$ such that $f(k) \geq cg(k)$ for all k . Also, $f(k) = \Theta(g(k))$ means that $f(k) = O(g(k))$ and $f(k) = \Omega(g(k))$ both hold; i.e., there exist $c_1 > 0$ and $c_2 < \infty$ such that $c_1 g(k) \leq f(k) \leq c_2 g(k)$ for all k .

A.4 Separate vs. Bundled Selling

We start with an example showing that the $1/k$ bound for k independent goods of Proposition 14 (i) is tight.

Example 27 $\text{BREV}(X_1, \dots, X_k) = (1/k + \varepsilon) \cdot \text{SREV}(X_1, \dots, X_k)$: Take a large M and let X_i have support $\{0, M^i\}$ with $\mathbb{P}[X_i = M^i] = M^{-i}$. Then $\text{REV}(X_i) = 1$ and so $\text{SREV}(X_1, \dots, X_k) = k$, while $\text{BREV}(X_1, \dots, X_k)$ is easily seen to be at most $\max_i M^i \cdot (M^{-i} + \dots + M^{-k}) \leq 1 + 1/(M - 1)$. Because $\text{SREV} \leq \text{REV}$ this also shows that bundling may yield no more than a $1/k$ fraction of the optimal revenue: $\text{BREV}(X_1, \dots, X_k) \leq (1/k + \varepsilon) \cdot \text{REV}(X_1, \dots, X_k)$.

We next prove that a GFOR of the order of $1/k$ is tight.

Lemma 28 *There exists a constant $c > 0$ such that for any $k \geq 2$ and any k independent goods X_1, X_2, \dots, X_k ,*

$$\text{BREV}(X_1, X_2, \dots, X_k) \geq \frac{c}{k} \text{REV}(X_1, X_2, \dots, X_k).$$

Proof. For k a power of two, we use (cf. the proof of Theorem C in Section 6) the decomposition of (8) to obtain by induction, starting from $\text{REV}(X_1) = \text{BREV}(X_1)$, the inequality $\text{REV}(X_1, \dots, X_k) \leq (3k - 2) \cdot \text{BREV}(X_1, \dots, X_k)$ (the induction step uses the fact that the bundled revenue from a subset of the goods is at most the bundled revenue from all of them, since all the X_i are nonnegative). Again, when k is not a power of 2 we can pad to the next power of 2 with goods that have value identically zero, which at most doubles k . ■

Next, we consider i.i.d. goods, where better bounds can be obtained: the bundling revenue cannot be much smaller than the separate revenue.

Lemma 29 *For any two i.i.d. goods X_1, X_2 ,*

$$\text{BREV}(X_1, X_2) \geq \frac{2}{3} \text{SREV}(X_1, X_2).$$

Proof. Let F be the distribution of the X_i , let p be the optimal one-good price for F , and put $\alpha := 1 - F(p)$; thus, $\text{REV}(X_i) = p\alpha$. If $\alpha \leq 2/3$ then the bundling mechanism can offer a price of p , and then the probability that the bundle will be sold is at least the probability that one of the goods by itself has value p , which is $2\alpha - \alpha^2 = \alpha(2 - \alpha) \geq 4\alpha/3$; the revenue is then at least $p \cdot 4\alpha/3 = (4/3)\text{REV}(X_i)$. If $\alpha \geq 2/3$ then the bundling mechanism can offer a price of $2p$, and then the probability that it will be accepted is at least the probability that both goods will get a value of at least p , which is α^2 ; the revenue is then $2p \cdot \alpha^2 \geq (4/3)p\alpha = (4/3)\text{REV}(X_i)$. In both cases the bundling revenue was at least $(4/3)\text{REV}(X_i) = (2/3)\text{SREV}(X_1, X_2)$. ■

This $2/3$ bound is tight.

Example 30 $\text{BREV}(X_1, X_2) = (2/3) \cdot \text{SREV}(X_1, X_2)$: Let X_i have support $\{0, 1\}$ with $\mathbb{P}[X_i = 1] = 2/3$; then $\text{REV}(X_i) = 2/3$ while $\text{BREV}(X_1, X_2) = 8/9$ (which is obtained both at price 1 and at price 2).³⁸

A.5 Many I.I.D. Goods

It is well known that when the goods are independent and identically distributed, and their number k tends to infinity, then the bundling revenue approaches the optimal revenue. Even more, essentially all the buyer's surplus can be extracted by selling optimally the bundle of all goods. The logic is quite simple: the law of large numbers tells us that there is almost no uncertainty about the sum of many i.i.d. random variables, and so the seller essentially knows this sum and may ask for it as the bundle price. For completeness we state this result and provide a short proof, which also covers the case where the expectation is infinite.

Theorem 31 (Armstrong 1999, Bakos and Brynjolfsson 1999) *Let X_i be i.i.d. one-good random valuations. Then*

$$\lim_{k \rightarrow \infty} \frac{\text{BREV}(X_1, X_2, \dots, X_k)}{k} = \lim_{k \rightarrow \infty} \frac{\text{REV}(X_1, X_2, \dots, X_k)}{k} = \mathbb{E}[X_1].$$

³⁸It can be checked that the optimal revenue is attained here by selling separately, i.e., $\text{REV}(X_1, X_2) = \text{SREV}(X_1, X_2) = 4/3$.

