Assignment 2

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1. Suppose G is a directed weighted complete graph. Let the weights of edges of G satisfy the following **triangle inequality**: for any three edges (i,j), (j,k), and (i,k) we have $w(i,j)+w(j,k) \geq w(i,k)$. Show that a potential of G exists if and only if $w(i,j)+w(j,i) \geq 0$.

   Answer: In the presence of triangle inequality, shortest path from i to j is the edge (i,j). Hence, $s(i,j) = w(i,j)$ for all $i,j \in N$. Now, consider a cycle $C = (i^1, i^2, \ldots, i^k, i^1)$. Suppose for all $i,j \in N$, we have $w(i,j)+w(j,i) \geq 0$. But $l(C) \geq s(i^1,i^k) + w(i^k,i^1) = w(i^1,i^k) + w(i^k,i^1) \geq 0$. Hence, a potential exists. If a potential exists, then $w(i,j)+w(j,i)$ is the length of cycle (i,j,i), and hence it should be non-negative.

2. Find a feasible solution or determine that there are no feasible solutions for the following system of difference inequalities.

$$
\begin{aligned}
x_1 - x_2 &\leq 4 \\
x_1 - x_5 &\leq 5 \\
x_2 - x_4 &\leq -6 \\
x_3 - x_2 &\leq 1 \\
x_4 - x_1 &\leq 3 \\
x_4 - x_3 &\leq 5 \\
x_4 - x_5 &\leq 10 \\
x_5 - x_3 &\leq -4 \\
x_5 - x_4 &\leq -8.
\end{aligned}
$$

   Answer: No solution exists. Check that a negative cycle exists of the form: $(1,4,2,3,5,1)$.

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3. For the directed graphs in Figure 1, verify if a potential exists. If a potential exists, then identify at least one potential.

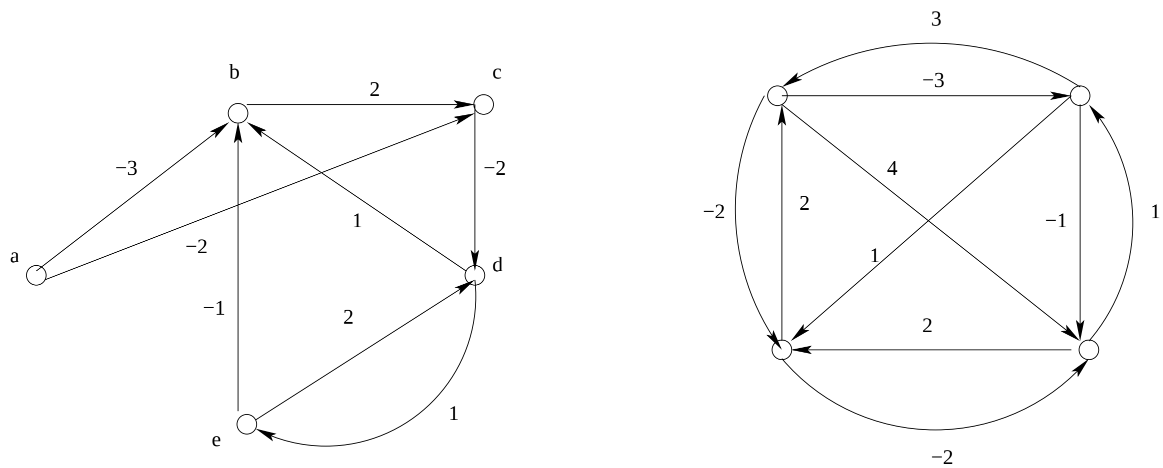

Figure 1: Two directed graphs

Answer: In both graphs, potential exists. Fix vertex a in the first graph, and assign it potential zero. Find shortest path from a to every other vertex to get the other potentials. For the second graph, one can fix any vertex, and apply the same shortest path method.

4. Suppose $G = (N, E, w)$ is a strongly connected digraph which has no cycle of negative length. Let p be a potential of G such that $p(i) = 0$ and $p(j) = s(i, j)$ for all $j \in N \setminus \{i\}$, where $s(i, j)$ is the shortest path from i to j in G. Let q be any other potential of G such that $q(i) = 0$. Show that $p(j) \geq q(j)$ for all $j \in N$.

   Answer: Consider any $j \in N$. Let the shortest path from i to j be $P = (i, i^1, \ldots, i^k, j)$. Then, $s(i, j) = l(P) = w(i, i^1) + \ldots + w(i^k, j) \geq [q(i^1) - q(i)] + [q(i^2) - q(i^1)] + \ldots + [q(j) - q(i^k)] = q(j) - q(i) = q(j)$. Using $s(i, j) = p(j)$, we get that $p(j) \geq q(j)$.

5. Find the maximum flow and the minimum capacity cut of the flow graph in Figure 2.

   Answer: Consider the following feasible flow for the flow graph in Figure 2.

$$
\begin{aligned}
&f(s,1) = 3,\ f(s,2) = 4\\
&f(1,3) = 3,\ f(2,3) = 2\\
&f(2,4) = 2,\ f(3,t) = 4\\
&f(3,4) = 1,\ f(4,t) = 3.
\end{aligned}
$$

This feasible flow is a maximum flow since the cut $(\{s, 1, 2, 3\}, \{4, t\})$ is a saturated cut for this flow, and is the minimum capacity cut.

