# Assignment 3


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1. Let $A \in \mathbb{R}^{m \times n}$. Define $K(\alpha) = \{b \in \mathbb{R}^m : b = Ax, \|x\| \leq \alpha\}$ for all $\alpha \in \mathbb{R}$. Show that $K(\alpha)$ is convex for all $\alpha \in \mathbb{R}$.

   **Answer:** Let $b^1, b^2 \in K(\alpha)$. By definition, $b^1 = Ax^1$ and $b^2 = Ax^2$ for some $x^1, x^2$ with $\|x^1\| \leq \alpha$ and $\|x^2\| \leq \alpha$. Let $b^3 = b^1\lambda + b^2(1-\lambda) = Ax^1\lambda + Ax^2(1-\lambda) = A[x^1\lambda + x^2(1-\lambda)]$. Let $x^3 = x^1\lambda + x^2(1-\lambda)$. By definition $\|x^3\| \leq \alpha$. Hence, $b^3 = Ax^3$ implies that $b^3 \in K(\alpha)$. Hence $K(\alpha)$ is convex.

2. A set $C \subseteq \mathbb{R}^n$ is bounded if there exists a $n-$dimensional ball $B(r) = \{x \in \mathbb{R}^n : \|x\| \leq r\}$ of some finite radius $r$ such that $C \subseteq B(r)$. Show that convex hull of $C$ is bounded.

   **Answer:** By Carathéodory theorem, every point in the convex hull of $C$ can be written as convex combination of $n+1$ points in $C$. Writing down the formula for convex combination gives you that this point will also lie in the $n-$dimensional ball $B(r)$.

3. Suppose $C \subseteq \mathbb{R}^n$ is any arbitrary set. Consider $x \in \mathbb{R}^n$ such that $x \notin H(C)$ (i.e., outside the convex hull of $C$). Can we strictly separate $x$ from $C$? If not, can you think of a condition on $C$ such that $x$ and $C$ can be separated?

   **Answer:** Clearly $x \notin C$ since $C \subseteq H(C)$. Of course $H(C)$ is convex. If $C$ is closed and bounded, then we know that $H(C)$ is closed and bounded. In this case, we can strictly separate $x$ from $H(C)$, and hence from $C$. If $H(C)$ is not closed, we cannot strictly separate. Even if $C$ is closed, $H(C)$ need not be closed (think of an unbounded set $C$).

4. Let $C \subseteq \mathbb{R}^n$. Then $C$ is a closed convex set if and only if $C = \cap\, \mathbb{F}$, where $\mathbb{F}$ is a collection (possibly infinite of them) of half-spaces. (Hint: Consider the intersection of half-spaces that contain $C$ and use separating hyperplane theorem)


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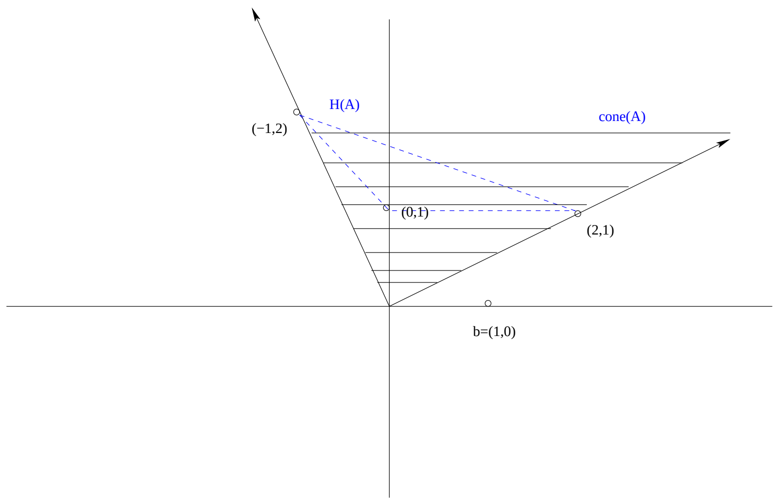


Figure 1: Cones generated by columns of $A$

**Answer:** Suppose $C = \cap\mathbb{F}$ is a collection of half-spaces. Since every half space is closed and convex, their intersection is also closed and convex. Hence, $C$ is closed and convex. For the other direction, suppose $C$ is closed and convex. Consider all the half spaces that contain $C$. Let $\mathbb{F}$ be the collection of all such half spaces. Let $C' = \cap\mathbb{F}$. Assume for contradiction $C \neq C'$. By definition $C \subset C'$. Consider a point $x \in C' \setminus C$. Since $C$ is closed and convex, $x$ can be separated from $C$ by a hyperplane. The half space corresponding to this hyperplane contains $C$, and hence belongs to $\mathbb{F}$, but does not contain $x \in C' \setminus C$. This is a contradiction by the definition of $\mathbb{F}$ and $C'$.

