Assignment 4 **Due date. 18 October, 2018**

1. Consider the linear program (**SP**)

$$
\begin{aligned}
Z = \max \sum_{j=1}^{n} c_j x_j & \\
\text{s.t.} & \qquad \textbf{(SP)} \\
\sum_{j=1}^{n} a_j x_j \leq b & \\
x_j \geq 0 \qquad \forall\, j \in \{1, \ldots, n\}. &
\end{aligned}
$$

   Assume that $c_j > 0$ and $a_j > 0$ for all $j \in \{1, \ldots, n\}$, and $b > 0$. Prove that $Z = b \max_{j \in \{1,\ldots,n\}} \frac{c_j}{a_j}$.

2. Consider the linear program (**P**).

$$
\begin{aligned}
\max \sum_{j=1}^{n} c_j x_j & \\
\text{s.t.} & \qquad \textbf{(P)} \\
\sum_{j=1}^{n} a_{ij} x_j \leq b_i & \qquad \forall\, i \in \{1, \ldots, m\} \\
x_j \geq 0 & \qquad \forall\, j \in \{1, \ldots, n\}. \qquad (1)
\end{aligned}
$$

   Show that if (**P**) is unbounded then there exists x_k ($k \in \{1, \ldots, n\}$) such that the following LP (**P**-k) is unbounded.

$$
\begin{aligned}
\max x_k & \\
\text{s.t.} & \qquad (\textbf{P}\text{-}k) \\
\sum_{j=1}^{n} a_{ij} x_j \leq b_i & \qquad \forall\, i \in \{1, \ldots, m\} \\
x_j \geq 0 & \qquad \forall\, j \in \{1, \ldots, n\}. \qquad (2)
\end{aligned}
$$

3. Consider the following linear program.

$$\max x_1 + x_2$$
$$\text{s.t.}$$
$$8x_1 + 5x_2 \leq 32$$
$$8x_1 + 6x_2 \leq 33$$
$$8x_1 + 7x_2 \leq 35$$
$$x_1, x_2 \geq 0.$$

Let x_3, x_4, x_5 be the slack variables corresponding to the first, second, and third constraints respectively. The optimal solution of this linear program was found using the simplex method: $x_1 = 0, x_2 = 5, x_3 = 7, x_4 = 3, x_5 = 0$.

(a) Identify the basic and non-basic variables in the final dictionary of the simplex method.

(b) Write down the final dictionary of the simplex method (you need not solve the linear program).

(c) Write down the dual and the optimal solution of the dual.

4. While solving for the optimal solution of a linear program, we encountered the following dictionary in the second phase of the simplex method.

$$x_2 = 5 + 2x_3 - x_4 - 3x_1$$
$$x_5 = 7 - 3x_4 - 4x_1$$
$$z = 5 + x_3 - x_4 - x_1.$$

(a) If x_1, x_2, x_3 are the original variables and x_4, x_5 are the slack variables, write the original linear program.

(b) Does the linear program have an optimal solution? If yes, find the optimal solution, else argue why it does not have an optimal solution.

(c) Write the dual of this linear program.

(d) Does the dual have an optimal solution? If yes, find the optimal solution, else argue why it does not have an optimal solution.