Proof. We always have $\text{BREV}(X_1, \dots, X_k) \leq \text{REV}(X_1, \dots, X_k) \leq k\mathbb{E}[X_1]$ (the second inequality follows from $s(x) = q(x) \cdot x - b(x) \leq \sum_i x_i$ by IR). Let us assume first that the X_i have finite expectation and finite variance. In this case if we charge a price of $(1 - \varepsilon)k\mathbb{E}[X_1]$ for the bundle, then, by Chebyshev's inequality, the probability that the bundle will not be bought is at most $\text{VAR}(X_1)/(\varepsilon^2\mathbb{E}[X_1]\sqrt{k})$, and this goes to zero as k increases.

If the expectation or variance is infinite, then consider the truncated distribution where values above a certain M are replaced by M , which has finite expectation and variance. We can choose the finite M so as to bring the expectation of the truncated distribution as close as we desire to the original one (including as high as we desire, if the original distribution has infinite expectation). ■

Despite the apparent strength of this result, it does not provide any guarantees for any *fixed value* of k . Indeed, we now show that for every large enough k we have $\text{GFOR}(\text{BUNDLED}; k \text{ i.i.d. goods}) \leq 57\%$ (recall that it is at least $1/4$ by Proposition 14 (ii)). Recently, Kupfer (2016) obtained the precise value of the limit of this GFOR as k increases; it turns out to be approximately 55.9%.

Example 32 *For every k large enough, a one-dimensional distribution F (which depends on k) such that $\text{BREV}(X_1, \dots, X_k) \leq 0.57 \cdot \text{SREV}(X_1, \dots, X_k)$, and thus $\text{BREV}(X_1, \dots, X_k) \leq 0.57 \cdot \text{REV}(X_1, \dots, X_k)$, where the X_i are i.i.d. F goods:* For each k consider the distribution F on $\{0, 1\}$ with $\mathbb{P}[X = 1] = c/k$ where $c \approx 1.256$ is the positive solution of $1 - e^{-c} = 2(1 - (1+c)e^{-c})$; the revenue from selling a single good is thus c/k , and so $\text{SREV}(X_1, \dots, X_k) = c$. The bundling mechanism should clearly offer an integral price. If it offers price 1 then the probability of selling is $1 - (1 - c/k)^k$, which converges to $1 - e^{-c} \approx 0.715$ as k increases. If it offers price 2 then the probability of selling is $1 - (1 - c/k)^k - k(c/k)(1 - c/k)^{k-1} \rightarrow 1 - (1+c)e^{-c}$, and the revenue is twice that, again ≈ 0.715 in the limit (recall the equation that c satisfies). If it offers price 3 then the probability of selling is $1 - (1 - c/k)^k - k(c/k)(1 - c/k)^{k-1} - \binom{k}{2}(c/k)^2(1 - c/k)^{k-2} \rightarrow 1 - (1+c+c^2/2!)e^{-c} \approx 0.13$, and the revenue is three times that, which is less than 0.715. For higher integral prices $m \geq 4$

the probability of selling converges to $1 - (1 + c + \dots + c^{m-1}/(m-1)!)e^{-c} \leq c^m/m!$ (because the corresponding remainder in the e^c series is bounded by $e^c c^m/m!$), and the revenue is thus $\leq c^m/(m-1)!$, which is even smaller. Therefore for all large enough k the optimal bundle price is either 1 or 2, and $\text{BREV}(X_1, \dots, X_k)/\text{SREV}(X_1, \dots, X_k)$ is close to $(1 - e^{-c})/c \approx 0.569$.

A.6 When Bundling Is Optimal

In this appendix we prove Theorem 16, stated in Section 7.2: for two i.i.d. goods, if the one-good distribution satisfies condition (9), then bundling is optimal.

Proof of Theorem 16. Let $X = (Y, Z)$ where Y, Z are i.i.d. with cumulative distribution F and probability density function f that satisfies (9). We will show that for every IC and IR mechanism μ there is a bundled mechanism $\hat{\mu}$ that yields at least as much revenue, i.e., $R(\hat{\mu}; X) \geq R(\mu; X)$; this proves that $\text{REV}(X) = \text{BREV}(X)$.

Let $\mu = (q, s)$ be an IC and IR mechanism with buyer payoff function b ; as in the proof of Theorem B in Appendix A.1, we assume without loss of generality that the mechanism is symmetric and satisfies NPT. Therefore

$$\begin{aligned} \mathbb{E}[s(Y, Z)] &= \mathbb{E}[Yq_1(Y, Z) + Zq_2(Y, Z) - b(Y, Z)] \\ &= \mathbb{E}[2Yq_1(Y, Z) - b(Y, Z)], \end{aligned}$$

because $\mathbb{E}[Zq_2(Y, Z)] = \mathbb{E}[Zq_1(Z, Y)] = \mathbb{E}[Yq_1(Y, Z)]$ by symmetry and then by interchanging the i.i.d. variables Y and Z . Truncating at M therefore yields (recall that s is nonnegative by NPT)

$$R(\mu; X) = \lim_{M \rightarrow \infty} r_M(b),$$

where

$$r_M(b) := \lim_{M \rightarrow \infty} \int_a^M \int_a^M (2yb_y(y, z) - b(y, z)) f(y)f(z) dy dz \quad (16)$$

(because $q_1(y, z) = b_y(y, z)$, the derivative of $b(y, z)$ with respect to its first variable, for almost every (y, z) by Proposition 5, and the distribution F is continuous).

