THEORY OF MECHANISM DESIGN

Midterm Examination

September 12, 2017; Duration: 2 hours; Total marks: **40**

All the questions assume quasilinear utility and independent private values model.

Write your answers clearly without unnecessary arguments.

1. Consider sale of a single object using a first-price auction with reserve price r. The values of all n buyers are symmetrically distributed using a differentiable distribution F with support $[0, \beta]$. Let $G(x) := [F(x)]^{n-1}$ for all $x \in [0, \beta]$. A Bayesian equilibrium of the first-price auction is the following: agent with value $x \geq r$ bids

$$x - \frac{1}{G(x)} \int_r^x G(y)dy$$

and agent with value $x < r$ bids zero.

   (a) Show that the equilibrium bid function is increasing, i.e., higher valued-agents bid higher amounts. **(4 marks)**

   (b) Write the direct revelation mechanism corresponding to this first-price auction. **(4 marks)**

   (c) Use this to argue that for every Vickrey auction with a reserve price, there is a first-price auction with a reserve price that generates the same expected revenue. **(4 marks)**

   (d) Use this to argue that there is a first-price auction with an optimally chosen reserve price which is an optimal mechanism. **(4 marks)**

2. There are three agents and one object. The values of the agents for the object are distributed independently and identically in $[0, 1]$ using a distribution G with pdf g. Consider the following allocation rule for allocating the object. At any valuation profile $v_1 \geq v_2 \geq v_3$, define

$$\gamma(v_1, v_2, v_3) = \frac{2}{3} + \frac{1}{3}\frac{v_3}{v_2}.$$

All the highest valued agents share the probability $\gamma(v_1, v_2, v_3)$ equally among themselves but the remaining probabilities $1 - \gamma(v_1, v_2, v_3)$ is not allocated to any agent.

   (a) Argue that f is implementable in dominant strategies. **(2 marks)**

(b) Design payment rules (p_1, p_2, p_3) such that (f, p_1, p_2, p_3) is dominant strategy incentive compatible and for every i and every v_{-i} with $v_j \geq v_k$, where $N \setminus \{i\} = \{j, k\}$,

$$p_i(0, v_{-i}) = -\frac{1}{3} v_k.$$

For a *generic* valuation profile $v_1 > v_2 > v_3$, what is the allocation probability and payment of each agent? **(4 marks)**

(c) Is (f, p_1, p_2, p_3) budget-balanced? **(2 marks)**

(d) Fix a generic valuation profile, $v_1 > v_2 > v_3$ and write down the welfare (sum of utilities of agents) in this mechanism, in the Green-Laffont mechanism, and in the Vickrey auction. Compare the welfares of these mechanisms. **(4 marks)**

(e) Compare the **ex-ante** expected welfare (i.e., expected total utility) of this mechanism to the dAGV mechanism and the Vickrey auction? **(4 marks)**

3. Suppose a seller is selling a single object to a single agent whose value is distributed in $[0, 1]$ using a distribution G with density g.

   (a) Write down the expression for expected payment of an incentive compatible and individually rational mechanism (f, p) in terms of virtual value of the agent (do not need to derive this). **(2 marks)**

   (b) Find the optimal mechanism if the function $H(x) = xG(x)$ for all $x \in [0, 1]$ is strictly convex. **(6 marks)**