5. Sketch the cone generated by the columns of following matrix:

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

What is the cone generated by just the first and third columns of the matrix? Suppose $b = (1, 0)$. Then decide if $Ax = b$ has a solution with $x \in \mathbb{R}^3_+$.

Draw $cone(A)$ and $H(A)$, the convex hull of columns of $A$. How are the points in $cone(A)$ and $H(A)$ related?

**Answer:** The cones generated by columns of $A$ and the first and third columns of $A$ are the same. It is shown in Figure 1.

As can be seen from Figure 1, $b \in cone(A)$. Convex hull of columns of $A$, $H(A)$ is shown in dotted lines. As can be seen from the figure $H(A) \subseteq cone(A)$.

6. Write down the Farkas alternative for the following system of constraints.

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &\le 5 \\
x_1 + 3x_2 - 2x_3 &\ge 7 \\
x_1 + x_2 + x_3 &\le 2 \\
x_1 - 2x_2 - 3x_3 &= 3 \\
x_1 &\ge 0.
\end{aligned}
$$

**Answer:** First we write the constraints in $\le$ or $=$ form.

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &\le 5 \\
-x_1 - 3x_2 + 2x_3 &\le -7 \\
x_1 + x_2 + x_3 &\le 2 \\
x_1 - 2x_2 - 3x_3 &= 3 \\
x_1 &\ge 0.
\end{aligned}
$$

To write the Farkas alternative, associate a variable for every constraint: $y_1, y_2, y_3, y_4$. Out of this, fourth constraints is equality. Hence $y_4$ must be free, and other variables are non-negative. Similarly, constraints corresponding to $x_1$ must be $\ge$ whereas constraints corresponding to $x_2, x_3$ must be $=$. The Farkas alternative is:

$$
\begin{aligned}
y_1 - y_2 + y_3 + y_4 &\ge 0 \\
2y_1 - 3y_2 + y_3 + 2y_4 &= 0 \\
3y_1 + 2y_2 + y_3 - 3y_4 &= 0 \\
5y_1 - 7y_2 + 2y_3 + 3y_4 &< 0 \\
y_1, y_2, y_3 &\ge 0
\end{aligned}
$$

7. Use Farkas Lemma to decide if the following system of equations have a solution.

$$
\begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}
$$

**Answer:** The Farkas alternative is:

$$
\begin{aligned}
-2y_1 + 3y_2 &< 0 \\
4y_1 + y_2 &= 0 \\
y_1 &= 0 \\
-2y_1 + 5y_2 &= 0.
\end{aligned}
$$

Clearly, this system has no solution since the last three equations imply $y_1 = y_2 = 0$. Hence, the original system has a solution.

8. Let $A$ be a $m \times n$ matrix. Define $F = \{x \in \mathbb{R}^n_+ : Ax = 0, \sum_{j=1}^n x_j = 1\}$ and $G = \{y \in \mathbb{R}^m : yA > 0\}$. Show that either $F \neq \emptyset$ or $G \neq \emptyset$ but not both.

   **Answer:** The Farkas alternative for $F$ is

$$\begin{aligned} yA + \delta &\geq 0 \\ \delta &< 0. \end{aligned}$$

   Suppose $F \neq \emptyset$. Then its Farkas alternative is empty. Hence for any $\delta < 0$, there exists no $y$ such that $yA + \delta \geq 0$. Hence, for every $y$ we have $yA \leq 0$. Hence, $G = \emptyset$. For the converse, suppose $G \neq \emptyset$. Then, we can find $\delta < 0$ such that the Farkas alternative has a solution. Hence, by Farkas lemma $F = \emptyset$.

9. Let $A$ be a $m \times n$ matrix. Prove that the system $Ax = 0$ has a **non-zero, non-negative** solution (i.e, $x \geq 0$ and $x \neq 0$) or there is a $y \in \mathbb{R}^m$ such that $yA > 0$, but not both. (Hint: Use the result from the previous question.)

   **Answer:** Suppose $Ax = 0$ has a non-zero, non-negative solution $x$. Define $\hat{x}_i = \frac{x_i}{\sum_j x_j}$. This is well-defined since $x$ is non-zero. Hence, $F = \{x \in \mathbb{R}^n_+ : Ax = 0, \sum_j x_j = 1\}$ has a solution. By the previous question, $y \in \mathbb{R}^m : yA > 0$ has no solution. Now, suppose $y \in \mathbb{R}^m : yA > 0$ has a solution. Then by previous question $F$ has no solution. This implies that $Ax = 0$ has no non-zero and non-negative solution.