For each fixed z integrate by parts the $2yb_y(y, z)f(y)$ term:

$$\begin{aligned} \int_a^M 2b_y(y, z)yf(y) \, dy &= [2b(y, z)yf(y)]_a^M - \int_a^M 2b(y, z)(f(y) + yf'(y)) \, dy \\ &= 2b(M, z)Mf(M) - 2b(a, z)af(a) \\ &\quad - \int_a^M 2b(y, z)(f(y) + yf'(y)) \, dy. \end{aligned}$$

Substituting this in (16) yields

$$\begin{aligned} r_M(b) &= 2Mf(M) \int_a^M b(M, z)f(z) \, dz \\ &\quad + 2 \int_a^M \int_a^M b(y, z) \left(-\frac{3}{2}f(y) - yf'(y) \right) f(z) \, dy \, dz \quad (17) \\ &\quad - 2af(a) \int_a^M b(a, z)f(z) \, dz. \end{aligned}$$

Define $\hat{b}(y, z) := b(y + z - a, a) = b(a, y + z - a)$ for every (y, z) with $y, z \geq a$. Then \hat{b} is a symmetric convex function on the quadrant $[a, \infty)^2$, it coincides with b on the boundaries $y = a$ and $z = a$, and is at least as large as b everywhere³⁹: indeed, the convexity of b yields

$$\begin{aligned} b(y, z) &\leq \frac{y - a}{y + z - 2a}b(y + z - a, a) + \frac{z - a}{y + z - 2a}b(a, y + z - a) \quad (18) \\ &= b(y + z - a, a) = \hat{b}(y, z) \end{aligned}$$

for every $(y, z) \in [a, \infty)^2$. Replacing b with \hat{b} can only increase the first and second terms of (17), since all the coefficients of b there are nonnegative (use (9)), while it does not affect the third term. Therefore $r_M(b) \leq r_M(\hat{b})$ for all M .

³⁹The function \hat{b} is in fact the *smallest* function satisfying these three properties (i.e., it is a convex function, coincides with b on the boundary, and is everywhere $\geq b$); see Hart (2012) for an interesting observation on this.

Define $\hat{q}(y, z) := (q_1(y + z - a, a), q_1(y + z - a, a)) \in [0, 1]^2$ and $\hat{s}(y, z) := \hat{q}(y, z) \cdot (y, z) - \hat{b}(y, z)$; then $\hat{q}(y, z)$ is a subgradient of⁴⁰ \hat{b} at (y, z) , and so $\hat{\mu} = (\hat{q}, \hat{s})$ is an IC and IR mechanism (by Proposition 5). Since $R(\hat{\mu}; X) = \lim_{M \rightarrow \infty} r_M(\hat{b})$ and $r_M(b) \leq r_M(\hat{b})$ for all M , we get $R(\mu; X) \leq R(\hat{\mu}; X)$. Finally, $\hat{\mu}$ is a bundled mechanism, since \hat{q} and \hat{s} are functions of $y + z$ (the corresponding one-good mechanism for $T := Y + Z$ is (\tilde{q}, \tilde{s}) with $\tilde{q}(t) := \hat{q}(t - a, a)$ and $\tilde{s}(t) := \hat{s}(t - a, a)$). This completes the proof. ■

A.7 Multiple Buyers

The generalization from one buyer to $n \geq 1$ buyers is as follows (recall Section 2.1). One seller is selling k goods to n buyers; these goods have no value or cost to the seller. For each buyer $j = 1, \dots, n$ and each good $i = 1, \dots, k$, buyer j 's value for good i is given by a nonnegative random variable X_i^j ; put $X^j = (X_i^j)_{i=1, \dots, k}$ for the \mathbb{R}_+^k -valued random valuation vector of buyer j , and $X_i = (X_i^j)_{j=1, \dots, n}$ for the \mathbb{R}_+^n -valued vector of values of good i ; put also $X = (X_i^j)_{j=1, \dots, n; i=1, \dots, k}$, and let F be the joint distribution of all these kn random variables. The distribution F is commonly known; in addition, each buyer j knows the realization of his random valuation X^j . Finally, the seller as well as all the buyers are risk-neutral and have quasi-linear utilities, and the valuation of a set of goods to each buyer is additive.