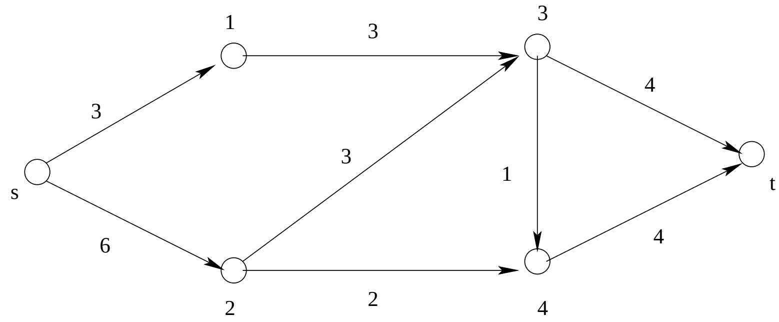

Figure 2: Maximum flow in a flow graph

6. Consider the flow graph in Figure 3. Compute the maximum flow of this flow graph using two approaches (both approaches run Ford-Fulkerson algorithm but asks you to pick a path from s to t in the residual graph in a particular way if there are more than one such path):

   - In the first approach, always pick an augmenting path from s to t in the residual graph which has the maximum number of edges.
   - In the second approach, always pick an augmenting path from s to t in the residual graph which has the minimum number of edges.

   Compare the number of iterations of Ford-Fulkerson algorithm in both approaches.

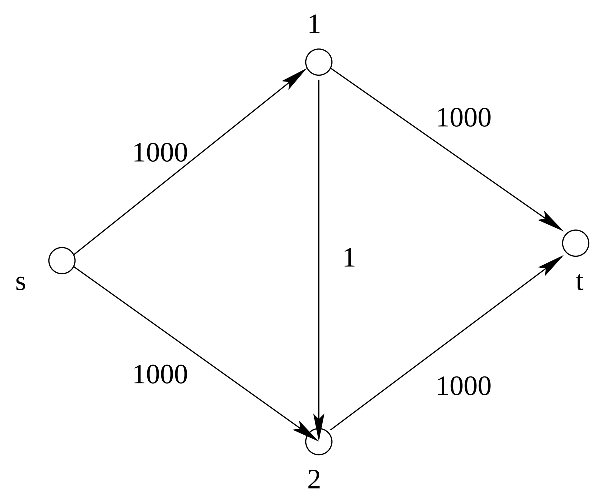

Figure 3: Maximum flow in a flow graph

Answer: If we start with zero flow on each edge, then the residual graph has three paths from s to t: $(s, 1, t), (s, 2, t)$, and $(s, 1, 2, t)$. If we pick the longer path, then we will only flow 1 unit extra through it. The next iteration, if we again pick the longer path, then we will pick $(s, 2, 1, t)$ with an increase of 1 unit of flow. So, it will take 2000 iterations. Instead, if we pick the shorter path every time, then, we will flow 1000 units in the first iteration and 1000 units more in the second iteration, and find a maximum flow (in just two iterations).

7. Let $G = (N, E, c)$ be a flow graph and f be a feasible flow of this flow graph. Assume that $c : E \to \mathbb{Z}_+$ (integer capacities) and $f : E \to \mathbb{R}_+$ (real flow). Show that there exists another feasible flow f' of this flow graph such that $\nu(f') = \lceil \nu(f) \rceil$ and $f'(i,j) \in \{\lfloor f(i,j) \rfloor, \lceil f(i,j) \rceil\}$ for all $(i,j) \in E$.

   Answer: Consider the maximum flow in a new flow graph (with same set of nodes and edges as G) where there are both upper and lower bound on the flow amount. For every $(i,j) \in E$, the upper bound is $\lceil f(i,j) \rceil$ and the lower bound is $\lfloor f(i,j) \rfloor$. Also, the capacity of the source node is set at $\lceil \nu(f) \rceil$. The current flow f satisfies these capacity bounds and is a feasible flow of the new flow graph.

   Now, a maximum flow of this graph is integral since the lower and upper bounds are integral (note that in the class we assumed the lower bound to be 0, but the same proofs/algorithms can be modified a little bit to show that any lower bound will work). Let this maximum flow be f^*. Because f is a feasible flow of the new flow graph, we get $\nu(f^*) \geq \nu(f)$.

   Also, $\nu(f^*) \geq \lceil \nu(f) \rceil$. This is because if $\nu(f^*) < \lceil \nu(f) \rceil$, then $\nu(f^*) < \nu(f)$ (since $\nu(f^*)$ is an integer), which contradicts the fact that is the maximum flow in the new flow graph.

   The proof is now complete by noting that the maximum flow of the new graph cannot exceed the source capacity constraint - so, $\nu(f^*) \leq \lceil \nu(f) \rceil$. Hence, $\nu(f^*) = \lceil \nu(f) \rceil$. So, f^* is a feasible flow of G such that $\nu(f^*) = \lceil \nu(f) \rceil$ and $f^*(i,j) \in \{\lfloor f(i,j) \rfloor, \lceil f(i,j) \rceil\}$ for all $(i,j) \in E$.