Assignment 1

Debasis Mishra*

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1. Prove that every cycle of a bipartite graph has even number of edges. Also, prove that if every cycle of an undirected graph has even number of edges then the graph must be bipartite.

2. Show whether a matching exists or not in the bipartite graphs in Figure 1. In each case, give explanation for your answer.

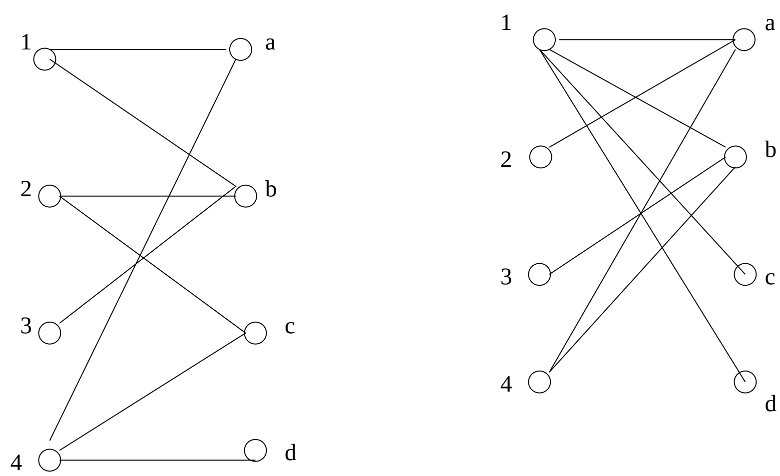

Figure 1: Two Bipartite Graphs

3. Find a maximum matching and a minimum vertex cover of the graphs in Figure 1.

4. Prove Hall's marriage theorem using König's theorem.

5. Consider any $m \times n$ matrix. Show that the maximum number of non-zero entries such that no two entries are in the same line (i.e., same column or row) is equal to the minimum number of lines that include all non-zero entries.

*Planning Unit, Indian Statistical Institute, 7 Shahid Jit Singh Marg, New Delhi 110016, India, E-mail: dmishra@isid.ac.in

6. Let (N, T) be a tree of an undirected graph $G = (N, E)$. Show the following:

$$\sum_{i \in N} \big[2 - d(i)\big] = 2,$$

where $d(i)$ is the degree of vertex i in tree (N, T).

7. Show that every tree which has a vertex of degree $m \geq 2$ has at least m vertices of degree 1.

8. Suppose G is an undirected weighted connected graph. An edge is called a **default edge** if it has the minimum weight over all edges in G. Show that a default edge belongs to some minimum cost spanning tree of G.

9. Find a minimum cost spanning tree of the graph in Figure 2.

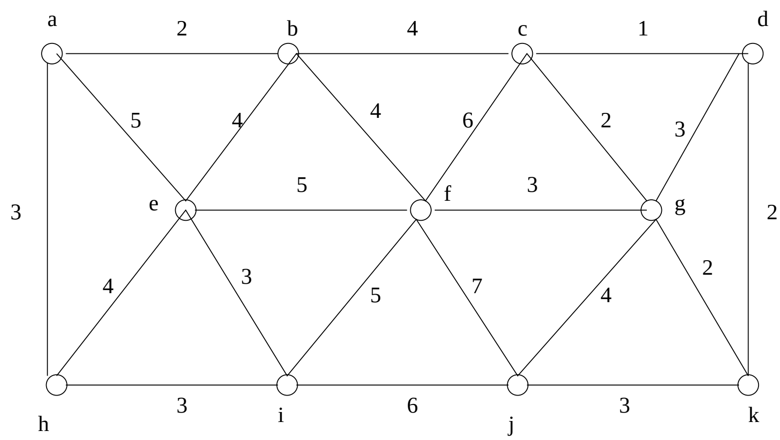

Figure 2: A connected graph with weights

10. Consider the following algorithm to find an MCST of any undirected graph $G = (N, E, w)$. Let $E = \{e_1, \ldots, e_k\}$ be the set of k edges. We order the set of edges such that $w(e_1) \leq w(e_2) \leq \ldots \leq w(e_k)$.

    The algorithm is iterative. In every iteration t, it maintains a set of edges E^t such that E^t contains no cycles. It adds an edge $e_j \in E \setminus E^t$ to E^t such that $E^t \cup \{e_j\}$ contains no cycle and e_j is a least weight edge satisfying this no cycle condition, i.e., for any edge $e_p \in E \setminus E^t$ such that $E \cup \{e_p\}$ has no cycles, we have that $w(e_j) \leq w(e_p)$. The algorithm terminates as soon as the number of edges in E^t is $n - 1$, where $n = \#N$ is the number of vertices in G. The output of the algorithm is (N, E^{t^*}), where t^* is the final iteration.

    The algorithm is initialized by setting $E^1 = \{e_1\}$, the least weight edge. Show that the algorithm produces an MCST.