A (direct) *mechanism* $\mu = (q^j, s^j)_{j=1, \dots, n}$ consists of an allocation function $q^j : \mathbb{R}_+^{kn} \rightarrow [0, 1]^k$ and a payment function $s^j : \mathbb{R}_+^{kn} \rightarrow \mathbb{R}$ for each buyer $j = 1, \dots, n$, where $\sum_{j=1}^n q_i^j(x) \leq 1$ for every good $i = 1, \dots, k$; the payoff of buyer j is $b^j(x) = q^j(x) \cdot x^j - s^j(x)$, and that of the seller is $S(x) := \sum_{j=1}^n s^j(x)$. Two standard equilibrium notions are used for the n -person game among the buyers (once the mechanism μ is given): “*dominant strategy*” (DS) and “*Bayesian Nash*” (BN), which yield the so-called ex-post and interim equilibria, respectively. The corresponding conditions are:

- In the dominant strategy case: *incentive compatibility* (IC-DS) re-

⁴⁰At points of differentiability $\hat{b}_y(y, z) = \hat{b}_z(y, z) = b_y(y + z - a, a)$.

quires that

$$b^j(x) = q^j(x) \cdot x^j - s^j(x) = \max_{\tilde{x}^j \in \mathbb{R}_+^k} [q^j(\tilde{x}^j, x^{-j}) \cdot x^j - s^j(\tilde{x}^j, x^{-j})]$$

for every $j = 1, \dots, n$ and $x \in \mathbb{R}_+^{kn}$; *individual rationality* (**IR-DS**) requires that $b^j(x) \geq 0$ for every j and $x \in \mathbb{R}_+^{kn}$.

- In the Bayesian Nash case: *incentive compatibility* (**IC-BN**) requires that⁴¹

$$\bar{b}^j(x^j) := \mathbb{E} [b^j(X) | X^j = x^j] = \max_{\tilde{x}^j \in \mathbb{R}_+^k} \mathbb{E} [q^j(\tilde{x}^j, X^{-j}) \cdot x^j - s^j(\tilde{x}^j, X^{-j}) | X^j = x^j]$$

for every $j = 1, \dots, n$ and $x^j \in \mathbb{R}_+^k$; *individual rationality* (**IR-BN**) requires that $\bar{b}^j(x^j) \geq 0$ for every j and $x^j \in \mathbb{R}_+^k$.

Let $R(\mu; X) := \mathbb{E} [S(X)] \equiv \mathbb{E} \left[\sum_{j=1}^n s^j(X) \right]$ denote the seller's expected revenue from the mechanism μ for the random valuations X . Let $\text{REV}^{DS}(X)$ stand for the maximal revenue obtained from k goods and n buyers with random valuations X using dominant strategy implementation, i.e., mechanisms that satisfy IC-DS and IR-DS; let $\text{REV}^{BN}(X)$ stand for the maximal revenue using Bayesian Nash implementation, i.e., mechanisms that satisfy IC-BN and IR-BN (in the one-buyer case, i.e., when $n = 1$, these two concepts clearly coincide).

Remarks. (a) *Bayesian Nash implementation for independent buyers.* In the Bayesian Nash case, when the buyers' random valuation vectors X^1, X^2, \dots, X^n are *independent*, only the expectations of allocations and payments, conditional on each buyer's own values, matter: replacing $q^j(x)$ with $\bar{q}^j(x^j) := \mathbb{E} [q^j(X) | X^j = x^j] = \mathbb{E} [q^j(x^j, X^{-j})]$ and $s^j(x)$ with $\bar{s}^j(x^j) := \mathbb{E} [s^j(X) | X^j = x^j] = \mathbb{E} [s^j(x^j, X^{-j})]$ throughout affects neither the IC-BN and IR-BN constraints

⁴¹The conditions in the Bayesian Nash case depend on the mechanism μ and (the distribution of) the valuations X , whereas in the dominant strategy case they depend only on the mechanism μ .

nor the revenue.⁴² We will thus assume without loss of generality that mechanisms in the Bayesian Nash case are given in this “reduced form” where q^j and s^j depend only on x^j (rather than on the entire x), for all j .

(b) *Non-positive transfer (NPT) and subdomain.* A mechanism $\mu = (q^j, s^j)_{j=1, \dots, n}$ for $n \geq 1$ buyers and $k \geq 1$ goods satisfies **NPT** if $s^j(x) \geq 0$ for every j and every x . The results (i)–(iv) of Proposition 6 in Section 2.1 extend to multiple buyers, as follows.

In the dominant strategy case, we consider separately each $j = 1, \dots, n$ and each $x^{-j} \in \mathbb{R}_+^{k(n-1)}$, and obtain, in particular, that NPT is equivalent to $s^j(0, x^{-j}) = 0$ for all j and all x^{-j} , that NPT can be assumed without loss of generality when maximizing revenue, and that the subdomain property holds: $\mathbb{E} \left[\sum_j s^j(X) \mathbf{1}_{X \in A} \right] \leq \text{REV}^{DS}(X \mathbf{1}_{X \in A}) \leq \text{REV}^{DS}(X)$ for every $A \subseteq \mathbb{R}_+^{kn}$.

