MICROECONOMICS 2 - FINAL EXAMINATION
TOTAL SCORE: 50

1. Consider the signaling game in Figure 1. There are three types of Sender: $\{t_1, t_2, t_3\}$. The probability that a Sender is of either of the three types is $\frac{1}{3}$. The Sender has two strategies L and R and the Receiver has two strategies u and d. The relevant payoffs for every strategy profile is shown in Figure 1 (first number indicates the payoff of the Sender and the second number indicates the payoff of the Receiver).

   - POOLING ON L: Specify all pure strategy pooling perfect Bayesian equilibria in which all Sender types play L. **7 marks**

     SOLUTION: Since the equilibrium strategy is pooling on L, the receiver must place probability $\frac{1}{3}$ on all types playing L. Also, when the receiver sees L, it is a dominant strategy for him to play u. Now, let us compute equilibrium beliefs for the receiver when he sees R. Let the belief on top node (R played by type t_1) be q_1, on middle node (R played by type t_2) be q_2, and on bottom node (R played by type t_3) be $1 - q_1 - q_2$. Then, the expected payoff from playing u is $q_1 + q_2$ and that from playing d is $1 - q_1 - q_2$. So, the receiver plays u if and only if $q_1 + q_2 \geq \frac{1}{2}$ with $0 \leq q_1 \leq 1$ and $0 \leq q_2 \leq 1$. Now, note that if the receiver plays d on seeing R, then type t_3 is better of deviating in equilibrium - he gets 2 by deviating to R but gets 1 by playing L. But if the receiver plays u on seeing R, nobody has any incentive to deviate. Hence, in equilibrium, the receiver must place $q_1 + q_2 \geq \frac{1}{2}$ beliefs and play u on seeing R and L.

   - Identify the set of all pure strategy pooling on L perfect Bayesian equilibria which satisfy the dominance criterion (i.e., beliefs on messages dominated for a type must be zero). **4 marks**

     SOLUTION: We see that only for type t_1 the minimum payoff from playing L dominates the maximum payoff from playing R. Hence, $q_1 = 0$ in any pooling on L equilibria satisfying dominance criterion. Thus, in equilibrium satisfying dominance criterion, the receiver must place the following beliefs: $q_1 = 0$, $q_2 \geq \frac{1}{2}$, and must play u on seeing R and L.

   - Identify the set of all pure strategy pooling on L perfect Bayesian equilibria which satisfy the Intuitive criterion. **4 marks**

     SOLUTION: Since $q_1 = 0$ due to dominance criterion, $q_1 = 0$ due to Intuitive criterion also. We also see that the equilibrium payoff of type t_2 is 2, but the maximum payoff type t_2 can obtain is 1. Hence, by Intuitive criterion we set $q_2 = 0$ also. But we know that for any pooling on L, we need $q_1 + q_2 \geq \frac{1}{2}$. So, the set of all pure strategy pooling on L equilibria satisfying the Intuitive criterion is an empty set.

