Assignment 2

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1. Let $A \in \mathbb{R}^{m \times n}$. Define $K(\alpha) = \{b \in \mathbb{R}^m : b = Ax, \|x\| \leq \alpha\}$ for all $\alpha \in \mathbb{R}$. Show that $K(\alpha)$ is convex for all $\alpha \in \mathbb{R}$.

2. A set $C \subseteq \mathbb{R}^n$ is bounded if there exists a $n-$dimensional ball $B(r) = \{x \in \mathbb{R}^n : \|x\| \leq r\}$ of some finite radius r such that $C \subseteq B(r)$. Show that convex hull of C is bounded.

3. Suppose $C \subseteq \mathbb{R}^n$ is any arbitrary set. Consider $x \in \mathbb{R}^n$ such that $x \notin H(C)$ (i.e., outside the convex hull of C). Can we strictly separate x from C? If not, can you think of a condition on C such that x and C can be separated?

4. Let $C \subseteq \mathbb{R}^n$. Then C is a closed convex set if and only if $C = \cap \mathbb{F}$, where $\mathbb{F}$ is a collection (possibly infinite of them) of half-spaces. (Hint: Consider the intersection of half-spaces that contain C and use separating hyperplane theorem)

5. Sketch the cone generated by the columns of following matrix:

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

   What is the cone generated by just the first and third columns of the matrix? Suppose $b = (1, 0)$. Then decide if $Ax = b$ has a solution with $x \in \mathbb{R}^3_+$.

   Draw $cone(A)$ and $H(A)$, the convex hull of columns of A. How are the points in $cone(A)$ and $H(A)$ related?

6. Write down the Farkas alternative for the following system of constraints.

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\begin{aligned}
x_1 + 2x_2 + 3x_3 &\leq 5 \\
x_2 + 3x_2 - 2x_3 &\geq 7 \\
x_1 + x_2 + x_3 &\leq 2 \\
x_1 - 2x_2 - 3x_3 &= 3 \\
x_1 &\geq 0.
\end{aligned}
$$

7. Use Farkas Lemma to decide if the following system of equations have a solution.

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\begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}
$$

8. Let A be a $m \times n$ matrix. Define $F = \{x \in \mathbb{R}^n_+ : Ax = 0, \sum_{j=1}^n x_j = 1\}$ and $G = \{y \in \mathbb{R}^m : yA > 0\}$. Show that either $F \neq \emptyset$ or $G \neq \emptyset$ but not both.

9. Let A be a $m \times n$ matrix. Prove that the system $Ax = 0$ has a **non-zero, non-negative** solution (i.e, $x \geq 0$ and $x \neq 0$) or there is a $y \in \mathbb{R}^m$ such that $yA > 0$, but not both. (HINT: Use the result from the previous question.)