In the Bayesian Nash case, when the buyers are independent (and each payment s^j depends only on x^j ; see Remark (a) above), we obtain in particular that NPT is equivalent to $\bar{s}^j(0) = 0$ for all j , that NPT can be assumed without loss of generality when maximizing revenue, and that $\mathbb{E} \left[\sum_j s^j(X) \mathbf{1}_{X \in A} \right] = \mathbb{E} \left[\sum_j \bar{s}^j(X^j) \mathbf{1}_{X \in A} \right] \leq \text{REV}^{BN}(X \mathbf{1}_{X \in A}) \leq \text{REV}^{BN}(X)$ for every⁴³ $A \subseteq \mathbb{R}_+^{kn}$.

We state the result separately for the two kinds of implementation, since in the dominant strategy case the result is stronger: the requirement that the buyers’ random valuations are independent is not needed (i.e., while there is independence between the two goods— X_1^j and X_2^ℓ are independent for any two buyers $j, \ell = 1, \dots, n$ —we allow, for each good $i = 1, 2$, the buyers’ values $X_i^1, X_i^2, \dots, X_i^n$ to be arbitrarily correlated). The two theorems below give Theorem 18.

⁴²However, the feasibility conditions $\sum_i q_\ell^i \leq 1$ on the allocations *cannot* be directly expressed in terms of the \bar{q}^i (this is known as the “implementability” condition; see Border 1991 for a necessary and sufficient condition for implementability, and Hart and Reny 2015b for a simple restatement and proof).

⁴³NPT need *not* hold in the Bayesian Nash case when the buyers’ random valuations are *not independent*. In this case, optimal mechanisms may make use of negative payments $s^j(x)$ (i.e., positive transfers); cf. Crémer and McLean (1988). In addition, the requirement that s^j depends only on x^j is needed because the restriction to a set A of values of X may introduce dependencies between the coordinates of $X \mathbf{1}_{X \in A}$, and then $\mathbb{E} \left[\sum_j s^j(X) \mathbf{1}_{X \in A} \right] = \mathbb{E} \left[\sum_j \bar{s}^j(X^j) \mathbf{1}_{X \in A} \right]$ need not hold.

Theorem 33 *In the case of n buyers, two goods, and dominant strategy implementation, if the random valuation vectors of the two goods $X_1 = (X_1^j)_{j=1,\dots,n}$ and $X_2 = (X_2^j)_{j=1,\dots,n}$ are independent, then $\text{GFOR}(\text{SEPARATE}) \geq 1/2$, i.e.,*

$$\text{REV}^{DS}(X_1) + \text{REV}^{DS}(X_2) \geq \frac{1}{2}\text{REV}^{DS}(X_1, X_2).$$

Theorem 34 *In the case of n independent buyers, two goods, and Bayesian Nash implementation, if the random valuation vectors of the two goods $X_1 = (X_1^j)_{j=1,\dots,n}$ and $X_2 = (X_2^j)_{j=1,\dots,n}$ are independent, then $\text{GFOR}(\text{SEPARATE}) \geq 1/2$, i.e.,*

$$\text{REV}^{BN}(X_1) + \text{REV}^{BN}(X_2) \geq \frac{1}{2}\text{REV}^{BN}(X_1, X_2).$$

Proof of Theorems 33 and 34. Put $Y = X_1 = (X_1^j)_{j=1,\dots,n}$ and $Z = X_2 = (X_2^j)_{j=1,\dots,n}$ for the random valuation vectors of good 1 and good 2, respectively; thus Y and Z are \mathbb{R}_+^n -valued random variables and $X = (Y, Z)$. Let $\mu = (q^j, s^j)_{j=1,\dots,n}$ be an NPT mechanism for the two goods that satisfies either IC-DS and IR-DS, or IC-BN and IR-BN; in the BN case, we assume in addition that it is in reduced form: q^j and s^j depend only on x^j (cf. Remark (a) above).⁴⁴

We split the total expected revenue from μ into two parts, according to which one of⁴⁵ $Y^{(1)} = \max_j Y^j$ and $Z^{(1)} = \max_j Z^j$ is higher, and show that

$$\mathbb{E}[S(Y, Z)\mathbf{1}_{Y^{(1)} \geq Z^{(1)}}] \leq 2\text{REV}(Y) \quad \text{and} \quad (19)$$

$$\mathbb{E}[S(Y, Z)\mathbf{1}_{Z^{(1)} \geq Y^{(1)}}] \leq 2\text{REV}(Z); \quad (20)$$

adding the two inequalities yields our result (recall that $S \geq 0$ by NPT).

To prove (19) (from which (20) follows by interchanging Y and Z), for every fixed vector of values $z \in \mathbb{R}_+^n$ of the n buyers for the second good define a mechanism $(\hat{q}, \hat{s}) \equiv (\hat{q}^z, \hat{s}^z)$ for the first good by $\hat{q}^j(y) := q_1^j(y, z)$ and $\hat{s}^j(y) := s^j(y, z) - q_2^j(y, z)z^j$ for every $y \in \mathbb{R}_+^n$ and $j = 1, \dots, n$, and put

⁴⁴Formally, put here $q^i(x) := \bar{q}^i(x^i)$ and $s^i(x) := \bar{s}^i(x^i)$; this allows the proof to apply mutatis mutandis to both implementations, dominant strategy and Bayesian Nash.