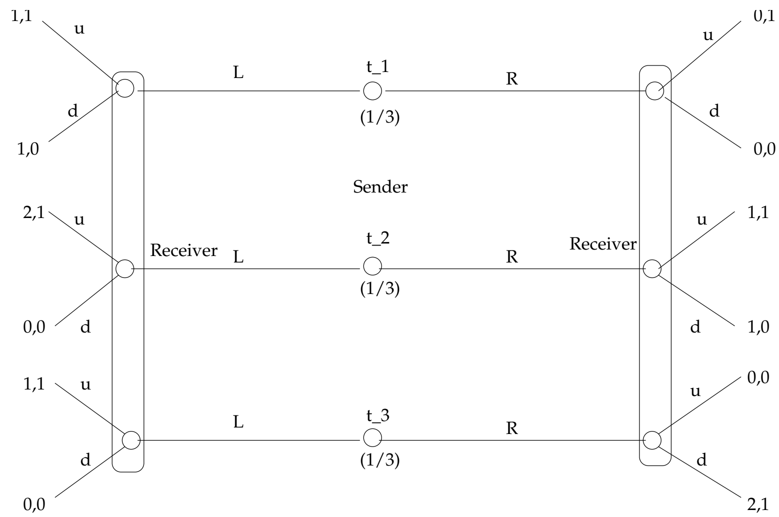

Figure 1: A Signaling Game

2. Find all (both pure and mixed) weak perfect Bayesian Nash equilibria of the game in Figure 2. **10 marks**

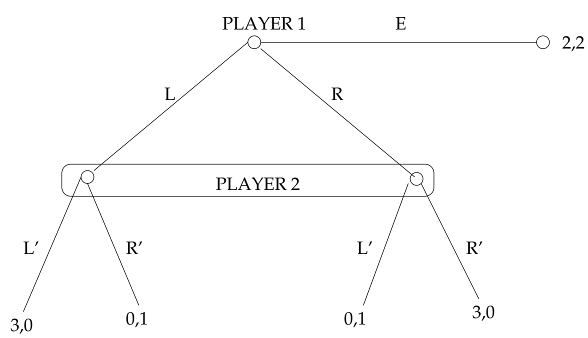

Figure 2: An Extensive Form Game

Solution: Suppose Player 2 attaches beliefs λ on left node and $(1-\lambda)$ on right node. Then, playing L' he gets payoff $\pi_2(L') = 1-\lambda$, and playing R' he gets payoff $\pi_2(R') = \lambda$. So, Player 2 randomizes between L' and R' if $\lambda = \frac{1}{2}$, plays L' if $\lambda < \frac{1}{2}$, and plays R' if $\lambda > \frac{1}{2}$. Consider the possibilities for Player 2.

Player 2 plays L'. Best strategy for Player 1 is to choose L. By Bayes' rule, $\lambda = 1$. But this means that Player 2 must play R'. So, no equilibrium is possible.

Player 2 plays R'. Best strategy for Player 1 is to choose R. By Bayes' rule, $\lambda = 0$, which means Player 2 must play L'. So, no equilibrium possible.

Player 2 mixes L' and R': Suppose it mixes with weight μ on L' and $(1-\mu)$ on R'. Note that $\mu \in (0,1)$. Also, we know that for Player 2 to mix, $\lambda = \frac{1}{2}$. Consider the payoff Player 1 from

$$\text{Playing } L\text{: } \pi_1(L) = 3\mu$$
$$\text{Playing } R\text{: } \pi_1(R) = 3(1-\mu)$$
$$\text{Playing } E\text{: } \pi_1(E) = 2.$$

Consider the possibilities for Player 1 now - since $\lambda = \frac{1}{2}$, note that he cannot play *only* L or *only* R or mix between L and E *only* or mix between R and E *only*.

Mixes L and R. Player 1 will mix L and R if he gets equal payoff from L and R, which is possible if $\mu = \frac{1}{2}$. But in that case, the best response of Player 1 is to play E. So, no equilibrium is possible.

Plays E. For Player 1 to play E, we must have $3\mu \leq 2$ and $3(1-\mu) \leq 2$. This gives $\mu \in [\frac{1}{3}, \frac{2}{3}]$. Also, $\lambda = \frac{1}{2}$ must be the belief of Player 2.

So, there is a class of weak perfect Bayesian equilibrium of this game, where Player 1 plays E, Player 2 plays $\mu L' + (1-\mu)R'$ for some $\mu \in [\frac{1}{3}, \frac{2}{3}]$, and Player 2 has belief that both the nodes in his information set will be reached with probability $\frac{1}{2}$.

3. Consider a model of adverse selection where the type of each worker is drawn uniformly from $[1,2]$. The type of the worker is unobservable to firms (firms are risk-neutral, price-taking, and use identical constant returns to scale technology). Each worker's home production function is $r(\theta) = \frac{\theta^2-1}{2}$.

   - If workers' types were perfectly observable, then what is the Pareto optimal allocation? **2 marks**

     Solution: A worker of type θ must be hired in a Pareto optimal allocation if $\theta \geq r(\theta) = \frac{\theta^2-1}{2}$, i.e., $\theta^2 \leq 2\theta + 1$. But $\theta \leq 2$, implies that $\theta^2 \leq 2\theta < 2\theta + 1$. Hence, all workers must be hired in a Pareto optimal allocation.

