Game Theory 2 - Mid Term Examination

September, 2011; Duration: 3 hours; Total marks: 50.

1. Suppose a single object is being sold by a seller to two bidders. The seller uses the following allocation rule f. Given a profile of valuations (v_1, v_2), f allocates the object to bidder 1 if $v_1^2 \geq v_2$, else it goes to bidder 2.

   (a) Explain why f is dominant strategy incentive compatible (DSIC). **2 marks**

   Answer: f is clearly monotone. By our characterization of DSIC allocation rules, the claim follows.

   (b) Find a payment scheme p which makes f DSIC, has the property that if a bidder does not win the object his payment is zero, and makes the mechanism (f, p) individually rational (for any payment scheme you propose, you need to show that these properties are satisfied). **6 marks**

   Answer: For any profile (v_1, v_2), if bidder 1 wins (equivalently $v_1^2 \geq v_2$) then the payments look as follows:

   $$p_1(v_1, v_2) = \sqrt{v_2},$$
   $$p_2(v_1, v_2) = 0.$$

   For any profile (v_1, v_2), if bidder 2 wins (equivalently $v_2 > v_1^2$) then the payments look as follows:

   $$p_1(v_1, v_2) = 0,$$
   $$p_2(v_1, v_2) = v_1^2.$$

   By definition, this payment scheme satisfies the property that if a bidder does not win the object then he pays zero. Also, a losing bidder makes zero net utility. A winning bidder always pays an amount less than his bid. As a result, this mechanism is individually rational.

   To see why p implements f, we can invoke the revenue equivalence theorem or use a direct argument. Consider bidder 1 at a profile (v_1, v_2). If $v_1^2 \geq v_2$, then his net utility from truth is $v_1 - \sqrt{v_2} \geq 0$. By deviating if he loses, then his net utility is zero. By deviating if he wins, then his payment remains $\sqrt{v_2}$, and hence, his net utility remains unchanged. If $v_2 > v_1^2$, then the net utility for bidder 1 from truth is zero. By deviating if he still loses, then his net utility remains zero. By deviating if he wins, he will pay $\sqrt{v_2} > v_1$, and hence, his net utility will be negative $(v_1 - \sqrt{v_2})$. So, no deviation is profitable. A similar argument works for bidder 2.

(c) What are the allocations and payments of the bidders at valuation profiles $(2,3)$ and $(3,2)$? **4 marks**

Answer: At $(2,3)$, bidder 1 wins and pays $\sqrt{3}$ and bidder 2 does not pay anything. At $(3,2)$, bidder 1 wins and pays $\sqrt{2}$ and bidder 2 does not pay anything.

2. Suppose a single object is being sold by a seller to two bidders. Further, suppose both the bidders draw their values using uniform distribution from intervals $[0,10]$ (for bidder 1) and $[5,15]$ (for bidder 2).

(a) Describe the optimal mechanism for the seller. **6 marks**

Answer: Denote by $[l_i, h_i]$ the type space of agent i. Fix a mechanism (a,p). We first use revenue equivalence to get the expression for net expected utility for agent $i \in N$ and for every t_i as:

$$u_i(t_i) = u_i(l_i) + \int_{l_i}^{t_i} \alpha_i(x_i)dx_i.$$

Simplifying this, for every $i \in N$ and for every t_i,

$$\pi_i(t_i) = -u_i(l_i) + \alpha_i(t_i)t_i - \int_{l_i}^{t_i} \alpha_i(x_i)dx_i.$$

Using the standard tricks, we can write down the revenue in the mechanism (a,p) as

$$\begin{aligned}\Pi(a,p) &= \sum_{i\in N}(-u_i(l_i)) + \sum_{i\in N}\int_{l_i}^{h_i} w_i(x_i)\alpha_i(x_i)f_i(x_i)dx_i \\ &= \sum_{i\in N}(-u_i(l_i)) + \int_{\times_{i=1}^{n}[l_i,h_i]} \big(\sum_{i\in N} w_i(x_i)a_i(x)\big)f(x)dx.\end{aligned}$$

Individual rationality and revenue maximization implies that $u_i(l_i) = 0$. Revenue maximization then implies point-wise maximization with respect to virtual valuations.

In our problem, $w_1(x_1) = 2x_1 - 10$ and $w_2(x_2) = 2x_2 - 15$. So, reserve price of bidder 1 is 5 and bidder 2 is 7.5. The optimal auction is then simple. Choose the bidders whose values are above their respective reserve prices. Out of these bidders, choose the bidder who has the highest virtual value and give him the object. If all the bidders have value less than their respective reserve prices, then the object is not sold.

The payments of the bidders are based on "cutoffs". A bidder who does not win the object pays zero. A bidder who wins the object pays the minimum amount he needs to bid to win the object given the value/bid of the other bidder. Note that this ensures that $u_i(l_i) = 0$ for all $i \in N$ - this is because if a bidder does not win an object when his value is l_i he does not pay anything and his net utility is zero; if he wins the object when his value is l_i then he pays l_i and his net utility is zero.