⁴⁵We write $a^{(1)} := \max_{j=1,\dots,n} a^j$ for the maximal coordinate of a vector $a = (a^j)_{j=1,\dots,n} \in \mathbb{R}^n$.

$\hat{S}(y) := \sum_j \hat{s}^j(y)$. The mechanism (\hat{q}, \hat{s}) is IC and IR for y , since (q, s) was IC and IR for (y, z) (for IC: only the constraints (\tilde{y}^j, z^j) vs. (y^j, z^j) matter; for IR, $\hat{b}^j(y) = b^j(y, z) \geq 0$). Then $S(y, z) = \sum_j s^j(y, z) = \sum_j \hat{s}^j(y) + \sum_j z^j q_2^j(y, z) \leq \sum_j \hat{s}^j(y) + z^{(1)} = \hat{S}(y) + z^{(1)}$ (the inequality obtains because $0 \leq z^j \leq z^{(1)}$ and $\sum_j q_2^j \leq 1$). Since Y is independent of Z we get

$$\begin{aligned} \mathbb{E} [S(Y, Z) \mathbf{1}_{Y^{(1)} \geq Z^{(1)}} | Z = z] &= \mathbb{E} [S(Y, z) \mathbf{1}_{Y^{(1)} \geq z^{(1)}}] \\ &\leq \mathbb{E} [\hat{S}(Y) \mathbf{1}_{Y^{(1)} \geq z^{(1)}}] + \mathbb{E} [z^{(1)} \mathbf{1}_{Y^{(1)} \geq z^{(1)}}]. \end{aligned}$$

The first term is the revenue from a subdomain of values of y , and so it is at most the maximal revenue $\text{REV}(Y)$ by Remark (b) above (in the BN case, \hat{s}^j depends only on y^j since s^j and q^j depend only on x^j); as for the second term,

$$\mathbb{E} [z^{(1)} \mathbf{1}_{Y^{(1)} \geq z^{(1)}}] = z^{(1)} \mathbb{P} [Y^{(1)} \geq z^{(1)}] \leq \text{REV}(Y), \quad (21)$$

since posting a price of $z^{(1)}$ and giving the good y to a buyer j with $y^j \geq z^{(1)}$, if there is any, constitutes an IC and IR mechanism for⁴⁶ y . Thus

$$\mathbb{E} [S(Y, Z) \mathbf{1}_{Y^{(1)} \geq Z^{(1)}} | Z = z] \leq 2\text{REV}(Y)$$

for every value z of Z ; taking expectation over z yields (19). ■

A.8 Summary of Results

The two tables below summarize the results of this paper; X stands for (X_1, X_2, \dots, X_k) , where X_1, X_2, \dots, X_k are k independent goods (i.e., one-dimensional nonnegative random variables).⁴⁷ Table 1 provides the bounds on the guaranteed fraction of optimal revenue for selling separately and for selling as one bundle (with the four main results in bold). Table 2 provides the comparisons between the separate and bundled revenues.⁴⁸

⁴⁶As in the Proof of Theorem A in Section 3, we are *not* using the characterization of optimal mechanisms in the one-good case (Myerson 1981), but only the fact that posting a price is IC and IR.

⁴⁷ $\mathcal{O}(1)$ means “converging to 0 as $k \rightarrow \infty$ ”; see footnote 37 for the \mathcal{O} , Ω , and Θ notations.

⁴⁸When comparing SREV and BREV we obtained tight bounds in almost all cases (unlike the results for GFOR). This is in part due to the fact that both SREV and BREV reduce

	$k = 2$ indep.	$k = 2$ i.i.d.	$k \geq 2$ indep.	$k \geq 2$ i.i.d.
$\forall X \frac{\text{SREV}(X)}{\text{REV}(X)} \geq$	$\frac{1}{2}$ (Th.A)	$\frac{e}{e+1} \approx 0.73$ (Th.B)	$\Omega\left(\frac{1}{\log^2 k}\right)$ (Th.C)	
$\exists X \frac{\text{SREV}(X)}{\text{REV}(X)} \leq$	$\frac{1}{1+w} \approx 0.78$ (Pr.15)		$O\left(\frac{1}{\log k}\right)$ (Co.26)	
$\forall X \frac{\text{BREV}(X)}{\text{REV}(X)} \geq$	$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ (Th.A+Pr.14(i))	$\frac{e}{e+1} \cdot \frac{2}{3}$ (Th.B+Le.29)	$\Omega\left(\frac{1}{k}\right)$ (Le.28)	$\Omega\left(\frac{1}{\log k}\right)$ (Th.D)
$\exists X \frac{\text{BREV}(X)}{\text{REV}(X)} \leq$	$\frac{1}{2} + \varepsilon$ (Ex.27)	$\frac{2}{3}$ (Ex.30)	$\frac{1}{k} + \varepsilon$ (Ex.27)	$\approx 0.57 + o(1)$ (Ex.32)