   - Suppose workers' types are not observable.

     – What is the competitive equilibrium wage and which workers are hired by firms? **4 marks**
       Solution: Workers' wage is given by $w^* = E[\theta : r(\theta) \leq w^*]$. This simplifies to $w^* = E[\theta : \theta \leq \sqrt{2w^*+1}] = 1 + \frac{\sqrt{2w^*+1}-1}{2}$. Solving for w^*, we get $w^* = 0$ or $w^* = \frac{3}{2}$.

The case of $w^* = 0$ is interesting. If we *assume* that workers who are indifferent between joining and not joining the firm join the firm, then at $w^* = 0$, only workers with $\theta = 1$ join. This implies expected productivity is 1, which does not equal zero, the wage. So, $w^* = 0$ is not an equilibrium in this case. However, if we assume workers who are indifferent between joining and not joining do not join the firm, then $w^* = 0$ is an equilibrium.
Note that $r(\theta) = \frac{\theta^2-1}{2} \leq \frac{3}{2}$ for all $\theta \in [1, 2]$. Hence, if $w^* = \frac{3}{2}$, then all the workers will be hired. If $w^* = 0$, then the set of workers hired are none (if $w^* = 0$ is equilibrium).

- Describe adverse selection in this model. **2 marks**
  SOLUTION: If $w^* = \frac{3}{2}$, then no adverse selection occurs in this model since the set of workers hired in equilibrium is the same set of workers hired in a Pareto optimal allocation. However, the other equilibrium involving $w^* = 0$ has nobody getting hired, resulting in adverse selection.
- Is this competitive equilibrium constrained Pareto optimum? **2 marks**
  SOLUTION: The equilibrium with $w^* = \frac{3}{2}$ is Pareto optimal, and hence constrained Pareto optimal. However, the equilibrium $w^* = 0$ can be Pareto improved by an intervention (where the government announces wage equal to $\frac{3}{2}$).

4. Consider the following hidden action principal agent model (moral hazard) with three possible actions $E = \{e_1, e_2, e_3\}$. There are two possible profit levels $\pi_H = 10$ and $\pi_L = 0$. The probabilities of π_H conditional on the effort levels are

$$f(\pi_H|e_1) = \frac{2}{3}, f(\pi_H|e_2) = \frac{1}{2}, f(\pi_H|e_3) = \frac{1}{3}.$$

The agent's effort cost function $g(\cdot)$ is given by

$$g(e_1) = \frac{5}{3}, g(e_2) = \frac{8}{5}, g(e_3) = \frac{4}{3}.$$

Finally, let the value of wage to the agent is given by $v(w) = \sqrt{w}$, and the reservation utility of the agent is $\bar{u} = 0$.

- What is the optimal contract when effort is observable? **3 marks**

  SOLUTION: Let w_H and w_L be the wages with high and low profits respectively. We know that with observable effort, the optimal effort level is one which maximizes $10f(\pi_H|e_1) - g(e)^2$. Computing these values for e_1, e_2, e_3, we get respectively $\frac{35}{9}, \frac{61}{25}, \frac{14}{9}$. This means e_1 is optimal effort level, with wage $\frac{25}{9}$.

- Show that if effort is not observable then e_2 is not implementable. **3 marks**

  SOLUTION: Denote $\sqrt{w_H}$ as w_1 and $\sqrt{w_L}$ as w_2. Now, consider the incentive compatibility constraints for implementing e_2.

$$\frac{1}{2}w_1 + \frac{1}{2}w_2 - \frac{8}{5} \geq \frac{2}{3}w_1 + \frac{1}{3}w_2 - \frac{5}{3}$$
$$\frac{1}{2}w_1 + \frac{1}{2}w_2 - \frac{8}{5} \geq \frac{1}{3}w_1 + \frac{2}{3}w_2 - \frac{4}{3}.$$

  Adding the two constraints, we get $3 \geq \frac{16}{5}$. This is a contradiction. Hence, e_2 cannot be implemented.

