Assignment 1

Due date. **August 7, 2018**

1. Prove that every cycle of a bipartite graph has even number of edges. Also, prove that if every cycle of an undirected graph has even number of edges then the graph must be bipartite.

2. Let (N, T) be a tree of an undirected graph $G = (N, E)$. Show the following:

$$\sum_{i \in N} \big[2 - d(i)\big] = 2,$$

   where $d(i)$ is the degree of vertex i in tree (N, T).

3. Show that every tree which has a vertex of degree $m \geq 2$ has at least m vertices of degree 1.

4. Suppose G is an undirected weighted connected graph. An edge is called a **default edge** if it has the minimum weight over all edges in G. Show that a default edge belongs to some minimum cost spanning tree of G.

5. Consider the following algorithm to find an MCST of any undirected connected graph $G = (N, E, w)$. Let $E = \{e_1, \ldots, e_k\}$ be the set of k edges. We order the set of edges such that $w(e_1) \leq w(e_2) \leq \ldots \leq w(e_k)$.

   The algorithm is iterative. In every iteration t, it maintains a set of edges E^t such that E^t contains no cycles. It adds an edge $e_j \in E \setminus E^t$ to E^t such that $E^t \cup \{e_j\}$ contains no cycle and e_j is a least weight edge satisfying this no cycle condition, i.e., for any edge $e_p \in E \setminus E^t$ such that $E \cup \{e_p\}$ has no cycles, we have that $w(e_j) \leq w(e_p)$. The algorithm terminates as soon as the number of edges in E^t is $n - 1$, where $n = \#N$ is the number of vertices in G. The output of the algorithm is (N, E^{t^*}), where t^* is the final iteration.

   The algorithm is initialized by setting $E^1 = \{e_1\}$, the least weight edge. Show that the algorithm produces an MCST.

6. Let $G = (N, E, w)$ be an undirected weighted connected graph. Assume that $w(e) \neq w(e')$ for every pair of edges $e, e' \in E$. For each $i \in N$, denote by e_i the edge which has the minimum weight over all the edges to which i is an endpoint. Let $\bar{E} = \cup_{i \in N} e_i$. Show that the subgraph $(N, \bar{E})$ is acyclic. Further, show that if $|\bar{E}| = |N| - 1$, then $(N, \bar{E})$ is an MCST of G.

7. Consider any $m \times n$ matrix. Show that the maximum number of non-zero entries such that no two entries are in the same line (i.e., same column or row) is equal to the minimum number of lines that include all non-zero entries.