Table 1. Summary of results for GFOR

	$k = 2$ indep.	$k = 2$ i.i.d.	$k \geq 2$ indep.	$k \geq 2$ i.i.d.
$\inf_X \frac{\text{SREV}(X)}{\text{BREV}(X)} =$	$\frac{1}{1+w} \approx 0.78$ (Pr.13(i)+Pr.12(iii))		$\Theta\left(\frac{1}{\log k}\right)$ (Pr.13(ii)+Pr.12(iv))	
$\inf_X \frac{\text{BREV}(X)}{\text{SREV}(X)} =$	$\frac{1}{2}$ (Pr.14(i)+Ex.27)	$\frac{2}{3}$ (Le.29+Ex.30)	$\frac{1}{k}$ (Pr.14(i)+Ex.27)	$\in \left[\frac{1}{4}, 0.57 + o(1)\right]$ (Pr.14(ii)+Ex.32)

Table 2. Summary of results for SREV vs. BREV

to one-good revenues, for which monotonicity holds (see Proposition 11 and the extensive use of ER goods; cf. Remark (b) after Proposition 12).

References

- Alaei, S., H. Fu, N. Haghpanah, J. Hartline, and A. Malekian (2012), “Bayesian Optimal Auctions via Multi- to Single-Agent Reduction,” *EC 2012: Proceedings of the 13th ACM Conference on Electronic Commerce*.
- Armstrong, M. (1999), “Price Discrimination by a Many-Product Firm,” *The Review of Economic Studies* 66, 151–168.
- Babaioff, M., Y. Gonczarowski, and N. Nisan (2017), “The Menu-Size Complexity of Revenue Approximation,” *STOC 2017: Proceedings of the 49th ACM Symposium on Theory of Computing*, forthcoming.
- Babaioff, M., N. Immorlica, B. Lucier, and S. M. Weinberg (2014), “A Simple and Approximately Optimal Mechanism for an Additive Buyer,” *FOCS 2014: Proceedings of the 55th Annual Symposium on Foundations of Computer Science*, 21–30.
- Bakos, Y. and E. Brynjolfsson (1999), “Bundling Information Goods: Pricing, Profits, and Efficiency,” *Management Science* 45, 1613–1630.
- Border, K. C. (1991), “Implementation of Reduced Form Auctions: A Geometric Approach,” *Econometrica* 59, 1175–1187.
- Briest, P., S. Chawla, R. Kleinberg, and M. Weinberg (2010), “Pricing Randomized Allocations,” *SODA 2010: Proceedings of the 21st Annual ACM–SIAM Symposium on Discrete Algorithms*, 585–597; *Journal of Economic Theory* 156 (2015), 144–174.
- Cai, Y., C. Daskalakis, and S. M. Weinberg (2012a), “An Algorithmic Characterization of Multi-Dimensional Mechanisms,” *STOC 2012: Proceedings of the 44th Annual ACM Symposium on Theory of Computing*, 459–478.
- Cai, Y., C. Daskalakis, and S. M. Weinberg (2012b), “Optimal Multi-Dimensional Mechanism Design: Reducing Revenue to Welfare Maximization,” *FOCS 2012: Proceedings of the 55th Annual Symposium on Foundations of Computer Science*.
- Chawla, S., J. D. Hartline, and R. D. Kleinberg (2007), “Algorithmic Pricing via Virtual Valuations,” *EC 2007: Proceedings of the 8th ACM Conference on Electronic Commerce*, 243–251.

- Chawla, S., J. D. Hartline, D. L. Malec, and B. Sivan (2010), “Multi-Parameter Mechanism Design and Sequential Posted Pricing,” *STOC 2010: Proceedings of the 42nd ACM Symposium on Theory of Computing*, 311–320.
- Chawla, S., D. L. Malec, and B. Sivan (2010), “The Power of Randomness in Bayesian Optimal Mechanism Design,” *EC 2010: Proceedings of the 11th ACM Conference on Electronic Commerce*, 149–158.
- Crémer, J. and R. P. McLean (1988), “Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions,” *Econometrica* 56, 1247–1257.
- Daskalakis, C., A. Deckelbaum, and C. Tzamos (2013), “Mechanism Design via Optimal Transport,” *EC 2013: Proceedings of the 14th ACM Conference on Electronic Commerce*, 269–286.
- Daskalakis, C., A. Deckelbaum, and C. Tzamos (2014), “The Complexity of Optimal Mechanism Design,” *SODA 2014: Proceedings of the 25th Annual ACM–SIAM Symposium on Discrete Algorithms*, 1302–1318.
- Daskalakis, C., A. Deckelbaum, and C. Tzamos (2017), “Strong Duality for a Multiple-Good Monopolist,” *Econometrica* 85, 735–767.
- Dughmi, S., L. Han, and N. Nisan (2014), “Sampling and Representation Complexity of Revenue Maximization,” *Lecture Notes in Computer Science* 8877, 277–291.
- Fang, H. and P. Norman (2006), “To Bundle or Not to Bundle,” *RAND Journal of Economics* 37, 946–963.
- Giannakopoulos, Y. (2014), “Bounding Optimal Revenue in Multiple-Items Auctions,” *arXiv* 1402.2382.
- Giannakopoulos, Y. and E. Koutsoupias (2014), “Duality and Optimality of Auctions for Uniform Distributions,” *EC 2014: Proceedings of the 15th ACM Conference on Electronic Commerce*, 259–276.
- Hart, S. (2012), “A Curious Property of Convex Functions and Mechanism Design” (mimeo).
- Hart, S. and N. Nisan (2012), “Approximate Revenue Maximization with Multiple Items,” The Hebrew University of Jerusalem, Center for Rationality DP-606; *arXiv* 1204.1846.