- What is the minimum expected wage for implementing e_1 and e_3 when effort is not observable? **6 marks**

  SOLUTION: Since e_2 cannot be implemented, we consider two possible cases of implementing e_1 and e_3 separately. As before, let $w_1 \equiv \sqrt{w_H}$ and $w_2 \equiv \sqrt{w_L}$.

  IMPLEMENTING e_1. The participation constraint is $\frac{2}{3}w_1 + \frac{1}{3}w_2 \geq \frac{5}{3}$, which is equivalent to saying

$$2w_1 + w_2 \geq 5 \tag{1}$$

  The incentive compatibility (IC) constraint for e_2 is: $\frac{2}{3}w_1+\frac{1}{3}w_2-\frac{5}{3} \geq \frac{1}{2}w_1+\frac{1}{2}w_2-\frac{8}{5}$. Rearranging this, we get $w_1-w_2 \geq \frac{2}{5}$. The IC constraint for e_3 is: $\frac{2}{3}w_1+\frac{1}{3}w_2-\frac{5}{3} \geq \frac{1}{3}w_1 + \frac{2}{3}w_2 - \frac{4}{3}$. Rearranging this we get $w_1 - w_2 \geq 1$. So, only one IC constraint need to hold, and that is:

$$w_1 - w_2 \geq 1. \tag{2}$$

  The optimization problem can now be written as

$$\min \frac{2}{3}w_1^2 + \frac{1}{3}w_2^2$$
$$\text{s.t.}$$
$$2w_1 + w_2 \geq 5$$
$$w_1 - w_2 \geq 1$$
$$w_1, w_2 \geq 0.$$

  Alternatively, it can be written as

$$\max -(2w_1^2 + w_2^2)$$
$$\text{s.t.}$$
$$2w_1 + w_2 \geq 5$$
$$w_1 - w_2 \geq 1$$
$$w_1, w_2 \geq 0.$$

Using Lagrange multipliers $\lambda \geq 0$ and $\mu \geq 0$ for first and second inequalities, we get the following optimization problem.

$$\max_{w_1, w_2 \geq 0} -(2w_1^2 + w_2^2) + \lambda(2w_1 + w_2 - 5) + \mu(w_1 - w_2 - 1).$$

Taking the first order conditions we get (with respect to w_1 and w_2 respectively)

$$2\lambda + \mu = 4w_1$$
$$\lambda - \mu = 2w_2.$$

Note that $\mu = \frac{4(w_1 - w_2)}{3} \geq \frac{4}{3} > 0$, where the inequality followed from Equation 2. Hence, $\lambda = \mu + 2w_2 > 0$. So, both the constraints must bind at optimality. So, solving for w_1, w_2 using $2w_1 + w_2 = 5$ and $w_1 - w_2 = 1$, we get $w_1 = 2$ and $w_2 = 1$. This gives $w_H = 4$ and $w_L = 1$, and expected wage paid is $\frac{2}{3}w_H + \frac{1}{3}w_L = 3$.

Implementing e_3. In this case, the manager can be given a fixed wage. Let $w_H = w_L = v^{-1}(g(e_3) + \bar{u}) = \frac{16}{9}$. Note that since the wage is fixed, it is optimal for the manager to choose the lowest effort level, i.e., e_3. So, the IC constraints hold. Further, we see that the participation constraint is binding (by definition of wage). Finally, this is an optimal wage since this equals the expected wage in the observed case. So, the expected wage given in this case is $\frac{16}{9}$.

- Determine the optimal contract when effort is not observable. **3 marks**

  Solution: Expected profit by implementing e_1 is

$$10\frac{2}{3} - 3 = \frac{11}{3}.$$

  Expected profit by implementing e_3 is

$$10\frac{1}{3} - \frac{16}{9} = \frac{14}{9}.$$

  So, it is optimal for the firm to implement e_1 at wage $w_H = 4$ and $w_L = 1$.