(b) Give a valuation profile where the optimal mechanism allocates the object efficiently and another valuation profile where the optimal mechanism allocates the object but does it inefficiently. **6 marks**

Answer: Consider the value profile $(5, 8)$. The virtual valuation profile is $(0, 1)$. So, bidder 2 wins the object, which is efficient. Consider the value profile $(7, 8)$. The virtual valuation profile is $(4, 1)$. So, bidder 1 wins the object, and this is inefficient.

3. Consider the sale of 2 identical units of a single object to three bidders who want at most one unit. Each bidder i gets a value $v_i \geq 0$ if he wins a unit and gets zero value if he does not get a unit. The units are sold efficiently. So, at any valuation profile, 2 out of 3 bidders win a unit.

(a) Is there a Groves payment scheme p where each bidder who wins a unit pays an amount equal to the difference between the value of the losing bidder and the other winning bidder? For instance, at a valuation profile (v_1, v_2, v_3) if bidders 1 and 2 win a unit each, then bidder 1 pays $v_3 - v_2$ (which may be negative) and bidder 2 pays $v_3 - v_1$. **6 marks**

Answer: At a valuation profile $v \equiv (v_1, v_2, v_3)$, let $f_i^e(v) \in \{1, 0\}$ denote if bidder i wins a unit or not. The Groves payment scheme at a valuation profile $v \equiv (v_1, v_2, v_3)$ is given by

$$p_i(v) = h_i(v) - \sum_{j \neq i} v_j f_j^e(v) \qquad \forall\, i \in N.$$

If bidder i wins at a valuation profile v, then the other unit must be given to the bidder with the highest value among the remaining two bidders (because of efficiency). Suppose j is the other winner at valuation profile v. Hence, the payment of bidder i can be written as

$$p_i(v) = h_i(v) - v_j.$$

The losing bidder is the bidder with the minimum value. If i is a winner, then we can write the value of the losing bidder as $\min_{k \neq i} v_k$. By setting $h_i(v) = \min_{k \neq i} v_k$, we get the desired Groves payment scheme.

(b) What should the losing bidder (the bidder who does not win any unit) pay in this Groves payment scheme? **4 marks**

Answer: The h_i function for each bidder i can be chosen differently. We were only asked to choose the h_i function for winning bidders in a certain way. But this puts no restriction on the h_i function of the losing bidder. Hence, if i is a losing bidder, then his Groves payment at a valuation profile v is given by

$$p_i(v) = h_i(v) - \sum_{j \neq i} v_j.$$

4. A seller is selling two objects a and b to three bidders. Each bidder has combinatorial valuations, i.e., for each subset of $\{a, b\}$, every bidder associates a non-negative value with zero value for $\emptyset$. A valuation profile (v_1, v_2, v_3) is given below.

$$v_1(\{a\}) = 5, v_1(\{b\}) = 5, v_1(\{a, b\}) = 13$$
$$v_2(\{a\}) = 7, v_2(\{b\}) = 4, v_2(\{a, b\}) = 9$$
$$v_3(\{a\}) = 0, v_3(\{b\}) = 10, v_3(\{a, b\}) = 10$$

(a) Describe the efficient allocation at valuation profile (v_1, v_2, v_3). **2 marks**

Answer: Efficient to give object a to bidder 2 and object b to bidder 3.

(b) What will be the payments of the bidders in the pivotal mechanism at valuation profile (v_1, v_2, v_3). **6 marks**

Answer: The payment of bidder 1 is zero. The payment of bidder 2 is: $15 - 10 = 5$. The payment of bidder 3 is $13 - 7 = 6$.

5. Consider the bilateral trading model with a single buyer whose value is distributed uniformly between $[0, 10]$ and a seller whose cost is uniformly distributed between $[5, 10]$.

- Describe the modified pivotal mechanism for this setting. **4 marks**

  Answer: The modified pivotal mechanism is efficient - so there is trade if and only if the value of the buyer is larger than the cost of the seller. If there is no trade, then nobody pays anything or is paid anything. If there is trade at a profile (v_b, v_s), then the payment of buyer is $W(0, v_s) + v_s = v_s$, where W is the total welfare function, and the payment of the seller is $W(v_b, -10) - v_b = \max(v_b - 10, 0) - v_b = 0 - v_b = -v_b$.

- Show a type profile where the modified pivotal mechanism is not budget-balanced. **4 marks**

Answer: For the modified pivotal mechanism to not balance the budget, there must be trade. The total payment at a profile (v_b, v_s) where there is trade is $v_s - v_b$ with the trade condition $v_b > v_s$. A type profile where the modified pivotal mechanism is not budget-balanced will be: $v_s = 1$ and $v_b = 2$. Here, there will be trade and the buyer pays 1 but the seller receives 2.