- Hart, S. and N. Nisan (2013), “The Menu-Size Complexity of Auctions,” The Hebrew University of Jerusalem, The Hebrew University of Jerusalem, Center for Rationality DP-637; *arXiv* 1304.6116.
- Hart, S. and P. J. Reny (2015a), “Maximal Revenue with Multiple Goods: Nonmonotonicity and Other Observations,” *Theoretical Economics* 10, 893–922.
- Hart, S. and P. J. Reny (2015b), “Implementation of Reduced Form Mechanisms: A Simple Approach and a New Characterization,” *Economic Theory Bulletin* 3, 1–8.
- Jehiel, P., M. Meyer-ter-Vehn, and B. Moldovanu (2007), “Mixed Bundling Auctions,” *Journal of Economic Theory* 134, 494–512.
- Krishna, V. (2010), *Auction Theory*, Second Edition, Academic Press.
- Kupfer, R. (2016), “A Note on the Ratio of Revenues between Selling in a Bundle and Separately,” *arXiv*1611.09613.
- Lev, O. (2011), “A Two-Dimensional Problem of Revenue Maximization,” *Journal of Mathematical Economics* 47, 718–727.
- Li, X. and A. C.-C. Yao (2013), “On Revenue Maximization for Selling Multiple Independently Distributed Items,” *Proceedings of the National Academy of Sciences* 110, 11232–11237.
- Manelli, A. M. and D. R. Vincent (2006), “Bundling as an Optimal Selling Mechanism for a Multiple-Good Monopolist,” *Journal of Economic Theory* 127, 1–35.
- Manelli, A. M. and D. R. Vincent (2007), “Multidimensional Mechanism Design: Revenue Maximization and the Multiple-Good Monopoly,” *Journal of Economic Theory* 137, 153–185.
- Manelli, A. M. and D. R. Vincent (2012), “Multidimensional Mechanism Design: Revenue Maximization and the Multiple-Good Monopoly. A Corrigendum,” *Journal of Economic Theory* 147, 2492–2493.
- McAfee, R. P. and J. McMillan (1988), “Multidimensional Incentive Compatibility and Mechanism Design,” *Journal of Economic Theory* 46, 335–354.

- Menicucci, D., S. Hurkens, and D.-S. Jeon (2015), “On the Optimality of Pure Bundling for a Monopolist,” *Journal of Mathematical Economics* 60, 33–42.
- Morgenstern, J. and T. Roughgarden (2016), “Learning Simple Auctions,” *JLMR: Workshop and Conference Proceedings* 49, 1–21.
- Myerson, R. B. (1981), “Optimal Auction Design,” *Mathematics of Operations Research* 6, 58–73.
- Pavlov, G. (2011), “Optimal Mechanism for Selling Two Goods,” *The B.E. Journal of Theoretical Economics: Advances* 11, Issue 1, Article 3.
- Pycia, M. (2006), “Stochastic vs Deterministic Mechanisms in Multidimensional Screening,” MIT (mimeo).
- Riley, J. G. and W. F. Samuelson (1981), “Optimal Auctions,” *American Economic Review* 71, 381–392.
- Riley, J. and R. Zeckhauser (1983), “Optimal Selling Strategies: When to Haggle, When to Hold Firm,” *Quarterly Journal of Economics* 98, 267–289.
- Rochet, J.-C. (1985), “The Taxation Principle and Multi-Time Hamilton–Jacobi Equations,” *Journal of Mathematical Economics* 14, 113–128.
- Rockafellar, T. R. (1970), *Convex Analysis*, Princeton University Press.
- Rubinstein, A. and S. M. Weinberg (2015), “Simple Mechanisms for a Sub-additive Buyer and Applications to Revenue Monotonicity,” *EC 2015: Proceedings of the Sixteenth ACM Conference on Economics and Computation*, 377–394.
- Shaked, M. and J. G. Shantikumar (2010), *Stochastic Orders*, Springer.
- Tang, P. and Z. Wang (2017), “Optimal Mechanisms with Simple Menus,” *Journal of Mathematical Economics* 69, 54–70.
- Thanassoulis, J. (2004), “Haggling over Substitutes,” *Journal of Economic Theory* 117, 217–245.
- Yao, A. C.-C. (2014), “An n -to-1 Bidder Reduction for Multi-Item Auctions and its Applications,” *Proceedings of the Twenty-Sixth Annual ACM–SIAM Symposium on Discrete Algorithms*, 92–109.

Zaliapin, I. V., Y. Y. Kagan, and F. P. Schoenberg (2005), "Approximating the Distribution of Pareto Sums," *Pure and Applied Geophysics* 162, 1187–1228.